Chapter 2

Commutativity Theorems for Nonassociative Rings
studied some results on commutativity of nonassociative rings with certain identities in the center. In this chapter, we generalize some commutativity results for 2-torsion free nonassociative rings. In section 2.1, we prove the commutativity of a 2-torsion free nonassociative ring $R$ with unity satisfying any one of the identities:

(i) $(xy)^2 + x^2y^2 \in Z(R)$  
(ii) $(xy)^2 + y^2x^2 \in Z(R)$  
(iii) $(xy)^2 + (xy^2)x \in Z(R)$  
(iv) $x^2y^2 + x(y^2x) \in Z(R)$  
(v) $(yx^2)y - (xy)x \in Z(R)$  
(vi) $(yx^2)y - x(yx) \in Z(R)$  
(vii) $x^2y^2 - x^2y - xy^2 - xy \in Z(R)$  
(viii) $x^2y^2 - x^2y - xy^2 - yx \in Z(R)$  
(ix) $(xy)^2 - x^2y - xy^2 - x^2y^2 \in Z(R)$  
(x) $(xy)^2 - x^2y - xy^2 - y^2x^2 \in Z(R)$

for all $x, y$ in $R$. Also we show that the commutativity of a 2-torsion free nonassociative ring $R$ with unity satisfying the identity $((xy)z)^2 + (xy)^2z^2 \in Z(R)$ or $((xy)z)^2 + (x^2y^2)z^2 \in Z(R)$, for all $x, y, z$ in $R$. In section 2.2, we prove the commutativity of a 2-torsion free nonassociative primitive ring with $(x, x, x) = 0$ satisfying any one of the following identities:

(i) $xy - x^2y^2 \in Z(R)$

(ii) $yx - x^2y^2 \in Z(R)$

(iii) $xy - (xy^2)x \in Z(R)$
(iv) $xy - y(x^2y) \in Z(R)$  
(v) $x^2y^2 - (xy^2)x - (yx^2)y - xy \in Z(R)$  

\[ x^2y^2 - (xy^2)x - (yx^2)y - xy \in Z(R), \quad \text{for all } x, y \in R. \]
In section 2.3, we prove the commutativity of a nonassociative ring satisfying the identity $[x, y^{n+1}] - [x^{n+1}, y] \in Z(R)$, for all $x, y \in R$.

2.1 Rings with $(xy)z^2 + (x^2y^2)z^2 \in Z(R)$

R.D.Giri and A.K.Modi [24] have proved that if $R$ is a 2-torsion free nonassociative ring with unity satisfying the condition $(xy)^2 - xy \in Z(R)$ for all $x, y \in R$, then $R$ is commutative. They also proved that a 2-torsion free nonassociativity ring $R$ with unity satisfying the identity $(xy)z^2 - (xy)z \in Z(R)$ for all $x, y, z \in R$, is commutative. R.D.Giri and R.R.Rakhunde[25] have proved that if $R$ is a nonassociative semi-simple ring with unity satisfying $x^2y^2 - x^2y - xy^2 + xy \in Z(R)$ for $x, y \in R$, then $R$ is commutative. We prove the similar results without using the semi-simple property of a ring.

In this section, we prove the commutativity of a 2-torsion free nonassociative ring $R$ with unity satisfying any one of the identities:

(i) $(xy)^2 + x^2y^2 \in Z(R)$  
(ii) $(xy)^2 + y^2x^2 \in Z(R)$  
(iii) $(xy)^2 + (xy^2)x \in Z(R)$  
(iv) $x^2y^2 + x(y^2x) \in Z(R)$  
(v) $(yx^2)y - (xy)x \in Z(R)$  
(vi) $(yx^2)y - x(yx) \in Z(R)$  
(vii) $x^2y^2 - x^2y - xy^2 - xy \in Z(R)$  
(viii) $x^2y^2 - x^2y - xy^2 - yx \in Z(R)$
(ix) \((xy)^2 - x^2y - xy^2 - x^2y^2 \in Z(R)\)

(x) \((xy)^2 - x^2y - xy^2 - y^2x^2 \in Z(R)\)

for all \(x, y\) in \(R\). Also we show that the commutativity of a 2-torsion free nonassociative ring \(R\) with unity satisfying the identity

\[ ((xy)z)^2 + (xy)^2z^2 \in Z(R) \]

or \( ((xy)z)^2 + (x^2y^2)z^2 \in Z(R) \), for all \(x, y, z\) in \(R\).

First we prove the following lemma:

**Lemma 2.1.1:** Let \(R\) be a 2-torsion free nonassociative ring with unity satisfying \((xy)^2 \in Z(R)\) for all \(x, y\) in \(R\), then \(R\) is Commutative.

**Proof:** By hypothesis, \((xy)^2 \in Z(R)\). By replacing \(x\) by \(x + 1\) in 2.1.1, we get

\[ ((x + 1)y)^2 \in Z(R). \]

\[(xy + y)^2 \in Z(R).\]

\[(xy)^2 + y^2 + (xy)y + y(xy) \in Z(R).\]

By using 2.1.1, we get,

\[ y^2 + (xy)y + y(xy) \in Z(R). \]

By replacing \(x\) by \(x + 1\), we get

\[ y^2 + ((x + 1)y)y + y((x + 1)y) \in Z(R). \]

\[ y^2 + (xy + y)y + y(xy + y) \in Z(R). \]

\[ y^2 + (xy)y + y^2 + y(xy) + y^2 \in Z(R). \]

By using 2.1.2, we get \(2y^2 \in Z(R)\).

Since \(R\) is 2-torsion free, we get
\[ y^2 \in Z(R) \]  \hspace{1cm} 2.1.3

By replacing \( y \) by \( y + 1 \) and using 2.1.3, we get \( 2y \in Z(R) \).

Since \( R \) is 2-torsion free, we get \( y \in Z(R) \).

