CHAPTER 3

COMMUTATIVITY OF ALTERNATIVE s-UNITAL RINGS
Tominaga and Yaqub [46], Abujabal, M.A.Khan and M.S. Khan [3], Abujabal [1], Hirano and Yaqub[34], Abujabal and M.S.Khan[2] studied the properties of commutativity of s-unital rings.

In this chapter, we generalize some results on commutativity of alternative s-unital rings. In section 3.1, we show that if \( R \) is an alternative ring satisfying (1) for every \( x, y \) in \( R \) there exist integers \( m = m(x, y) > 1, n = n(x, y) \geq 0 \) and \( k = k(x, y) \geq 0 \) such that \([x, x^n y + y^m x^k] = 0\), then \( R \) is normal. Also if \( R \) satisfies (2) for each \( y \in R \), there exists an integer \( m = m(y) > 1 \) such that \([x, x^n y + y^m x^k] = 0 = [x, x^n y^m + y^{m^2} x^k]\) for all \( x \in R \), where \( k \geq 0, n \neq 1 \) is a fixed non-negative integer, then the commutator ideal \( C(R) \subseteq Z(R) \). Using these results it is shown that if \( R \) is an alternative left or right s-unital ring, then the following statements are equivalent: (i) \( R \) is commutative (ii) \( R \) satisfies (1) and there exist a subset \( A(R) \) of \( N(R) \) for which \( R \) satisfies (3) for each \( x \in R \), either \( x \in Z(R) \) or there exists a polynomial \( f(t) \) in \( Z[t] \) such that \( x - x^2 f(x) \in A(R) \). (iii) \( R \) satisfies (2). In section 3.2, we show that if \( R \) satisfies (i) \( (x - x^n)(y - y^n) = 0 \), then the set of nilpotent elements \( N(R) \) forms an ideal of \( R \) with \( N^2(R) = 0 \) and \([a, x^{k+n}] = [a, x^{k+1}]\), for all \( a \in N(R), x \in R \) and \( k \geq 0 \).

Also if a subdirectly irreducible ring \( R \) with 1 satisfies (i) and (ii) \((xy)^{n+1} - y^{n+1}x^{n+1} \in Z(R)\), then \( R \) is a local ring with radical \( N(R) \) and \( R/N(R) \) is a finite field. We use this to prove the commutativity of an alternative s-unital ring \( R \). In section 3.3, we prove that if \( n > 1, m, r \) are fixed non-negative integers and an alternative ring \( R \) with unity 1 satisfies the polynomial identity (i) \( x[x^n, y] = y^r [x, y^m]y \) for all \( x, y \) in \( R \), then \( C(R) \) is
null and if $R$ is $n$-torsion free, then $N(R) \subseteq Z(R)$. Also we show that an alternative left s-unital ring $R$ satisfying the polynomial identity (i) is commutative.

3.1: s-unital rings with $[x, x^n y + y^m x^k] = 0 = [x, x^n y^m + y^m x^k]$

There is a number of conditions each of which implies the commutativity of certain rings. The equivalence of few such conditions to that of commutativity of rings was established by Tominaga and Yaqub[46]. The list of these equivalent conditions was further enlarged by these authors in [48]. In this section we extend the work of Tominaga and Yaqub[46], Abujabal, M.A.Khan and M.S. Khan[3], Abujabal[1] for rings satisfying more general polynomial identities.

Throughout this section $R$ denotes an alternative s-unital ring, $A(R)$ a nonempty subset of $R$, $V_R(A(R))$ the centralizer of a subset $A(R)$ of $R$ and $Z[t]$ the set of polynomials in $t$ with coefficients in the ring of integers $Z$.

We consider the following ring properties:

(I-A(R)): For each $x \in R$, there exists a polynomial $f(t)$ in $Z[t]$ such that $x - x^2 f(x) \in A(R)$.

(II-A(R)): For each $x \in R$, either $x \in Z(R)$ or there exists a polynomial $f(t)$ in $Z[t]$ such that $x - x^2 f(x) \in A(R)$.

(III-A(R)): For every $a \in A(R)$ and $x \in R$, $[[a, x], x] = 0$. 
(IV): For every $y \in R$, there exist integers $m = m(x, y) > 1$, $n = n(x, y) \geq 0$ and $k = k(x, y) \geq 0$ such that $[x, x^n y + y^m x^k] = 0$.

(V): For each $y \in R$, there exists an integer $m = m(y) > 1$ such that $[x, x^n y + y^m x^k] = 0 = [x, x^n y^m + y^{m^2} x^k]$ for all $x \in R$, where $k \geq 0$, $n \neq 1$ is a fixed non-negative integer.

(VI): For every $x, y \in R$, there exist integers $m > 1$, $k \geq 0$ and $n \geq 0$ such that $[x, x^n y + y^m x^k] = 0$.

To prove the main results, we need the following well-known results:

**Lemma 3.1.1**: Let $R$ be a ring such that $[x, [x, y]] = 0$ for all $x$ and $y$ in $R$, then $[x^k, y] = k x^{k-1} [x, y]$ for any positive integer $k$.

**Proof**: We prove this by induction on $k$.

The identity $[x^k, y] = k x^{k-1} [x, y]$ is true for integer $k = 1$.

Suppose we assume that $[x^k, y] = k x^{k-1} [x, y]$.

Consider $[x^{k+1}, y] = [x^k x, y]$

$$= x^k [x, y] + [x^k, y] x$$

$$= x^k [x, y] + k x^{k-1} [x, y] x$$

$$= x^k [x, y] + k x^k [x, y],$$ since $[x, [x, y]] = 0$. 


\[(k + 1)x^k \leq [x, y], \text{ for all } k > 1.\]

Therefore by induction for all positive integers \(k\), \([x^k, y] = kx^{k-1}[x, y]\). □

**Lemma 3.1.2[6, Lemma 2]:** Let \(R\) be ring with unity 1 and let \(x\) and \(y\) be elements of \(R\). If \(kx^m[x, y] = 0\) and \((k + 1)^m[x, y] = 0\), for some integers \(m \geq 1\) and \(k \geq 1\), then necessarily \(k[x, y] = 0\).

