CHAPTER 5

ON A NEW I-CONVERGENT SEQUENCE SPACE

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CHAPTER-5

ON A NEW l-CONVERGENT SEQUENCE SPACE

5.1. INTRODUCTION

One hundred years ago mathematics was undergoing a revolution. The Kantian dictate that the Euclidean Geometry is the only rationally conceivable basis for the physical universe had been debunked. Numerous alternative geometries, each self-consistent, were being discovered, axiomatized and developed. Felix Klein found a unifying principle for relating and classifying the various geometrical Invariant Theory. The key idea is to classify mathematical structures by the transformations under which they are invariant. Invariant Theory has achieved wide influence in mathematics, physics (including relativity and quantum mechanics), and computer science. The calculus developed here is based upon relatively simple aspects of Invariant Theory.

Let \( v \) denote the space of sequences of bounded variation, that is

\[
 v = \{ x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, x_{-1} = 0 \}. 
\]

\( v \) is a Banach space normed by

\[
 \|x\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}|.
\]

Let \( \sigma \) be a mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional \( \phi \) on \( l_\infty \) is said to be an invariant mean or \( \sigma \)-mean if and only if

\[
\phi \text{ is invariant under } \sigma.
\]

"The moving power of Mathematical invention is not reasoning but imagination." - Demorgan.
[i] \( \phi(x) \geq 0 \) when the sequence \( x = (x_k) \) has \( x_k \geq 0 \) for all \( k \),

[ii] \( \phi(e) = 1 \), where \( e = \{1, 1, 1, \ldots\} \),

[iii] \( \phi(x_{\sigma(n)}) = \phi(x) \) for all \( x \in l_\infty \).

In case \( \sigma \) is the translation mapping \( n \rightarrow n+1 \), a \( \sigma \)-mean is often called a Banach limit (See [2]) and \( V_\sigma \), the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

If \( x = (x_k) \) write \( Tx = (Tx_k) = (x_{\sigma(k)}) \). It can be shown that

\[
V_\sigma = \left\{ x = (x_k) : \sum_{m=1}^{\infty} t_{m,k}(x) = L \text{ uniformly in } k, L = \sigma - \lim x \right\}
\]

where \( m \geq 0, k > 0 \).

\[ t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + \ldots + x_{\sigma^m(k)}}{m+1} \quad \text{and} \quad t_{-1,k} = 0; \quad [5.2] \]

where \( \sigma^m(k) \) denote the \( m \)th iterate of \( \sigma(k) \) at \( k \). In the case \( \sigma \) is the translation mapping, \( \sigma(k) = k + 1 \) is often called a Banach limit and \( V_\sigma \), the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequence. The special case of [5.1] in which \( \sigma(n) = n + 1 \) was given by Lorentz [58], Theorem 1, and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on \( c \).

**Theorem 5.1.1.** [62, Theorem 1.1.] A \( \sigma \)-mean extends the limit functional on \( c \) in the sense that \( \phi(x) = \lim x \) for all \( x \in c \) if and only if \( \sigma \) has no finite orbits, that is to say, if and only if, for all \( k \geq 0, j \geq 1 \)

\( \sigma^j(k) \neq k \).
Put
\[ \phi_{m,k}(x) = \ell_{m,k}(x) - \ell_{m-1,k}(x), \]
assuming that \( \ell_{-1,k} = 0 \). A straightforward calculation shows (See[10]) that
\[
\phi_{m,k}(x) = \begin{cases} 
\frac{1}{m(m+1)} \sum_{j=1}^{m} J(x_{j+1}(k) - x_{j}(k)) & (m \geq 1) \\
x_{k} & (m = 0)
\end{cases}
\]
For any sequence \( x, y \) and scalar \( \lambda \) we have
\[
\phi_{m,k}(x + y) = \phi_{m,k}(x) + \phi_{m,k}(y) \quad \text{and} \quad \phi_{m,k}(\lambda x) = \lambda \phi_{m,k}(x).
\]

**Definition 5.1.2.** [62, Definition 2.1.] A sequence \( x \in l_\infty \) is of \( \sigma \)-bounded variation if and only if

[i] \( \sum_{k=0}^{\infty} |\phi_{m,k}(x)| \) converges uniformly in \( n \),

[ii] \( \lim_{m \to \infty} \ell_{m,k}(x) \), which must exist, should take the same value for all \( k \).

We denote by \( BV_\sigma \), the space of all sequences of \( \sigma \)-bounded variation (see[61])
\[
BV_\sigma = \{ x \in l_\infty : \sum_{m} |\phi_{m,k}(x)| < \infty, \text{uniformly in } n \}.
\]

**Theorem 5.1.3.** [62, Theorem 2.2.] \( BV_\sigma \) is a Banach space normed by
\[
||x|| = \sup_{k} \sum_{m=0}^{\infty} |\phi_{m,k}(x)|.
\]

Subsequently invariant means have been studied by Ahmad and Mursaleen[1], J.P.King[20], Raimi[67], Kantthan and Gupta[19] and many others.