Hence \( R \) is Commutative. \[ \square \]

**Theorem 2.1.1**: Let \( R \) be a 2-torsion free nonassociative ring with unity satisfying

1. \((xy)^2 + x^2 y^2 \in Z(R)\)
2. \((xy)^2 + y^2 x^2 \in Z(R)\)
3. \((xy)^2 + (xy^2)x \in Z(R)\)
4. \(x^2 y^2 + x(y^2 x) \in Z(R)\)
5. \((yx^2)y - (xy)x \in Z(R)\)
6. \((yx^2)y - x(yx) \in Z(R)\)
7. \(x^2 y^2 - x^2 y - xy^2 - xy \in Z(R)\)
8. \(x^2 y^2 - x^2 y - xy^2 - yx \in Z(R)\)
9. \((xy)^2 - x^2 y - xy^2 - x^2 y^2 \in Z(R)\)
10. \((xy)^2 - x^2 y - xy^2 - y^2 x^2 \in Z(R)\) for all \( x, y \) in \( R \), then \( R \) is Commutative.

**Proof**: (i) By hypothesis

\[ (xy)^2 + x^2 y^2 \in Z(R). \]  \hspace{1cm} 2.1.4

By replacing \( x \) by \( x + 1 \) in 2.1.4, we get

\[ ((x + 1)y)^2 + (x + 1)^2 y^2 \in Z(R). \]

\[ (xy)^2 + y^2 + (xy)y + y(xy) + x^2 y^2 + y^2 + 2xy^2 \in Z(R). \]

By using 2.1.4, we get

\[ 2y^2 + (xy)y + y(xy) + 2xy^2 \in Z(R). \]  \hspace{1cm} 2.1.5

By replacing \( x \) by \( x + 1 \) in 2.1.5, we get

\[ 2y^2 + ((x + 1)y)y + y((x + 1)y) + 2(x + 1)y^2 \in Z(R). \]
By using 2.1.5, we get $4y^2 \in Z(R)$.

Since $R$ is 2-torsion free, we get
\[ y^2 \in Z(R). \]  

By applying the same argument as in (i), $R$ is commutative.

(ii) By hypothesis
\[ (xy)^2 + y^2x^2 \in Z(R). \]  

By replacing $x$ by $x + 1$ in 2.1.7, we get
\[ ((x + 1)y)^2 + y^2(x + 1)^2 \in Z(R). \]
\[ (xy)^2 + y^2 + (xy)y + y(xy) + y^2x^2 + y^2 + 2y^2x \in Z(R). \]

By using 2.1.7, we get
\[ 2y^2 + (xy)y + y(xy) + 2y^2x \in Z(R). \]  

By replacing $x$ by $x + 1$ in 2.1.8, we get
\[ 2y^2 + ((x + 1)y)y + y((x + 1)y) + 2y^2(x + 1) \in Z(R). \]
\[ 2y^2 + (xy + y)y + y(xy + y) + 2y^2x + 2y^2 \in Z(R). \]
\[ 2y^2 + (xy)y + y^2 + y(xy) + y^2 + 2y^2x + 2y^2 \in Z(R). \]

By using 2.1.8, we get $4y^2 \in Z(R)$.

By applying the same argument as in (i), $R$ is commutative.

(iii) By hypothesis
\[(xy)^2 + (xy^2)x \in Z(R). \] 2.1.9

By replacing \(x\) by \(x + 1\) in 2.1.9, we get

\[(x + 1)(y)^2 + ((x + 1)y^2)(x + 1) \in Z(R).\]

\[(xy + y)^2 + (xy^2 + y^2)(x + 1) \in Z(R).\]

\[(xy)^2 + y^2 + (xy)y + y(xy) + (xy^2)x + xy^2 + y^2x + y^2 \in Z(R).\]

By using 2.1.9, we get

\[2y^2 + (xy)y + y(xy) + xy^2 + y^2x \in Z(R).\] 2.1.10

By replacing \(x\) by \(x + 1\) in 2.1.10, we get

\[2y^2 + ((x + 1)y)y + y((x + 1)y) + (x + 1)y^2 + y^2(x + 1) \in Z(R).\]

\[2y^2 + (xy)y + y^2 + y(xy) + y^2 + xy^2 + y^2 + y^2x + y^2 \in Z(R).\]

By using 2.1.10, we get \(4y^2 \in Z(R).\)

By applying same argument as in (i), \(R\) is commutative.

(iv) By hypothesis \(x^2y^2 + x(y^2x) \in Z(R).\) 2.1.11

By replacing \(x\) by \(x + 1\) in 2.1.11, we get

\[(x + 1)^2y^2 + (x + 1)(y^2(x + 1)) \in Z(R).\]

\[x^2y^2 + y^2 + 2xy^2 + x(y^2x) + xy^2 + y^2x + y^2 \in Z(R).\]

By using 2.1.11, we get

\[2y^2 + 3xy^2 + y^2x \in Z(R).\] 2.1.12

By replacing \(x\) by \(x + 1\) in 2.1.12, we get

\[2y^2 + 3(x + 1)y^2 + y^2(x + 1) \in Z(R).\]

\[2y^2 + 3xy^2 + 3y^2 + y^2x + y^2 \in Z(R).\]
By using 2.1.12, we get $4y^2 \in Z(R)$.

By applying the same argument as in (i), $R$ is commutative.

(v) By hypothesis $(yx^2)y - (xy)x \in Z(R)$.  \hspace{1cm} 2.1.13

By replacing $x$ by $x + 1$ in 2.1.13, we get

$$(y(x + 1)^2)y - ((x + 1)y)(x + 1) \in Z(R).$$

$$(yx^2 + y + 2yx)y - (xy + y)(x + 1) \in Z(R).$$

$$(yx^2)y + y^2 + 2(yx)y - (xy)x - xy - yx - y \in Z(R).$$

By using 2.1.13, we get

$$y^2 + 2(yx)y - xy - yx - y \in Z(R).$$  \hspace{1cm} 2.1.14

By replacing $x$ by $x + 1$ in 2.1.14, we get

$$y^2 + 2(y(x + 1))y - (x + 1)y - y(x + 1) - y \in Z(R).$$

$$y^2 + 2(yx)y + 2y^2 - xy - y - yx - y - y \in Z(R).$$

By using 2.1.14, we get $2y^2 - 2y \in Z(R)$.

Since $R$ is 2-torsion free, we get

$$y^2 - y \in Z(R).$$  \hspace{1cm} 2.1.15

By replacing $y$ by $y + 1$ in 2.1.15, we get $2y \in Z(R)$.

Since $R$ is 2-torsion free, we get $y \in Z(R)$.

Hence $R$ is commutative.

(vi) By hypothesis $(yx^2)y - x(yx) \in Z(R)$.