**Lemma 3.1.3[46]:**

(i) Let \(\Phi\) be a ring homomorphism of \(R\) onto \(R^*\). If \(R\) satisfies (I-A(R)), (II-A(R)) or (III-A(R)) then \(R^*\) satisfies (I- \(\Phi\)(A(R))), (II- \(\Phi\)(A(R))) or (III- \(\Phi\)(A(R))) respectively.

(ii) If \(A(R)\) is commutative and \(R\) satisfies (II-A(R)), then \(N(R)\) is a commutative nil ideal of \(R\) containing \(C(R)\) and is contained in \(V_R(A(R))\). In particular, \((N(R))^2 \subseteq Z(R)\).

(iii) If there exists a commutative subset \(A(R)\) of \(N(R)\) for which \(R\) satisfies (II-A(R)) and (III-A(R)), then \(R\) is commutative.

**Lemma 3.1.4[10]:** Let \(R\) be a left (resp. right) s-unital ring. If for each pair of elements \(x\) and \(y\) in \(R\), there exists a positive integer \(k = k(x, y)\) and an element \(e = e(x, y)\) of \(R\) such that \(x^ke = x^k\) and \(y^ke = y^k\) (resp. \(ex^k = x^k\) and \(ey^k = y^k\)), then \(R\) is s-unital.
Lemma 3.1.5[42, Lemma 3]: Let $R$ be a ring with unity and let $k$ and $m$ be natural numbers. If $(1 - y^k)x = 0$ then $(1 - y^{km})x = 0$, for all $x, y$ in $R$.

Lemma 3.1.6[37, Theorem]: Let $f$ be a polynomial in $n$ non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with relatively prime integral coefficients. Then the following are equivalent:

(a) Every ring satisfying the polynomial identity $f = 0$ has a nil commutator ideal.

(b) Every semiprime ring satisfying $f = 0$ is commutative.

(c) For every prime $p$, $(GF(p))_2$, the ring of $2 \times 2$ matrices over the Galois field $GF(p)$, fails to satisfy $f = 0$.

Lemma 3.1.7[33, Theorem 3]: If $R$ is a ring with center $Z(R)$ such that for every $a \in R$ there exists a polynomial $p_a(t)$ such that $a - a^2p_a(t) \in Z(R)$, then $R$ is commutative.

Lemma 3.1.8[29, Theorem 19]: Let $R$ be a ring and let $n = n(x) > 1$ be an integer depending on $x$. If $x^n - x \in Z(R)$ for all $x \in R$, then $R$ is commutative.

The following Lemmas are essential in proving our results.

Lemma 3.1.9: Let $R$ be a ring satisfying (IV). Then $R$ is normal. Proof: Given an idempotent element $e$ and an element $x$ in $R$, then

there exist integer $m = m(e, e + ex(1 - e)) > 1$ and
\[ n = n(e, e + ex(1 - e)) \geq 0 \text{ and } k = k(e, e + ex(1 - e)) \geq 1 \text{ such that } x = e \text{ and } y = e + ex(1 - e), \text{ we have} \]

\[ [e, e^n(e + ex(1 - e)) + (e + ex(1 - e))^m e^k] = 0. \]

So, \( e^{n+1}(e + ex(1 - e)) + e(e + ex(1 - e))^m e^k - e^n(e + ex(1 - e))e - (e + ex(1 - e))^m e^{k+1} = 0. \)

As \( e^k = e \) for all \( k \geq 1 \), we get \( ex(1 - e) + ex(e - e^2) = 0. \)

Hence \( ex(1 - e) = 0 \), that is \( ex = exe. \)

Similarly \( xe = exe. \) Therefore \( ex = xe, \text{ for all } x \in R. \)

Thus \( R \) is Normal. \( \square \)

**Lemma 3.1.10**: Let \( R \) be an alternative ring with unity 1 satisfying (IV). Then \( N(R) \subseteq Z(R). \)

**Proof**: If \( a \in N(R) \) and \( x \in R \), then there exist integers \( m_1 = m(x, a) > 1, n_1 = n(x, a) \geq 0 \) and \( k_1 = k(x, a) \geq 0 \) such that \( x^{n_1}[x, a] = [a^{m_1}, x]x^{k_1}, \) for all \( x \in R. \)

If \( m_2 = m(x, a^{m_1}) > 1, n_2 = n(x, a^{m_1}) \geq 0 \) and \( k_2 = k(x, a^{m_1}) \geq 0, \) then \( x^{n_2}[x, a^{m_1}] = [(a^{m_1})^{m_2}, x]x^{k_2} = [a^{m_1m_2}, x]x^{k_2}, \) for all \( x \in R. \) Thus \( x^{n_1+n_2}[x, a] = -[a^{m_1m_2}, x]x^{k_1+k_2}, \) for all \( x \in R. \)
Hence, for any positive integer $t$,

$$x^{n_1+n_2+\cdots+n_t}[x,a] = (-1)^{t-1}[a^{m_1m_2\cdots m_t}, x]x^{k_1+k_2+\cdots+k_t},$$
for all $x \in R$.

But $a$ is nilpotent, then $a^{m_1m_2\cdots m_t} = 0$, for sufficiently large $t$. So $x^{n_1+n_2+\cdots+n_t}[x,a] = 0$ for all $x \in R$. Let $n'(x) = n_1 + n_2 + \cdots + n_t$.

Then $x^{n'(x)}[x,a] = 0$, for all $x \in R$. For $n' = \max \{n'(x), n'(x+1)\}$, we have $x^{n'}[x,a] = 0$ and $(x+1)^{n'}[x,a] = 0$. Now by Lemma 3.1.2 yields

$$[x,a] = 0, \text{ for all } x \in R.$$ Therefore, $a \in Z(R)$ and thus $N(R) \subseteq Z(R)$. ■

By using Lemma 3.1.7, we have the following:

**Lemma 3.1.11[1]**: Let $R$ be an alternative ring with unity $1$ satisfying (IV) and (II-A(R)) for a subset $A(R)$ of $N(R)$. Then $R$ is commutative.