5.2 Main Results.
In this chapter we introduce the sequence space

$$BV_\omega^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |\phi_{m,k}(x) - L| \geq \epsilon\} \in I, \text{ for some } L \in \mathbb{C}\}.$$ 

**Theorem 5.2.1.** $BV_\omega^I$ is a linear space.

**Proof.** Let $(x_k), (y_k) \in BV_\omega^I$ and $\alpha, \beta$ be two scalars. Then for a given $\epsilon > 0$, we have

$$\{\{k \in \mathbb{N} : |\phi_{m,k}(x) - L_1| \geq \frac{\epsilon}{2}\} \in I, \text{ for some } L_1 \in \mathbb{C}\}$$

$$\{\{k \in \mathbb{N} : |\phi_{m,k}(y) - L_2| \geq \frac{\epsilon}{2}\} \in I, \text{ for some } L_2 \in \mathbb{C}\}.$$ 

Now let,

$$A_1 = \{\{k \in \mathbb{N} : |\phi_{m,k}(x) - L_1| < \frac{\epsilon}{2}\} \in I, \text{ for some } L_1 \in \mathbb{C}\}$$

$$A_2 = \{\{k \in \mathbb{N} : |\phi_{m,k}(y) - L_2| < \frac{\epsilon}{2}\} \in I, \text{ for some } L_2 \in \mathbb{C}\}$$

be such that $A_1, A_2 \in I$. Then

$$A_3 = \{k \in \mathbb{N} : |(\alpha \phi_{m,k}(x) + \beta \phi_{m,k}(y)) - (\alpha L_1 + \beta L_2)| < \epsilon\}$$

$$\sup \{k \in \mathbb{N} : |\alpha| \phi_{m,k}(x) - L_1| < \epsilon\} \cap \{k \in \mathbb{N} : |\beta| \phi_{m,k}(y) - L_2| < \epsilon\}.$$ 

Thus $A_3 = A_1 \cap A_2 \in I$. Hence

$$(\alpha(x_k) + \beta(y_k)) \in BV_\omega^I.$$ 

Therefore $BV_\omega^I$ is a linear space.

**Theorem 5.2.2.** The space $BV_\omega^I$ is a paranormed space, paranormed by

$$g(x_k) = \sup_k |\phi_{m,k}(x)|.$$ 

**Proof.** For $x = 0$, $g(x_k) = 0$ is trivial.
[i] \( g(x) = \sup_k |\phi_{m,k}(x)| \geq 0 \), for all \( x \in BV^I_\sigma \).

[ii] \( g(-x) = \sup_k |\phi_{m,k}(-x)| = \sup_k |\phi_{m,k}(x)| = \sup_k |\phi_{m,k}(x)| = g(x) \) for all \( x \in BV^I_\sigma \).

[iii] \( g(x + y) = \sup_k |\phi_{m,k}(x + y)| \leq \sup_k |\phi_{m,k}(x)| + \sup_k |\phi_{m,k}(y)| = g(x) + g(y) \).

[iv] Let \( (\lambda_k) \) be a sequence of scalars with \( \lambda_k \to \lambda \) \( (k \to \infty) \) and \( \phi_{m,k}(x), L \in BV^I_\sigma \) such that
\[
\phi_{m,k}(x) \to L \quad (k \to \infty),
\]
in the sense that
\[
g(\phi_{m,k}(x) - L) \to 0 \quad (k \to \infty).
\]

Therefore
\[
g(\lambda_k \phi_{m,k}(x) - \lambda L) \leq g(\lambda_k \phi_{m,k}(x)) - g(\lambda L)
= \lambda_n g(\phi_{m,k}(x)) - \lambda g(L) \to 0 \quad (k \to \infty).
\]

Hence \( BV^I_\sigma \) is a paranormed space.

**Theorem 5.2.3.** \( BV^I_\sigma \) is a closed subspace of \( l^I_\infty \).

**Proof.** Let \( (x^{(n)}_k) \) be a cauchy sequence in \( BV^I_\sigma \) such that \( x^{(n)} \to x \). We show that \( x \in BV^I_\sigma \). Since \( (x^{(n)}_k) \in BV^I_\sigma \), then there exists \( a_n \) such that
\[
\{ k \in \mathbb{N} : |\phi_{m,k}(x^{(n)}) - a_n| \geq \epsilon \} \in I.
\]

We need to show that

[i] \( (a_n) \) converges to \( a \).