By applying the same argument as in (v), $R$ is commutative.

(vii) By hypothesis $x^2y^2 - x^2y - xy^2 - xy \in Z(R)$.  \hspace{1cm} 2.1.16
By replacing $x$ by $x + 1$ in 2.1.16, we get

$$(x + 1)^2 y^2 - (x + 1)^2 y - (x + 1)y^2 - (x + 1)y \in Z(R).$$

$x^2 y^2 + y^2 + 2xy^2 - x^2 y - y - 2xy - xy^2 - y^2 - xy - y \in Z(R)$.

By using 2.1.16, we get $2(xy^2 - xy - y) \in Z(R)$.

Since $R$ is 2-torsion free, we get

$$xy^2 - xy - y \in Z(R). \quad 2.1.17$$

By replacing $x$ by $x + 1$ in 2.1.17, we get

$$(x + 1)y^2 - (x + 1)y - y \in Z(R).$$

$xy^2 + y^2 - xy - y - y \in Z(R)$.

By using 2.1.17, we get

$$y^2 - y \in Z(R). \quad 2.1.18$$

By replacing $y$ by $y + 1$ in 2.1.18, we get $2y \in Z(R)$.

Since $R$ is 2-torsion free, we get $y \in Z(R)$.

Hence $R$ is commutative.

(viii) By hypothesis $x^2 y^2 - x^2 y - xy^2 - xy \in Z(R)$.

By applying the same argument as in vii), $R$ is commutative.

(ix) By hypothesis $(xy)^2 - x^2 y - xy^2 - x^2 y^2 \in Z(R). \quad 2.1.19$

By replacing $x$ by $x + 1$ in 2.1.19, we get

$$((x + 1)y)^2 - (x + 1)^2 y - (x + 1)y^2 - (x + 1)^2 y^2 \in Z(R).$$

$$(xy + y)^2 - (x^2 + 1 + 2x)y - xy^2 - y^2 - (x^2 + 1 + 2x)y^2 \in Z(R).$$
(xy)^2 + y^2 + (xy)y + y(xy) - x^2y - y - 2xy - xy^2 - y^2 - x^2y^2 - y^2 - 2xy^2 \in Z(R).

By using 2.1.19, we get

(xy)y + y(xy) - y - 2xy - y^2 - 2xy^2 \in Z(R).

By replacing x by x + 1 in 2.1.22, we get

((x + 1)y)y + y((x + 1)y) - y - 2(x + 1)y - y^2 - 2(x + 1)y^2 \in Z(R).

(xy)y + y^2 + y(xy) + y^2 - y - 2xy - 2y - y^2 - 2xy^2 - 2y^2 \in Z(R).

By using 2.1.20, we get 2y \in Z(R).

Since R is 2-torsion free, we get y \in Z(R).

Hence R is commutative.

(x) By hypothesis (xy)^2 - x^2y - xy^2 - y^2x^2 \in Z(R).

By applying the same argument as in (ix), R is commutative.

We give an example showing that the unity is essential:

Example 2.1.1: The ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in Z_4 \right\}$ is not of 2-torsion free and with unity. However it satisfies the identities (i), (ii) of Theorem 2.1.1 But R is not commutativity. This shows that the conditions of being 2-torsion free and having unity are essential.

Example 2.1.2: The ring $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in Z_4 \right\}$ is not of 2-torsion free and with unity. However it satisfies the identities (i), (ii) of Theorem 2.1.1 But R is not
commutativity. This shows that the conditions of being 2-torsion free and having unity are essential.

**Theorem 2.1.2:** Let $R$ be a 2-torsion free nonassociative ring with unity satisfying

(i) $((xy)z)^2 + (xy)^2 z^2 \in Z(R)$

(ii) $((xy)z)^2 + (x^2 y^2)z^2 \in Z(R)$ for all $x,y,z$ in $R$, then $R$ is commutative.

**Proof:** (i) By hypothesis $((xy)z)^2 + (xy)^2 z^2 \in Z(R)$. 2.1.21

By replacing $z$ by $z + 1$ in 2.1.24, we get

$((xy)(z + 1))^2 + (xy)^2 (z + 1)^2 \in Z(R)$.

$((xy)z + xy)^2 + (xy)^2 (z^2 + 1 + 2z) \in Z(R)$.

$((xy)z)^2 + (xy)^2 + ((xy)z)(xy) + (xy)((xy)z) + (xy)^2 z^2 + (xy)^2 + 2(xy)^2 z$

$\in Z(R)$.

By using 2.1.21, we get

$2(xy)^2 + ((xy)z)(xy) + (xy)((xy)z) + 2(xy)^2 z \in Z(R)$. 2.1.22

By replacing $z$ by $z + 1$ in 2.1.22, we get

$2(xy)^2 + ((xy)(z + 1))(xy) + (xy)((xy)(z + 1))$

$+ 2(xy)^2 (z + 1) \in Z(R)$.

$2(xy)^2 + ((xy)z + xy)(xy) + (xy)((xy)z + xy)$

$+ 2(xy)^2 z + 2(xy)^2 \in Z(R)$.

$2(xy)^2 + ((xy)z)(xy) + (xy)(xy) + (xy)((xy)z) + (xy)(xy) + 2(xy)^2 z + 2(xy)^2$

$\in Z(R)$. 
By using 2.1.22, we get

\[(xy)(xy) + (xy)(xy) + 2(xy)^2 \in Z(R)\].

\[4(xy)^2 \in Z(R)\].

Since \(R\) is 2-torsion free, we get \((xy)^2 \in Z(R)\).

By using Lemma 2.1.1, \(R\) is commutative.

(ii) By hypothesis \(((xy)z)^2 + (x^2y^2)z^2 \in Z(R)\). 2.1.23

By replacing \(z\) by \(z + 1\) in 2.1.23, we get

\[((xy)(z + 1))^2 + (x^2y^2)(z + 1)^2 \in Z(R)\].

\[((xy)z + xy)^2 + (x^2y^2)(z^2 + 1 + 2z) \in Z(R)\].

\[((xy)z)^2 + (xy)^2 + ((xy)z)(xy) + (xy)((xy)z) + (x^2y^2)z^2 + x^2y^2 + 2(x^2y^2)z \in Z(R)\].