**Lemma 3.1.12**: Let $R$ be an alternative ring with unity $1$ satisfying (V). Then $C(R) \subseteq Z(R)$.

**Proof**: Let $n \geq 0$ be a fixed integer. Then for any $y \in R$, there exists an integer $m = m(y) > 1$ and $k = k(y) > 1$ such that condition (V) can be rewritten as $x^n[x,y] = \left[y^m, x\right]x^k$, for all $x \in R$ \hspace{1cm} 3.1.1

and $x^n[x,y^m] = \left[y^{m^2}, x\right]x^k$, for all $x \in R$. \hspace{1cm} 3.1.2

Now replacing $x$ by $x+1$ in 3.1.1, we get
\[(x + 1)^n [x, y] x^k = [y^m, x] (x + 1)^k x^k\]

\[= x^n [x, y] (x + 1)^k, \text{ for all } x, y \in R.\]

By Lemma 3.1.6, we observe that \(C(R)\) is a nil ideal, that is \(C(R) \subseteq Z(R)\). Since \(x = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) and \(y = e_{12} + e_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) fail to satisfy the identity \((x + 1)^n [x, y] x^k - x^n [x, y] (x + 1)^k = 0\) in \((GF(p))_2\), for a prime \(p\). Hence, by Lemma 3.1.10, \(C(R) \subseteq N(R) \subseteq Z(R)\).

\[\text{Lemma 3.1.13:}\] Let \(R\) be an alternative ring with unity 1 satisfying (V). Then \(R\) is commutative.

**Proof:** For \(n = 0 = k\), we get \([x, y] = [y^m, x]\), for all \(x, y \in R\). So by the commutativity of \(R\) follows from Lemma 3.1.8. Now, we suppose that \(n > 1\) or \(k > 1\). If \(k = n\), then \(x^n [x, y] = [y^m, x] x^n\), and by Lemma 3.1.10, we get \(x^n [x, y] = x^n [x, y^m]\). Therefore, \(x^n [x, y - y^m] = 0\) and \((x + 1)^n [x, y - y^m] = 0\), for all \(x, y \in R\). By Lemma 3.1.2, we have \([x, y - y^m] = 0\), for all \(x, y \in R\). Therefore, \(R\) is commutative by Lemma 3.1.8. Without loss of generality, we suppose that \(n > k\).

Let \(t = 2^{n+1} - 2^{k+1}\). Then \(t > 0\) for \(n > k\). By using 3.1.1, we get

\[tx^n [x, y] = (2^{n+1} - 2^{k+1}) x^n [x, y]\]

\[= 2^{n+1} x^n [x, y] - 2^{k+1} x^n [x, y]\]

\[= (2x)^n [(2x), y] - 2^{k+1} [y^m, x] x^k\]
\[(2x)^n[(2x), y] - [y^m, (2x)](2x)^k\]

\[= 0.\]

By replacing \(x\) by \(x + 1\) and using Lemma 3.1.2 gives \(t[x, y] = 0\). Again, Lemma 3.1.1 and Lemma 3.1.12 together imply that \([x^t, y] = tx^{t-1}[x, y] = 0\), for all \(x, y \in R\). So, \(x^t \in Z(R)\), for all \(x \in R\).

Further, using 3.1.1, 3.1.2 and the fact that \(C(R) \subseteq Z(R)\). We see that

\[(1 - y^{(m-1)^2})[x, y]x^{2n-k} = [x, y]x^{2n-k} - y^{(m-1)^2}[x, y]x^{2n-k}\]

\[= [y^m, x]x^n - y^{(m-1)^2}[y^m, x]x^n\]

\[= -x^n[x, y^m] - my^{m-1}y^{(m-1)^2}[y, x]x^n\]

\[= -x^n[x, y^m] + my^{m(m-1)}x^n[x, y]\]

\[= -x^n[x, y^m] + my^{m(m-1)}[y^m, x]x^k\]

\[= -x^n[x, y^m] + [y^{m^2}, x]x^k = 0.\]

This implies that \((1 - y^{(m-1)^2})[x, y]x^{2n-k} = 0\), for all \(x, y \in R\).

By replacing \(x\) by \(x + 1\), we get \((1 - y^{(m-1)^2})[x, y](x + 1)^{2n-k} = 0\), for all \(x, y \in R\). By Lemma 3.1.2, we obtain

\[(1 - y^{(m-1)^2})[x, y] = 0, \text{ for all } x, y \in R.\] 3.1.3
But since, $x^t \in Z(R)$ for all $x \in R$, we get

$$[x, y - y^t(m-1)^2+1] = (1 - y^t(m-1)^2)[x, y] = 0,$$

for all $x, y \in R$.

Thus $y^t(m-1)^2+1 \in Z(R)$, so $m = m(y) > 1$. Therefore, $R$ is commutative by Lemma 3.1.8.

**Theorem 3.1.1**: An alternative ring $R$ is commutative if and only if $R$ satisfies (IV) and (II-A(R)) for a commutative subset $A(R)$ of $N(R)$.

**Proof**: It is easy to see that a commutative ring $R$ satisfies the conditions given in the theorem. Now, let $R$ be an alternative ring satisfying the hypothesis of our theorem. If $R$ has unity 1, then the result follows from the Lemma 3.1.11. So we suppose that $R$ does not contain unity 1. In view of Lemma 3.1.3(i), $R$ can be assumed to be a subdirectly irreducible ring without unity 1. Let $x \in R \setminus Z(R)$ be an arbitrary element. By hypothesis, $R$ satisfies (II-A(R)) for a commutative subset $A(R)$ of $N(R)$, and thus, there exists an element $y \in < x >$, the subring generated by $x$, and an integer $m > 1$ such that $x^m = x^{m+1}y$. Clearly, $e = x^m y^m$ is idempotent with $x^m = x^m e$, and also $e$ is central by Lemma 3.1.5. Since $R$ has no unity, $e = 0$. Again by Lemma 3.1.3(ii), $x$ is in the commutative ideal $N(R)$ and $[x, [x, a]] = 0$ for all $a \in A(R)$. Hence $R$ is commutative by Lemma 3.1.3(iii).
Theorem 3.1.2: If $R$ is an alternative left or right s-unital ring, then the following statements are equivalent:

(i) $R$ is commutative

(ii) $R$ satisfies (IV) and there exists a subset $A(R)$ of $N(R)$ for which $R$ satisfies (II-A(R)).

(iii) $R$ satisfies (V).

Proof: If $R$ is a commutative left or right s-unital ring, then clearly, $R$ satisfies (ii) and (iii). Now, suppose that $R$ satisfies (ii). First, we show that $R$ is s-unital. Let $R$ be a right s-unital ring, and let $x$ and $y$ be arbitrary elements of $R$. Then we can find an element $e \in R$ such that $xe = x$ and $ye = y$. Further, there exist integers $m = m(x, e) > 1$, $n = n(x, e) \geq 1$ and $k = k(x, e) \geq 0$ such that $e^m x^{k+1} = -[x, x^n e + e^m x^k] + x^{k+1} = x^{k+1}$. Similarly, there exist integers $m' = m'(y, e) > 1$, $n' = n'(y, e) \geq 1$ and $k' = k'(x, e) \geq 0$ such that $e^{m'} y^{k'+1} = y^{k'+1}$. Therefore, $e^{mm'} x^{k'+1} = x^{k'+1}$ and $e^{mm'} y^{k'+1} = y^{k'+1}$. Hence, $R$ is an s-unital by Lemma 3.1.4.

Now, suppose that $R$ is a left s-unital ring. Let $x, y \in R$. Then there exists an element $e \in R$ such that $ex = x$ and $ey = y$. Also, there exists integers $m = m(x, e) > 1$, $n = n(x, e) \geq 1$ and $k = k(x, e) \geq 0$ such that $x^{n+1} e = [x, x^n e + e^m x^k] + x^{n+1} = x^{n+1}$. Similarly, if $m' = m(y, e) > 1$, $k' = k(y, e) \geq 0$ and $n' = n(y, e) \geq 1$, then we have $y^{n'+1} e = y^{n'+1}$. Hence, $x^{n+n'+1} e = x^{n+n'+1}$ and $y^{n+n'+1} e = y^{n+n'+1}$. Again, by Lemma 3.1.4, $R$ is an s-unital ring.
In view of proposition 1 of [35], we assume that $R$ has unity 1. Hence $R$ is commutative by Lemma 3.1.11. Thus (ii) implies (i).

Finally, if $R$ satisfies (iii), then as argued above, we assume that $R$ has unity 1. Hence, $R$ is commutative by Lemma 3.1.13.

Example 3.1.1: Theorem 3.1.1 need not be true if we drop the condition that $A(R)$ is commutative. For this, consider

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in GF(2) \right\}.$$

Then $R$ is a nilpotent ring of index 3 and also $N(R) = R$. Further, $R$ satisfies (IV). However, with $A(R) = N(R)$, $R$ also satisfies (II-A(R)). But $R$ is not commutative.

Example 3.1.2: This example shows that both conditions (IV) and (II-A(R)) in Theorem 3.1.2(ii) are essential for the ring $R$ with unity 1 to be commutative. Let $R = \left\{ aI + S : S = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b, c, d \in GF(2) \right\}$.

Then, it is easy to check that $N(R) = S$ and $R$ does not satisfy (IV). Let $A(R) = N(R)$. Then for all $x \in R$, we have $x - x^2 f(x) \in A(R)$. However, $R$ is not commutative.

3.2: s-unital rings with $(xy)^{n+1} - y^{n+1}x^{n+1} \in Z(R)$
A well-known theorem of Jacobson asserts that if $R$ is a ring with the property that for every $x$ in $R$ there exists an integer $n > 1$ such that $x^n = x$, then $R$ is commutative. With this motivation, Hirano and Yaqub[34] considered the structure of a ring $R$ which satisfies the identity $(x - x^n)(y - y^n) = 0$, for all $x, y$ in $R$, $n > 1$ is fixed. In this section, we show that if $R$ satisfies (i) $(x - x^n)(y - y^n) = 0$, then the set of nilpotent elements $N(R)$ forms an ideal of $R$ with $N^2(R) = 0$ and $[a, x^{k+n}] = [a, x^{k+1}]$, for all $a \in N(R), x \in R$ and $k \geq 0$. Also if a subdirectly irreducible ring $R$ with 1 satisfies (i) and (ii) $(xy)^{n+1} - y^{n+1}x^{n+1} \in Z(R)$, then $R$ is a local ring with radical $N(R)$ and $R/N(R)$ is a finite field. We use this to prove the commutativity of an alternative s-unital ring $R$.

In order to prove our main results, we first state a number of well-known results.

**Lemma 3.2.1[5]**: Suppose that i) for every $x$ in $R$, $x - x^n \in N(R)$ with some integer $n > 1$, where $N(R)$ is the set of nilpotents in $R$. ii) $N(R)$ is commutative and iii) for all $x$ in $R$ and $a$ in $N(R)$, $[x, [x, a]] = 0$, then $R$ is commutative.

**Lemma 3.2.2[40]**: Let $R$ be a ring with identity and suppose that $[x^h, y^h] = 0$ and $[x^k, y^k] = 0$ for all $x, y$ in $R$, where $h$ and $k$ are fixed relatively prime positive integers, then $R$ is commutative.

We now prove the following Lemmas which are essential in proving our results:
Lemma 3.2.3: Suppose that $R$ satisfies $(x - x^n)(y - y^n) = 0$ for all $x, y$ in $R$, then $N(R)$ forms an ideal of $R$ with $N^2(R) = 0$ and $[a, x^{k+n}] = [a, x^{k+1}]$, for all $a \in N(R), x \in R$ and $k \geq 0$. In particular, if $R$ has 1 and $x$ is invertible then $[a, x^{n-1}] = 0$.

Proof: By 3.2.1, $(a - a^n)^2 = 0$, namely $a^2 = a^2(2a^{n-1} - a^{2(n-1)})$. Since $n - 1 > 0$, we can easily that $a^2 = 0$. Now let $b \in N(R)$, then $0 = (a - a^n)(b - b^n) = ab$. Hence $N(R)$ forms an ideal by [47, Lemma 1(1)]. By using 3.2.1, we get $(x - x^n) \in N(R)$.