By using 2.1.23, we get

\[(xy)^2 + ((xy)z)(xy) + (xy)((xy)z) + x^2y^2 + 2(x^2y^2)z \in Z(R)\]. 2.1.24

By replacing \(z\) by \(z + 1\) in 2.1.24, we get

\[(xy)^2 + ((xy)(z + 1))(xy) + (xy)((xy)(z + 1)) + x^2y^2 + 2(x^2y^2)(z + 1) \in Z(R)\].

\[(xy)^2 + ((xy)z + xy)(xy) + (xy)((xy)z + xy) + x^2y^2 + 2(x^2y^2)z + 2(x^2y^2) \in Z(R)\].

\[(xy)^2 + ((xy)z)(xy) + (xy)(xy) + (xy)((xy)z) + (xy)(xy) + x^2y^2 + 2(x^2y^2)z + 2(x^2y^2) \in Z(R)\].

By using 2.1.24, we get \((xy)(xy) + (xy)(xy) + 2(x^2y^2) \in Z(R)\).
\[ 2(xy)^2 + 2(x^2y^2) \in Z(R). \]

Since \( R \) is 2-torsion free, we get \((xy)^2 + (x^2y^2) \in Z(R)\).

By applying same argument as in Theorem 2.1.1(i), \( R \) is commutative. \( \blacksquare \)

### 2.2 Primitive rings with \( x^2y^2 - (xy^2)x - (yx^2)y - xy \in Z(R) \)

Asharf and Quadri [10] proved that an associative ring \( R \) with unity satisfying \([xy - x^m y^m, x] = 0\), for all \( x, y \) in \( R \), fixed integers \( m > 1 \), \( n > 1 \), is commutative.

R.D. Giri and R.R. Rakhunde [25] generalized this result by choosing \( m = n = 2 \) and \( R \) a nonassociative ring with unity satisfying the condition \( xy - x^2y^2 \in Z(R) \) is commutative. We generalize this result for commutativity of 2-torsion free nonassociative primitive ring.

**Theorem 2.2.1:** If \( R \) is a 2-torsion free nonassociative primitive ring with \((x, x, x) = 0\) satisfying

(i) \( xy - x^2y^2 \in Z(R) \)  
(ii) \( yx - x^2y^2 \in Z(R) \)  
(iii) \( xy - (xy^2)x \in Z(R) \)  
(iv) \( xy - y(x^2y)x \in Z(R) \)  
(v) \( x^2y^2 - (xy^2)x - (yx^2)y - xy \in Z(R) \)  
(vi) \( x^2y^2 - (xy^2)x - (yx^2)y - xy \in Z(R) \) for all \( x, y \) in \( R \), then \( R \) is commutative.

**Proof:** (i) By hypothesis \( xy - x^2y^2 \in Z(R) \). \( \quad \) 2.2.1

By replacing \( x \) by \( x + y \) in 2.2.1, we get

\[(x + y)y - (x + y)^2y^2 \in Z(R).\]

\[xy + y^2 - (x^2 + y^2 + xy + yx)y^2 \in Z(R).\]

\[xy + y^2 - x^2y^2 - y^4 - (xy)y^2 - (yx)y^2 \in Z(R).\]

By using 2.2.1, we get
$y^2 - y^4 - (xy)y^2 - (yx)y^2 \in Z(R)$. \hspace{1cm} 2.2.2

By replacing $x$ by $y$ in 2.2.1, we get

$y^2 - y^4 \in Z(R)$. \hspace{1cm} 2.2.3

By using 2.2.2 and 2.2.3, we get

$(xy)y^2 + (yx)y^2 \in Z(R)$. \hspace{1cm} 2.2.4

By replacing $x$ by $x + y$ in 2.2.4, we get

$((x + y)y)y^2 + (y(x + y)y^2 \in Z(R)$.

$(xy + y^2)y^2 + (yx + y^2)y^2 \in Z(R)$.

$(xy)y^2 + y^4 + (yx)y^2 + y^4 \in Z(R)$.

By using 2.2.4 and 2-torsion free condition, we get

$y^4 \in Z(R)$. \hspace{1cm} 2.2.5

By using 2.2.5 and 2.2.3, we get

$y^2 \in Z(R)$. \hspace{1cm} 2.2.6

By replacing $y$ by $x + y$ in 2.2.6, we get $(x + y)^2 \in Z(R)$.

$x^2 + y^2 + xy + yx \in Z(R)$. \hspace{1cm} 2.2.7

By using 2.2.6 and 2.2.7, we get

$(x^2 + y^2 + xy + yx)y^2 \in Z(R)$.

$x^2y^2 + y^4 + (xy)y^2 + (yx)y^2 \in Z(R)$. \hspace{1cm} 2.2.8

By using 2.2.4, 2.2.5 and 2.2.8, we get

$x^2y^2 \in Z(R)$. \hspace{1cm} 2.2.9
By using 2.2.1 and 2.2.9, we get
\[ xy \in Z(R). \quad 2.2.10 \]

By using 2.2.6 and 2.2.7, we get \[ xy + yx \in Z(R). \quad 2.2.11 \]

So, \( yx \in Z(R). \)

Therefore, \( xy - yx \in Z(R). \)

Since \( R \) is a primitive ring, \( R \) has no maximal right ideal which contains no nonzero ideal of \( R \). Consequently, we obtain \( (xy - yx)R = 0 \) which further yields \( xy - yx = 0 \) due to the primitivity of \( R \).

Hence \( R \) is commutative.

(ii) By hypothesis \( yx - x^2y^2 \in Z(R). \)

By applying the same argument as in (i), \( R \) is commutative.

(iii) By hypothesis \( xy - (xy^2)x \in Z(R). \quad 2.2.12 \)

By replacing \( x \) by \( x + y \) in 2.2.12, we get
\[
(x + y)y - ((x + y)y^2)(x + y) \in Z(R).
\]
\[
xy + y^2 - (xy^2 + y^3)(x + y) \in Z(R).
\]
\[
xy + y^2 - (xy^2)x - y^3x - (xy^2)y - y^4 \in Z(R).
\]

By using 2.2.12, we get
\[
y^2 - y^3x - (xy^2)y - y^4 \in Z(R). \quad 2.2.13
\]

By replacing \( x \) by \( y \) in 2.2.12, we get
\[
y^2 - y^4 \in Z(R). \quad 2.2.14
\]

By using 2.2.13 and 2.2.14, we get
By replacing \( x \) by \( x + y \) in 2.2.15, we get
\[
y^3(x + y) + ((x + y)y^2)y \in Z(R).
\]
\[
y^3x + y^4 + (xy^2 + y^3)y \in Z(R).
\]
\[
y^3x + y^4 + (xy^2)y + y^4 \in Z(R).
\]
By using 2.2.15 and 2.2.16, we get \( 2y^4 \in Z(R) \).