Now $[a, x^{k+1}] - [a, x^{k+n}] = ax^{k+1} - x^{k+1}a - ax^{k+n} + x^{k+n}a$

$= a(x - x^n)x^k - x^k(x - x^n)a$

$= 0$. ■

Lemma 3.2.4: Let $R$ be a ring with 1. If $R$ satisfies $(xy)^{n+1} - y^n x^{n+1} \in Z(R)$, for all $x, y$ in $R$, then $R$ is normal.

Proof: Let $e$ be an idempotent in $R$ and $r \in R$. By putting $x = e + er(1 - e)$ and $y = 1 - e$ in 3.2.2, we get

$0 = [1 - e, (er(1 - e))^{n+1} - er(1 - e)] = er(1 - e)$. Hence $er = ere$. Similarly, we get $re = ere$. Therefore $R$ is normal. ■

Lemma 3.2.5: Let $R$ be a subdirectly irreducible ring with 1. If $R$ satisfies $(x - x^n)(y - y^n) = 0$ and $(xy)^{n+1} - y^{n+1}x^{n+1} \in Z(R)$, for all $x, y$ in $R$, then $R$ is a local ring with radical $N(R)$ and $R/N(R)$ is a finite field.
Proof: Let $R$ be an arbitrary element in $R/N(R)$. By using 3.2.1, $r^2 = r^3 f(r)$, for some $f(t) \in Z[t]$. Obviously, $e = (rf(r))^2$ is an idempotent with $r^2e = r^2$. Hence $e$ is a nonzero central idempotent by Lemma 3.2.4. Since $R$ is subdirectly irreducible, $e = 1$ and $r$ is invertible. Thus we have thus seen that $R$ is a local ring with radical $N(R)$. Since $R/N(R)$ satisfies the identity $(x - x^n) = 0$, by Jacobson’s theorem, we see that $R/N(R)$ is a finite field.

Theorem 3.2.1: Let $R$ be an alternative s-unital ring with center $Z(R)$ and $n > 1$ a fixed integer. Suppose $(x - x^n)(y - y^n) = 0$ and $(xy)^n - y^n x^n \in Z(R)$, for all $x, y$ in $R$. 3.2.3 If for all $x$ in $R$ and $a$ in $N(R)$, $(n - 1)[x, a] = 0$ implies $[x, a] = 0$ then $R$ is commutative, where $N(R)$ is the set of nilpotents in $R$.

Proof: Let $x$ in $R$, $a$ in $N(R)$ and choose a pseudo identity $e$ of $\{a, x\}$, then by Lemma 3.2.3, we have

$$(n - 1)[x, a] = (n - 1)[x^n, a]$$

$$= \{(e + a)x^n - x^n(e + a)\} - ((x + e)^n - (e + a)^n x^n) \in Z(R)$$

So, $(n - 1)[y, [x, a]] = 0$, for all $y$ in $R$.

Hence $[y, [x, a]] = 0$ and therefore $R$ is commutative.
**Theorem 3.2.2:** Let $R$ be an alternative s-unital ring with center $Z(R)$ and $n > 1$ a fixed integer. Suppose $(x - x^n)(y - y^n) = 0$ and

$$(xy)^{n+1} - y^{n+1}x^{n+1} \in Z(R),$$

for all $x, y$ in $R$, then $R$ is commutative.

**Proof:** In view of [35, proposition 1], we assume that $R$ has 1. By Lemma 3.2.3 and Lemma 3.2.4, we assume further that $R$ is a local ring with radical $N(R)$, $N^2(R) = 0$ and $R/N(R) = GF(p^a)$ with some prime $p$. Let $u, v$ be units in $R$, $a$ in $N(R)$ and $x, y$ in $R$.

By using Lemma 3.2.3, we get

$$n[x^2,a] = n[x^{n+1},a] =$$

$$\{(1+a)x^{n+1} - (1+a)^{n+1}x^n\} - \{(x(1+a))^{n+1} - x^{n+1}(1+a)^{n+1}\}.$$

Since $R/N(R)$ is commutative and $N^2(R) = 0$, we have

$$[(uv)^n - v^n u^n, x] = [v^{-1}((vu)^{n+1} - v^{n+1}u^{n+1})u^{-1}, x]$$

$$= ((vu)^{n+1} - v^{n+1}u^{n+1})[v^{-1}u^{-1}, x] = 0.$$

So, $(uv)^n - v^n u^n \in Z(R)$. 

Hence \((n + 1)[u, a] = n[u^n, a] =\)
\[
\left\{ ((1 + a)x)^{n+1} - (1 + a)^{n+1}x^{n+1} \right\} - \left\{ (x(1 + a))^{n+1} - x^{n+1}(1 + a)^{n+1} \right\} \in Z(R).
\]

Since both \(n[u^2, a]\) and \((n + 1)[u^2, a]\) are in \(Z(R)\), we get \([u^2, a] \in Z(R)\). Therefore, \([x^2, a] \in Z(R)\).

If \(p \neq 2\) then \(2[u, a] = [(1 + u)^2 - u^2, a] \in Z(R)\) and \(p^2[u, a] = 0\) implies \([u, a] \in Z(R)\). On the other hand, if \(p = 2\) then \(u^2a - u \in N(R)\), and so \([u, a] = [u, a] - [u - u^2a, a] = [u^2a, a] \in Z(R)\). Thus in either case, \([x, a] \in Z(R)\) and therefore \(R\) is commutative by Lemma 3.2.1.

Consider the following examples:

**Example 3.2.1**: Consider the ring

\[
R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : 0, 1 \in GF(2) \right\}.
\]

The condition \((x - x^n)(y - y^n) = 0\) holds in \(R\) for all positive integers \(n\) but \(R\) is not commutative.