Since \( R \) is 2-torsion free, we get
\[
y^4 \in Z(R).
\]

By using 2.2.14 and 2.2.17, we get
\[
y^2 \in Z(R).
\]

By replacing \( y \) by \( x + y \) in 2.2.18, we get \( (x + y)^2 \in Z(R) \),
\[
x^2 + y^2 + xy + yx \in Z(R).
\]
By using 2.2.18, we get
\[
xy + yx \in Z(R).
\]
By replacing \( y \) by \( y^2 \) in 2.2.19, we get
\[
xy^2 + y^2x \in Z(R).
\]
Since \( y^2 \in Z(R) \), \( (y^2, x) = 0 \), for \( x \) in \( R \).

That is, \( y^2x - xy^2 = 0 \).

So, \( y^2x = xy^2 \).

By using 2.2.20 and 2.2.21 and 2-torsion free condition, we get
\( y^2x \in Z(R). \) \hspace{1cm} 2.2.22

By replacing \( y \) by \( x + y^2 \) in 2.2.22, we get \( (x + y^2)^2x \in Z(R). \)

\( (x^2 + y^4 + xy^2 + y^2x)x \in Z(R). \)

\( x^3 + y^4x + (xy^2)x + (y^2x)x \in Z(R). \) \hspace{1cm} 2.2.23

By replacing \( y \) by \( y^4 \) in 2.2.19, we get

\( xy^4 + y^4x \in Z(R). \) \hspace{1cm} 2.2.24

Since \( y^4 \in Z(R) \), we get \( y^4x = xy^4. \) \hspace{1cm} 2.2.25

By using 2.2.24 and 2.2.25 and 2-torsion free condition, we get

\( y^4x \in Z(R). \) \hspace{1cm} 2.2.26

By replacing \( y \) by \( x \) in 2.2.22, we get

\( x^3 \in Z(R). \) \hspace{1cm} 2.2.27

By using 2.2.26, 2.2.27 and 2.2.23, we get \( (xy^2)x + (y^2x)x \in Z(R). \)

By using 2.2.21 and 2-torsion free condition, we get

\( (xy^2)x \in Z(R). \) \hspace{1cm} 2.2.28

By using 2.2.12 and 2.2.28, we get \( xy \in Z(R). \)

By using 2.2.19, we get \( yx \in Z(R). \)

So, \( xy - yx \in Z(R). \)

By applying the same argument as in (i), \( R \) is commutative.
(iv) By hypothesis, \( xy - y(x^2y) \in Z(R) \). \( \text{2.2.29} \)

By replacing \( y \) by \( x + y \) in 2.2.29, we get

\[ x(x + y) - (x + y)(x^2(x + y)) \in Z(R). \]

\[ x^2 + xy - (x + y)(x^3 + x^2y) \in Z(R). \]

\[ x^2 + xy - x^4 - x(x^2y) - yx^3 - y(x^2y) \in Z(R). \]

By using 2.2.29, we get

\[ x^2 - x^4 - x(x^2y) - yx^3 \in Z(R). \] \( \text{2.2.30} \)

By replacing \( y \) by \( x \) in 2.2.29, we get

\[ x^2 - x^4 \in Z(R). \] \( \text{2.2.31} \)

By using 2.2.30 and 2.2.31, we get

\[ x(x^2y) + yx^3 \in Z(R). \] \( \text{2.2.32} \)

By replacing \( y \) by \( x + y \) in 2.2.32, we get

\[ x(x^2(x + y)) + (x + y)x^3 \in Z(R). \]

\[ x(x^3 + x^3y) + x^4 + yx^3 \in Z(R). \]

\[ x^4 + x(x^2y) + x^4 + yx^3 \in Z(R). \]

By using 2.2.32 and 2-torsion free condition, we get

\[ x^4 \in Z(R). \] \( \text{2.2.33} \)

By using 2.2.31 and 2.2.33, we get

\[ x^2 \in Z(R). \] \( \text{2.2.34} \)

By replacing \( x \) by \( x + y \) in 2.2.34, we get \((x + y)^2 \in Z(R)\).
By using 2.2.34, we get

\[ xy + yx \in Z(R). \]  \hspace{1cm} 2.2.35

By replacing \( x \) by \( x^2 \) in 2.2.35, we get

\[ x^2 y + yx^2 \in Z(R). \]  \hspace{1cm} 2.2.36

Since \( x^2 \in Z(R), x^2 y = yx^2. \)  \hspace{1cm} 2.2.37

By using 2.2.36, 2.2.37 and 2-torsion free condition, we get

\[ yx^2 \in Z(R). \]  \hspace{1cm} 2.2.38

By replacing \( x \) by \( x^2 + y \) in 2.2.38, we get

\[ y(x^2 + y)^2 \in Z(R). \]

\[ y(x^4 + y^2 + x^2 y + yx^2) \in Z(R). \]

\[ yx^4 + y^3 + y(x^2 y) + y(yx^2) \in Z(R). \]  \hspace{1cm} 2.2.39

By replacing \( x \) by \( x^2 \) in 2.2.38, we get \( yx^4 \in Z(R). \)  \hspace{1cm} 2.2.40

By replacing \( x \) by \( y \) in 2.2.38, we get \( y^3 \in Z(R). \)  \hspace{1cm} 2.2.41

By using 2.2.39, 2.2.40 and 2.2.41, we get \( y(x^2 y) + y(yx^2) \in Z(R). \)

By using 2.2.37 and 2-torsion free condition, we get \( y(x^2 y) \in Z(R). \)

By using 2.2.29, we get \( xy \in Z(R). \)

By using 2.2.35, we get \( yx \in Z(R). \)

Therefore, \( xy - yx \in Z(R). \)

Applying the same argument as in (i), \( R \) is commutative.