**Example 3.2.2**: Consider the ring

\[
R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : 0, 1 \in GF(2) \right\}.
\]

Obvisously, \(R\) is not s-unital but satisfies 3.2.1, 3.2.2 and 3.2.3 for \(n = 2\). This example shows that in Theorem 3.2.1 and
Theorem 3.2.2: The hypothesis that \( R \) is s-unital cannot be deleted.

\[ \]

3.3: Left s-unital rings with \( x[x^n, y] = y^r[x, y^m]y \)

Abujabal and M.S. Khan [2] studied the commutativity of a left s-unital ring \( R \) satisfying the polynomial identity \( x^t[x^n, y] = y^r[x, y^m]y^s \), for all \( x, y \) in \( R \). In this section, we prove that if \( n>1, m, r \) are fixed nonnegative integers and an alternative ring \( R \) with unity 1 satisfies the polynomial identity (i) \( x[x^n, y] = y^r[x, y^m]y \) for all \( x, y \) in \( R \), then \( C(R) \) is nil and if \( R \) is \( n \)-torsion free, then \( N(R) \subseteq Z(R) \). Also we show that an alternative left s-unital ring \( R \) satisfying the polynomial identity (i) is commutative.

Throughout this section \( R \) denotes an alternative left s-unital ring, \( Z(R) \) the center of \( R \), \( C(R) \) the commutator ideal of \( R \), \( N(R) \) the set of all nilpotent elements of \( R \), \( N'(R) \) the set of all zero divisors in \( R \), \( GF(p) \) the Galois field with \( p \) elements and \((GF(p))_2\) the ring of all 2x2 matrices over \( GF(p) \).

In order to prove our results, we shall require the following well-known results.

**Lemma 3.3.1[11, Lemma 2]:** Let \( R \) be a ring with unity 1, and let \( x \) and \( y \) be elements in \( R \). If \( kx^m[x, y] = 0 \) and \( k(x + 1)^m[x, y] = 0 \), for some integers \( m \geq 1 \) and \( k \geq 1 \), then necessarily \( k[x, y] = 0 \).
Lemma 3.3.2 [42, Lemma 3]: Let $R$ be a ring with unity 1, and let $x$ and $y$ be elements in $R$. If $(1 - y^k)x = 0$, then $(1 - y^{km})x = 0$, for some integers $k > 0$ and $m > 0$.

Lemma 3.3.3 [2]: Let $x$ and $y$ be elements in a ring $R$. Suppose that there exists relatively prime positive integers $m$ and $n$ such that $m[x, y] = 0$ and $n[x, y] = 0$ then $[x, y] = 0$.

Lemma 3.3.4 [14, Theorem 4(c)]: Let $R$ be a ring with unity 1. Suppose that for each $x$ in $R$ there exists a pair $n$ and $m$ of relatively prime positive integers for which $x^n \in Z(R)$ and $x^m \in Z(R)$, then $R$ is commutative.

Lemma 3.3.5 [27, Theorem 18]: Let $R$ be a ring and let $n>1$ be an integer. Suppose that $(x^n - x) \in Z(R)$, for all $x$ in $R$, then $R$ is commutative.

Lemma 3.3.6 [29, Theorem]: If for every $x$ and $y$ in a ring $R$ we can find a polynomial $p_{x,y}(t)$ with integral coefficients which depends on $x$ and $y$ such that $[x^2 p_{x,y}(x) - x, y] = 0$, then $R$ is commutative.

We first prove the following Lemmas:

Lemma 3.3.7: Let $n>0$, $m$ and $r$ be fixed non negative integers such that $(r,n,m) \neq (0,1,1)$ and let $R$ be an alternative left $s$-unital ring satisfying the polynomial identity

$$x[x^n, y] = y^r[x^{ym}]y, \text{ for all } x, y \text{ in } R, \quad 3.3.1$$

then $R$ is an $s$-unital ring.
Proof: Let $x$ and $y$ be arbitrary elements in $R$. Suppose that $R$ is an alternative s-unital ring. Then there exists an element $e \in R$ such that $ex = x$ and $ey = y$. By replacing $x$ by $e$ in 3.3.1, we get $e[e^n, y] = y^r[e, y^m]y$

$$e(e^n y - ye^n) = y^r(ey^m - y^m e)y$$

$$e(y - ye^n) = y^r(y^m - y^m e)y$$

$$ey - e ye^n = (y^{r+m} - y^{r+m} e)y$$

$$y - ye^n = y^{r+m+1} - y^{r+m} ey$$

$$y - ye^n = y^{r+m+1} - y^{r+m+1}$$

$$y - ye^n = 0.$$ 

So $y = ye^n \in yR$, for all $y$ in $R$.

Thus $R$ is an s-unital ring. ■

Lemma 3.3.8: Let $n \geq 0$, $r$, $m$ be fixed non-negative integers and let $R$ be an alternative ring satisfying the polynomial identity $x[x^n, y] = y^r[x, y^m]y$, for all $x, y$ in $R$, then $C(R)$ is nil.

Proof: Let $x = e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $x$ and $y$ fail to satisfy the polynomial identity whenever $n > 0$ except for $r = 0, m = 1$. 
In this later case we can choose \( x = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( y = e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). Hence Lemma 3.1.6 ensures that \( C(R) \subseteq N(R) \).

**Lemma 3.3.9**: Let \( n > 1, m \) and \( R \) be fixed non-negative integers and let \( R \) be an alternative ring with unity 1. Suppose that \( R \) satisfies the polynomial identity \( x[x^n, y] = y^r[x, y^m]y \), for all \( x, y \) in \( R \). Further, if \( R \) is \( n \)-torsion free then \( N(R) \subseteq Z(R) \).

**Proof**: Let \( a \in N(R) \) then there exists a positive integer \( p \) such that \( a^k \in Z(R) \) for all \( k \geq p \) and \( p \) minimal. \( \triangleright \)

If \( p = 1 \) then \( a \in Z(R) \).

Now suppose that \( p > 1 \) and \( b = a^{p-1} \).

By replacing \( x \) by \( b \) in the polynomial identity, we get \( b[b^n, y] = y^r[b, y^m]y \), for all \( x, y \) in \( R \).

By using 3.3.3 and the fact that \( (p-1)n \geq p \) for \( n > 1 \),

we get \( a^{p-1}[a^{(p-1)n}, y] = y^r[a^{p-1}, y^m]y \)

\[ = y^r[b, y^m]y = 0, \text{ for all } y \in R. \] \( \triangleright \)
By replacing $x$ by $1+b$ in the polynomial identity, we get

$$(1 + b)[(1 + b)^n, y] = y^r [1 + b, y^m]y,$$ for all $y$ in $R$.