(v) By hypothesis \( x^2 y^2 - (xy^2)x - (yx^2)y - xy \in Z(R). \)  \hspace{1cm} 2.2.42
By replacing $x$ by $x + y$ in 2.2.42, we get

$$(x + y)^2 y^2 - ((x + y)y^2)(x + y) - (y(x + y)^2)y - (x + y)y \in Z(R).$$

$$(x^2 + y^2 + xy + yx)y^2 - (xy^2 + y^3)(x + y) - (y(x^2 + y^2 + xy + yx))y$$

$$- xy - y^2 \in Z(R).$$

$$x^2 y^2 + y^4 + (xy)y^2 + (yx)y^2 - (xy^2)x - (xy^2)y - y^3x - y^4$$

$$- (yx^2)y - y^4 - y(xy)y - (y^2x)y - xy - y^2 \in Z(R).$$

By using 2.2.42, we get

$$(xy)y^2 + (yx)y^2 - (xy^2)y - y^3x - y^4 - y(xy)y$$

$$- (y^2x)y - y^2 \in Z(R).$$

2.2.43

By replacing $x$ by $y$ in 2.2.42, we get

$$-y^4 - y^2 \in Z(R).$$

2.2.44

By using 2.2.43 and 2.2.44, we get

$$(xy)y^2 + (yx)y^2 - (xy^2)y - y^3x - y(xy)y - (y^2x)y \in Z(R).$$

By replacing $x$ by $y$ and using 2-torison free condition, we get

$$-y^4 \in Z(R).$$

2.2.45

By using 2.2.44, we get

$$y^2 \in Z(R).$$

2.2.46

By replacing $y$ by $x + y$ in 2.2.46, we get

$$(x + y)^2 \in Z(R)$$

$$x^2 + y^2 + xy + yx \in Z(R).$$
By using 2.2.46, we get

\[ xy + yx \in Z(R). \]  \hspace{1cm} 2.2.47

By replacing \( y \) by \( y^2 \) in 2.2.47, we get

\[ xy^2 + y^2x \in Z(R). \]  \hspace{1cm} 2.2.48

Since \( y^2 \in Z(R) \), \( xy^2 = y^2x \).  \hspace{1cm} 2.2.49

By using 2.2.47 and 2-torsion free condition, we get

\[ y^2x \in Z(R). \]  \hspace{1cm} 2.2.50

By replacing \( y \) by \( x + y^2 \) in 2.2.50, we get \( (x + y^2)^2x \in Z(R) \).

\[ (x^2 + y^4 + xy^2 + y^2x)x \in Z(R). \]  \hspace{1cm} 2.2.51

By replacing \( y \) by \( y^2 \) in 2.2.50, we get \( y^4x \in Z(R) \).  \hspace{1cm} 2.2.52

By replacing \( y \) by \( x \) in 2.2.50, we get

\[ x^3 \in Z(R). \]  \hspace{1cm} 2.2.53

By using 2.2.51 and 2.2.52, 2.2.53, we get \( (xy^2)x + (y^2x)x \in Z(R) \).

By using 2.2.48 and 2-torsion free condition, we get

\[ (xy^2)x \in Z(R). \]  \hspace{1cm} 2.2.54

By replacing \( x \) by \( x^2 \) in 2.2.47, we get

\[ x^2y + yx^2 \in Z(R). \]  \hspace{1cm} 2.2.55

Since \( x^2 \in Z(R) \), \( x^2y = yx^2 \).  \hspace{1cm} 2.2.56

By using 2.2.55 and 2-torsion free condition, we get

\[ x^2y \in Z(R). \]  \hspace{1cm} 2.2.57
By replacing $x$ by $x^2 + y$ in 2.2.57, we get

$$(x^2 + y)^2 y \in Z(R).$$

$$(x^4 + y^2 + x^2 y + yx^2)y \in Z(R).$$

$$x^4 y + y^3 + (x^2 y)y + (yx^2)y \in Z(R).$$

By replacing $x$ by $x^2$ in 2.2.57, we get

$$x^4 y \in Z(R).$$

By replacing $x$ by $y$ in 2.2.57, we get

$$y^3 \in Z(R).$$

By using 2.2.58, 2.2.59 and 2.2.60, we get

$$(x^2 y)y + (yx^2)y \in Z(R).$$

By using 2.2.56 and 2-torsion free condition, we get

$$(yx^2)y \in Z(R).$$

By using 2.2.54, 2.2.62 and 2.2.42, we get

$$x^2 y^2 - xy \in Z(R).$$

By applying the same argument as in (i), $R$ is commutative.

(vi) By hypothesis, $x^2 y^2 - (xy^2)x - (yx^2)y - yx \in Z(R)$.

By applying the same argument as in (v), $R$ is commutative. $\square$

**Example 2.2.1** : The identities (i) and (ii) of Theorem 2.2.1 along with the common primitivity of $R$ are indispensable. This can be checked in the ring

$$R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} : a, b, c, d \in Z \right\}.$$ $\square$

2.3: **Rings with** $[x,y^{n+1}] - [x^{n+1},y]$ **in the center**
Ram Awtar [45] studied nonassociative rings with unity satisfying the identity 
\[(xy)^n = x^n y^n \]. In this section we prove that if \( R \) is a semisimple nonassociative ring with 
unity satisfying \([x, y^{n+1}] - [x^{n+1}, y] \) in the center, then \( R \) is commutative.

**Lemma 2.3.1:** For any positive integers \( n \) and \( m \) the following relation hold:

\[
\sum_{k=0}^{m} (-1)^k \binom{m}{k} (m+1-k)^n = \begin{cases} 
0 & \text{if } m > n \\
n! & \text{if } m = n
\end{cases}
\]

**Proof:** We prove the Lemma by using Induction on \( m \).

**Case 1:** Consider \( m > n \).

If \( m = 1 \) then \( L.H.S. = \sum_{k=0}^{1} (-1)^k \binom{1}{k} (2-k)^n \).

Since \( m > n \), we have \( n = 0 \) because \( m = 1 \)

\[
L.H.S = \sum_{k=0}^{1} (-1)^k \binom{1}{k} (2-k)^0 = (-1)^0 \binom{1}{0} + (-1)^1 \binom{1}{1} = 1 \cdot 1 + (-1) \cdot 1 = 1 - 1 = 0.
\]

Therefore the result is true for \( m = 1 \).

Now we assume that the result is true for \( m \).

i.e., \( \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m+1-k)^n = \begin{cases} 
0 & \text{if } m > n \\
n! & \text{if } m = n
\end{cases} \).  \quad 2.3.1

Now consider \( \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} (m+2-k)^n \). 


\[
\sum_{k=0}^{m} (-1)^{k} \left( \begin{array}{c} m+1 \\ k \end{array} \right) (m+2-k)^n + (-1)^{m+1} \left( \begin{array}{c} m+1 \\ m+1 \end{array} \right) (m+2-(m+1))^n 
\]

\[
= \sum_{k=0}^{m} (-1)^{k} \frac{(m+1)!}{k! (m+1-k)!} (m+1-k)^n + (-1)^{m+1}
\]

\[
= \sum_{k=0}^{m} (-1)^{k} \frac{(m+1)m!}{k!(m+1-k)(m-k)!} \left( \begin{array}{c} m+1 \\ m+1 \end{array} \right) (m+1-k)^n + \left( \begin{array}{c} n \\ 1 \end{array} \right) (m+1-k)^{n-1} + \ldots + 1 \right) + (-1)^{m+1}
\]

\[
= \sum_{k=0}^{m} (-1)^{k} \frac{(m+1)(m)}{k!(m-k)!} \left( \begin{array}{c} m+1 \\ m+1 \end{array} \right) \left( (m+1-k)^n + n(m+1-k)^{n-2} + \ldots + \frac{1}{(m+1-k)} \right) + (-1)^{m+1}
\]

\[
= (m+1) \sum_{k=0}^{m} (-1)^{k} \left( \begin{array}{c} m \\ k \end{array} \right) \left( (m+1-k)^n + \ldots + \frac{1}{(m+1-k)} \right) + (-1)^{m+1}. \quad 2.3.2
\]

Now by 2.3.1,
\[
\sum_{k=0}^{m} (-1)^{k} \left( \begin{array}{c} m \\ k \end{array} \right) (m+1-k)^{n-1} + \ldots + 1} = 0, \text{ since } n-1, n-2, \ldots < m.
\]

Then 2.3.2 is
\[
(m+1) \sum_{k=0}^{m} (-1)^{k} \left( \begin{array}{c} m \\ k \end{array} \right) \frac{1}{m+1-k} + (-1)^{m+1}
\]

\[
= \sum_{k=0}^{m} (-1)^{k} \frac{(m+1)}{(m+1-k)} \left( \begin{array}{c} m \\ k \end{array} \right) + (-1)^{m+1}
\]

\[
= \sum_{k=0}^{m} (-1)^{k} \frac{(m+1)m!}{(m+1-k)k!(m-k)!} + (-1)^{m+1}
\]

\[
= \sum_{k=0}^{m} (-1)^{k} \frac{(m+1)!}{k!(m+1-k)!} + (-1)^{m+1}
\]

\[
= \sum_{k=0}^{m} (-1)^{k} \left( \begin{array}{c} m+1 \\ k \end{array} \right) + (-1)^{m+1}
\]

\[
= (-1)^0 \left( \begin{array}{c} m+1 \\ 0 \end{array} \right) + (-1)^1 \left( \begin{array}{c} m+1 \\ 1 \end{array} \right) + (-1)^2 \left( \begin{array}{c} m+1 \\ 2 \end{array} \right) + \ldots + (-1)^m \left( \begin{array}{c} m+1 \\ m \end{array} \right) + (-1)^{m+1}
\]

\[
= 1 - \left( \begin{array}{c} m+1 \\ 1 \end{array} \right) - \ldots + (-1)^m \left( \begin{array}{c} m+1 \\ m \end{array} \right) + (-1)^{m+1} \left( \begin{array}{c} m+1 \\ m+1 \end{array} \right)
\]
Thus the induction completes.

**Case 2:** For $m = n$, we adopt the same technique of induction.

**Theorem 2.3.1:** If $R$ is a $(n + 1)!$-torsion free semisimple nonassociative ring with unity satisfying $[x,y^{n+1}] - [x^{n+1},y] \in Z(R)$ for all $x,y$ in $R$, then $R$ is commutative.

**Proof:** By hypothesis $[x,y^{n+1}] - [x^{n+1},y] \in Z(R)$. Then

$$xy^{n+1} - y^{n+1}x - x^{n+1}y + yx^{n+1} \in Z(R).$$  

2.3.3

By replacing $y$ with $y + 1$ in 2.3.3, we get

$$x(y + 1)^{n+1} - (y + 1)^{n+1}x - x^{n+1}(y + 1) + (y + 1)x^{n+1} \in Z(R).$$

$$x(y + 1)^{n+1} - (y + 1)^{n+1}x - x^{n+1}y - x^{n+1} + yx^{n+1} + x^{n+1} \in Z(R).$$

$$x(y + 1)^{n+1} - (y + 1)^{n+1}x - x^{n+1}y + yx^{n+1} \in Z(R).$$

2.3.4

From 2.3.3 and 2.3.4, we get

$$x(y + 1)^{n+1} - (y + 1)^{n+1}x - xy^{n+1} + y^{n+1}x \in Z(R).$$

2.3.5 By applying Binomial expansion for 2.3.5, we get

$$x\left(\binom{n+1}{0}y^{n+1} + \binom{n+1}{1}y^n + \binom{n+1}{2}y^{n-1} + \cdots + \binom{n+1}{n}y + \binom{n+1}{n+1}\right) - \left(\binom{n+1}{0}y^{n+1} + \binom{n+1}{1}y^n + \binom{n+1}{2}y^{n-1} + \cdots + \binom{n+1}{n}y + \binom{n+1}{n+1}\right)x - xy^{n+1} + y^{n+1}x \in Z(R).$$
So, \[ xy^{n+1} + x \left[ \binom{n+1}{1} y^n + \binom{n+1}{2} y^{n-1} + \ldots + \binom{n+1}{n} y^1 \right] + x - y^{n+1} x \]

\[ + \left[ \binom{n+1}{1} y^n + \binom{n+1}{2} y^{n-1} + \ldots + \binom{n+1}{n} y^1 \right] x - x - xy^{n+1} + y^{n+1} x \in Z(R). \]

On simplification, we get

\[ x \left[ \binom{n+1}{1} y^n + \binom{n+1}{2} y^{n-1} + \ldots + \binom{n+1}{n} y^1 \right] - \left[ \binom{n+1}{1} y^n + \binom{n+1}{2} y^{n-1} + \ldots + \binom{n+1}{n} y^1 \right] x \in Z(R). \]