As $(1+b)$ is invertible and using 3.3.4, we get

$$[(1 + b)^n, y] = 0,$$ for all $y$ in $R$.  \hspace{1cm} 3.3.5

By using 3.3.3 and 3.3.5, we get $[(1 + b)^n, y] = 0$.

That is, $[1 + nb), y] = 0$.

So, $n[b, y] = 0$, for all $y$ in $R$.

Since $R$ is $n$-torsion free, we get $[b, y] = 0$, for all $y$ in $R$.

So, $b \in Z(R)$.

That is, $a^{p-1} \in Z(R)$.

This contradicts the minimality of $p$.

So we conclude that $p = 1$ and hence $a \in Z(R)$.

Therefore, $N(R) \subseteq Z(R)$.  \hspace{1cm} 3.3.6

Combining 3.3.2 and 3.3.6, we get

$C(R) \subseteq N(R) \subseteq Z(R)$.  \hspace{1cm} 3.3.7
Theorem 3.3.1: Let \( n > 1, m, r \) be fixed non-negative integers and let \( R \) be an alternative left unital ring satisfying the polynomial identity \( x[x^n, y] = y^r[x, y^m]y \), for all \( x, y \) in \( R \). Further, if \( R \) is \( n \)-torsion free, then \( R \) is commutative.

Proof: According to Lemma 3.3.7, \( R \) is an s-unital ring.

Therefore, in view of proposition 1 of [35], it is sufficient to prove the theorem for \( R \) with unity.

If \( m = 0 \), then 3.3.1 gives \( x[x^n, y] = 0 \), for all \( x, y \) in \( R \).

Hence \( nx^n[x, y] = 0 \), for all \( x, y \) in \( R \).

By replacing \( x \) by \( x+1 \) and applying Lemma 3.3.1, we obtain \( n[x,y] = 0 \), for all \( x, y \) in \( R \).

Since \( R \) is \( n \)-torsion free, we get \( [x,y] = 0 \), for all \( x, y \) in \( R \).

Therefore, \( R \) is commutative.

Now, we consider \( m \geq 1 \). Let \( q = (2^{n+1} - 2) \). Then from 3.3.1 we have

\[
qx[x^n, y] = (2^{n+1} - 2) x[x^n, y]
\]

\[
= 2^{n+1} x[x^n, y] - 2x[x^n, y]
\]

\[
= (2x) [(2x)^n, y] - 2y^r[x, y^m]y
\]

\[
= (2x) [(2x)^n, y] - y^r[(2x), y^m]y
\]
Therefore, \( q x [x^n, y] = 0 \).

So, \( qn x^n [x, y] = 0 \), for all \( x, y \in R \).

By replacing \( qn \) by \( k \) and using Lemma 3.3.1, we obtain \( k [x, y] = 0 \), for all \( x, y \in R \).

Thus \( [x^k, y] = k x^{k-1} [x, y] = 0 \), for all \( x, y \in R \).

So \( x^k \in Z(R) \), for all \( x, y \in R \). \hspace{1cm} 3.3.8

Here we distinguish between the two cases.

**Case (a)**: Let \( m > 1 \). Then from 3.3.1 and 3.3.7 we have,

\[
x^n [x^n, y] = m [x, y] y^{r+m}, \text{ for all } x, y \in R.
\]

By replacing \( y \) by \( y^m \), we get \( x^n [x^n, y^m] = m [x, y^m] y^{m(r+m)} \).

So, \( m x^n [x, y] y^{m-1} = m [x, y^m] y^{m(r+m)} \), for all \( x, y \in R \).

By using 3.3.1, we get \( m y^r [x, y^m] y^m = m [x, y^m] y^{m(r+m)} \).

\[
m [x, y^m] y^{m+r} - m [x, y^m] y^{m(r+m)} = 0.
\]

\[
m [x, y^m] y^{r+m} (1 - y^{(m-1)(r+m)}) = 0, \text{ for all } x, y \in R.
\]

By using Lemma 3.3.2, we get
\(m[x, y^m]y^{r+m} (1 - y^{k(m-1)(r+m)}) = 0\), for all \(x, y\) in \(R\). \hspace{1cm} 3.3.9

Now by using 3.3.6 the polynomial identity 3.3.1 becomes

\[nx^n[x, y] = my^{r+m}[x, y] = m[x, y]y^{r+m}.\] 3.3.10

It is well known that \(R\) is isomorphic to a subdirect sum of subdirectly irreducible rings \(R_i, i \in I\), the Index set. Each \(R_i\) satisfies 3.3.1, 3.3.7, 3.3.8, 3.3.9 and 3.3.10 but not necessarily \(n\)-torsion free.

We consider the ring \(R_i, i \in I\). Let \(S\) be the intersection of all nonzero ideals of \(R_i\), then \(S \neq (0)\) and \(Sd = 0\), for any central zero-divisor \(d\).

Let \(a \in N'(R_i)\), the set of all zero-divisors of \(R\) then by using 3.3.9, we have

\[m[x, a^m]a^{r+m} (1 - a^{k(m-1)(r+m)}) = 0\], for all \(x\) in \(R_i\).

Suppose \(m[x, a^m]a^{r+m} \neq 0\), for \(x\) in \(R_i\).

So, \(a^{k(m-1)(r+m)}\) and \(1 - a^{k(m-1)(r+m)}\) are central zerodivisors.

That is, \((0) = S(1 - a^{k(m-1)(r+m)}) = S \neq (0)\), which is a contradiction.

Therefore \(m[x, a^m]a^{r+m} = 0\), for all \(x\) in \(R_i\). \hspace{1cm} 3.3.11

From 3.3.10 and 3.3.11, we have \(nx^n[x, a^m] = m[x, a^m]a^{m(r+m)} = 0\).

Therefore by Lemma 3.3.1, we get \(n[x, a^m] = 0\), for all \(x\) in \(R_i\).
Hence \( nm[x, a]a^{m-1} = 0 \), for all \( x \) in \( R \).

Now by Lemma 3.1.1, we have
\[
n^2x^n[x, a] = n(nx^n[x, a])
\]
\[
= nm[x, a]a^{r+m}, \text{for all } x \text{ in } R.
\]

By replacing \( x \) by \( x+1 \) and applying Lemma 3.3.1, we get
\[
n^2[x, a] = 0, \text{ for all } x \text{ in } R.
\]

But \([x^{n^2}, a] = n^2x^{n^2-1}[x, a]\).

Therefore \([x^{n^2}, a] = 0\), for all \( x \) in \( R \), and \( a \) in \( N'(R) \). \(3.3.12\)

Let \( c \in Z(R) \). Then by 3.3.1, we have
\[
(c^{n+1} - c)x[x^n, y] = c^{n+1}x[x^n, y] - cx[x^n, y].
\]
\[
= (cx)[(cx)^n, y] - cy^r[x, y^n]y.
\]
\[
= (cx)[(cx)^n, y] - y^r[(cx), y^m]y.
\]
\[
= 0, \text{ for all } x, y \text{ in } R.
\]

By applying Lemma 3.1.1, we obtain \( n(c^{n+1} - c)x^n[x^n, y] = 0 \), for all \( x, y \) in \( R \).

By using Lemma 3.3.1, we obtain \( n(c^{n+1} - c)[x, y] = 0 \) which implies
\[
(c^{n+1} - c)[x^n, y] = 0, \text{ for all } x, y \text{ in } R, \text{ and } c \in Z(R) \). \(3.3.13\)
In particular, by 3.3.8, we have

\[(y^{k(n+1)} - y^k)[x^n, y] = 0, \text{ for all } x, y \text{ in } R_i\]  \hspace{1cm} 3.3.14

Consider \(y \in R_i\). If \([x^n, y] = 0\) then clearly \([x^{n^2}, y^j - y] = 0\), for all positive integers \(j\) and \(x\) in \(R_i\).

If \([x^{n^2}, y] \neq 0\) then \([x^n, y] \neq 0\). For \([x^n, y] = 0\) implies that \([x^{n^2}, y] = 0\), which is a contradiction.

Since \([x^n, y] \neq 0\), then by 3.3.14 \((y^{k(n+1)} - y^k)\) is a zerodivisor.

Therefore \((y^{kn+1} - y)\) is also a zerodivisor.

Hence by 3.3.12, \([x^{n^2}, y^{kn+1} - y] = 0\), for all \(x, y \text{ in } R_i\).  \hspace{1cm} 3.3.15

As each \(R_i\) satisfies 3.3.15, the original ring \(R\) also satisfies 3.3.15. But \(R\) is \(n\)-torsion free. Therefore combining 3.3.15 with Lemma 3.1.1, we finally obtain \([x, y^{kn+1} - y] = 0\), for all \(x, y \text{ in } R\).

Thus \(R\) is commutative by Lemma 3.3.5.

**Case (b) :** Let \(m = 1\), Then we get \(x[x^n, y] = y^r[x, y]y\), for all \(x, y \in R\).

Thus \(nx^n[x, y] = [x, y]y^{r+1}\), for all \(x, y \in R\).  \hspace{1cm} 3.3.16
By replacing $x$ by $x^n$ in 3.3.16, we get
\[ nx^{n^2}[x^n, y] = [x^n, y]y^{r+1} \]
\[ = nx^{n-1}[x, y]y^{r+1} \]
\[ = nx^n[x^n, y], \text{ for all } x, y \text{ in } R. \]

Therefore, \( n(1 - x^{(n-1)n})x^n[x^n, y] = 0, \text{ for all } x, y \text{ in } R. \)

By using Lemma 3.3.2, we get
\[ n(1 - x^{k(n-1)n})x^n[x^n, y] = 0, \text{ for all } x, y \text{ in } R. \] 3.3.17

As in case (a), if \( a \in N'(R_i) \) then by 3.3.17, we obtain
\[ n(1 - a^{k(n-1)n})a^n[a^n, y] = 0, \text{ for all } y \in R_i. \]

By similar argument as in case (a), we can prove that
\[ na^n[a^n, y] = 0, \text{ for all } y \in R_i. \] 3.3.18

Now we have \([a^n, y]y^{r+1} = na^{n^2}[a^n, y] = 0.\)

By using Lemma 3.3.1, we get \([a^n, y] = 0, \text{ for all } y \text{ in } R_i.\)

Therefore, \([a, y]y^{r+1} = a[a^n, y] = 0.\)
So \( [a, y] = 0 \), for all \( y \) in \( R_i \) and \( a \in N(R_i) \).

If \( c \in Z(R_i) \), then as in case (a), we obtain \( (c^{n+1} - c) [x, y] = 0 \), for all \( x, y \) in \( R_i \).

In particular by 3.3.8, we have \( (x^{k(n+1)} - x^k) [x, y] = 0 \), for all \( x, y \) in \( R_i \).

If \( [x, y] = 0 \) for all \( x, y \) in \( R_i \), then \( R \) satisfies \( [x, y] = 0 \), for all \( x, y \) in \( R \). Therefore, \( R \) is commutative.

Now if for each \( x, y \) in \( R_i \), \( [x, y] \neq 0 \) then \( (x^{kn+1} - x) \in N(R_i) \) and hence

\[
(x^{kn+1} - x) \in N(R) .
\]

But the identity 3.3.19 is satisfied by the original ring \( R \).

Therefore, \( (x^{kn+1} - x, y) = 0 \), for all \( x, y \) in \( R \).

Hence \( R \) is commutative by Lemma 3.3.5.

In Theorem 3.3.1, \( n \)-torsion free property is essential. Consider the following example:

**Example 3.3.1** : Let 
\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]
be the elements of the ring of all 3x3 matrices over \( Z_2 \), the ring of integers mod 2. If \( R \) is the ring generated by the matrices \( A, B, C \), then using Dooroh construction with \( Z_2 \), we
obtain with unity 1. Then $R$ is not commutative and satisfies $[x^2, y] = [x, y^2]$, for all $x, y$ in $R$. ■