By continuing the same process as above for \((t-1)\) times, we obtain

\[
E_r + E_{r+1} + \ldots + E_{n+1} - E'_{r+1} - \ldots - E'_{n+1} \in Z(R). \tag{2.3.6}
\]

where for \(r = t, t + 1, t + 2, \ldots, n + 1\), we have

\[
E_r = (t-1) \left( \binom{n+1}{r-1} \sum_{k=0}^{r-2} (-1)^k \binom{t-2}{k} (t-k-1)^{-2} (xy^{n+2-r}) \right)
\]

and

\[
E'_r = (t-1) \left( \binom{n+1}{r-1} \sum_{k=0}^{r-2} (-1)^k \binom{t-2}{k} (t-k-1)^{-2} (y^{n+2-r} x) \right) \tag{2.3.7}
\]

Now again by replacing \(y\) with \(y + 1\) in 2.3.6 and subtracting 2.3.6 from the resulting expression, we obtain

\[
G_{r+1} + G_{r+2} + \ldots + G_{n+1} - G'_{r+1} - G'_{r+2} - \ldots - G'_{n+1} \in Z(R), \tag{2.3.8}
\]

where for \(r = t + 1, t + 2, \ldots, n + 1\), we have
\[ G_r = t \left( \frac{n+1}{r-1} \right) \sum_{k=0}^{r-1} (-1)^k \left( \frac{t-1}{k} \right) (t-k)^{r-2} (xy^{n-r+2}) \]

and

\[ G'_r = t \left( \frac{n+1}{r-1} \right) \sum_{k=0}^{r-1} (-1)^k \left( \frac{t-1}{k} \right) (t-k)^{r-2} (y^{n-r+2} x). \]

The coefficients of \( xy^{n-r+2} \) and \( y^{n-r+2} x \) \((t + 1 \leq m \leq n + 1)\) are

\[ t \left( \frac{n+1}{m-1} \right) \sum_{k=0}^{m-1} (-1)^k \left( \frac{t-1}{k} \right) (t-k)^{m-2}. \]

If we replace \( t \) with \( n \) in 2.3.6 then \( E_{n+1} - E'_{n+1} \in Z(R) \).

Therefore,

\[
\begin{align*}
 n \left( \frac{n+1}{n} \right) \sum_{k=0}^{n-1} (-1)^k \frac{n-1}{k} (n-k)^{n-2} (xy^{n-1}) - n \left( \frac{n+1}{n} \right) \sum_{k=0}^{n-1} (-1)^k \frac{n-1}{k} (n-k)^{n-1} (y^{n-1} x) & \in Z(R), \\
n \left( \frac{n+1}{n} \right) \sum_{k=0}^{n-1} (-1)^k \frac{n-1}{k} (n-k)^{n-2} (x^{n-1} y) - n \left( \frac{n+1}{n} \right) \sum_{k=0}^{n-1} (-1)^k \frac{n-1}{k} (n-k)^{n-1} (x^{n-1}) & \in Z(R), \\
n \left( \frac{n+1}{n} \right) \sum_{k=0}^{n-1} (-1)^k \frac{n-1}{k} (n-k)^{n-2} (x^{n-1}) & \in Z(R). \quad 2.3.9
\end{align*}
\]

Now by using Lemma 2.3.1, we have

\[
\sum_{k=0}^{m} (-1)^k \binom{m}{k} (m+1-k)^n \begin{cases} 
0 & \text{if } m > n \\
n! & \text{if } m = n.
\end{cases}
\]

So we have

\[
\sum_{k=0}^{n-1} (-1)^k \frac{n-1}{k} (n-k)^{n-1} = \sum_{k=0}^{n-1} (-1)^k \frac{n-1}{k} (n-1+1-k)^{n-1} = (n-1)!. 
\]

and we know that \( \binom{n+1}{n} = \binom{n+1}{1} = n+1 \).
Using these, 2.3.9 can be written as

\[ n(n + 1)(n - 1)! (xy - yx) \in Z(R). \]

That is, \((n + 1)n (n - 1)! (xy - yx) \in Z(R).\)

So, \((n + 1)! (xy - yx) \in Z(R).\) \hspace{1cm} 2.3.10

Since \(R\) is \((n + 1)!\) -torsion free, we have \((xy - yx) \in Z(R).\)

Now if \(Z(R) = 0\) or \(Z(R) = R\), then \(R\) is essentially a commutative ring. Therefore, suppose \(0 \neq Z(R) \neq R\). We take \(R\) to be a simple ring. Consider the principle ideal \((xy - yx)R\).

Since \(Z(R) \neq R\) and \(R\) is simple, \((xy - yx)R = 0.\) Now if \(R\) is a division ring then \(xy - yx = 0.\) Hence \(R\) is commutative. If \(R\) is a simple ring which is not a division ring, then \(R\) is homomorphic to \(D_2\), the complete matrix of 2x2 matrices over a division ring \(D\) which must satisfy the condition \([x,y^{n+1}] - [x^{n+1},y] \in Z(R).\) Infact, if we choose \(x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \(y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\) then this condition fails. Hence \(R\) is commutative.

Now semisimple ring is a subdirect sum of simple rings, each of which is shown to be commutative. Hence semisimple ring \(R\) is also commutative under given hypothesis, because a subdirect sum of simple rings will also satisfy the identity satisfied by the simple rings. ■

We present two examples to show that the existence of a unity in \(R\) is necessary in the hypothesis of the Theorem 2.3.1.

**Example 2.3.1:** Let \(R\) be the subring generated by the matrices
in the ring of all 3x3 matrices over $\mathbb{Z}_2$, the ring of integers mod 2. For each integer $n \geq 1$ and all $x, y \in R$, $[x, y^{n+1}] = [x^{n+1}, y]$ holds, however $R$ is not commutative.

**Example 2.3.2:** Let $R$ be the subring generated by the matrices

$$
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

in the ring of all 3x3 matrices over $\mathbb{Z}_2$, the ring of integers mod 2. For each integer $n \geq 1$ and all $x, y \in R$, $[x, y^{n+1}] = [x^{n+1}, y]$ is satisfied in $R$, but $R$ is not commutative.