[ii] If \( U = \{ k \in \mathbb{N} : |x_k - a| < \epsilon \} \), then \( U^c \in I \).
[i] Since \((x^{(n)}_k)\) is a cauchy sequence in \(BV^{I}_{\sigma}\) then for a given \(\epsilon > 0\), there exists \(k_0 \in \mathbb{N}\) such that
\[
\sup_k |\phi_{m,k}(x^{(n)}_k) - \phi_{m,k}(x^{(i)}_k)| < \frac{\epsilon}{3}, \quad \text{for all } n, i \geq k_0.
\]
For a given \(\epsilon > 0\), we have
\[
B_{ni} = \{k \in \mathbb{N} : |\phi_{m,k}(x^{(n)}_k) - \phi_{m,k}(x^{(i)}_k)| < \frac{\epsilon}{3}\};
\]
\[
B_i = \{k \in \mathbb{N} : |\phi_{m,k}(x^{(i)}_k) - a_i| < \frac{\epsilon}{3}\};
\]
\[
B_n = \{k \in \mathbb{N} : |\phi_{m,k}(x^{(n)}_k) - a_n| < \frac{\epsilon}{3}\}.
\]
Then
\[
B^c_{ni}, B^c_i, B^c_n \in I.
\]
Let
\[
B^c = B^c_{ni} \cap B^c_i \cap B^c_n,
\]
where
\[
B = \{k \in \mathbb{N} : |a_i - a_n| < \epsilon\}.
\]
Then \(B^c \in I\). We choose \(k_0 \in B^c\), then for each \(n, i \geq k_0\), we have
\[
\{k \in \mathbb{N} : |a_i - a_n| < \epsilon\} \supseteq \{k \in \mathbb{N} : |\phi_{m,k}(x^{(i)}_k) - a_i| < \frac{\epsilon}{3}\}
\]
\[
\cap \{k \in \mathbb{N} : |\phi_{m,k}(x^{(n)}_k) - \phi_{m,k}(x^{(i)}_k)| < \frac{\epsilon}{3}\}
\]
\[
\cap \{k \in \mathbb{N} : |\phi_{m,k}(x^{(n)}_k) - a_n| < \frac{\epsilon}{3}\}.
\]
Then \((a_n)\) is a cauchy sequence of scalars in \(\mathcal{C}\) so there exists a scalar \(a \in \mathcal{C}\) such that \((a_n) \to a\), as \(n \to \infty\).

[ii] Let \(0 < \delta < 1\) be given. Then we show that if
\[
U = \{k \in \mathbb{N} : |\phi_{m,k}(x) - a| < \delta\},
\]
then \(U^c \in I\). Since \(\phi_{m,k}(x^{(n)}) \to \phi_{m,k}(x)\), then there exists \(q_0 \in \mathbb{N}\) such that
\[
P = \{k \in \mathbb{N} : |\phi_{m,k}(x^{(q_0)}) - \phi_{m,k}(x)| < \frac{\delta}{3}\} \quad [5.3]
\]
which implies that \( P^c \in I \). The number \( q_0 \) can be so chosen that together with [5.3], we have

\[
Q = \left\{ k \in \mathcal{N} : |a_{q_0} - a| < \frac{\delta}{3} \right\}
\]

such that \( Q^c \in I \). Since

\[
\left\{ k \in \mathcal{N} : |\phi_{m,k}(x^{(q_0)}_k) - a_{q_0}| \geq \delta \right\} \in I.
\]

Then we have a subset \( S \) of \( \mathcal{N} \) such that \( S^c \in I \), where

\[
S = \left\{ k \in \mathcal{N} : |\phi_{m,k}(x^{(q_0)}_k) - a_{q_0}| < \frac{\delta}{3} \right\}.
\]

Let \( U^c = P^c \cap Q^c \cap S^c \), where \( U = \left\{ k \in \mathcal{N} : |\phi_{m,k}(x) - a| < \delta \right\} \).

Therefore for each \( k \in U^c \), we have

\[
\left\{ k \in \mathcal{N} : |\phi_{m,k}(x) - a| < \delta \right\} \supseteq \left\{ k \in \mathcal{N} : |\phi_{m,k}(x^{(q_0)}_k) - \phi_{m,k}(x)| < \frac{\delta}{3} \right\},
\]

\[
\cap \left\{ k \in \mathcal{N} : |\phi_{m,k}(x^{(q_0)}_k) - a_{q_0}| < \frac{\delta}{3} \right\}
\]

Then the result follows. Since the inclusion \( BV^I_\delta \subset l^I_\infty \) is strict so in view of Theorem 5.2.3 we have the following result.

**Theorem 5.2.4.** The space \( BV^I_\delta \) is nowhere dense subsets of \( l^I_\infty \).

**Theorem 5.2.5.** The space \( BV^I_\delta \) is solid and monotone.

**Proof.** Let \((x_k) \in BV^I_\delta \) and \( \alpha_k \) be a sequence of scalars with

\[
|\alpha_k| \leq 1, \text{ for all } k \in \mathcal{N}.
\]

Then we have

\[
|\alpha_k \phi_{m,k}(x)| \leq |\alpha_k||\phi_{m,k}(x)| \leq |\phi_{m,k}(x)|, \text{ for all } k \in \mathcal{N}.
\]
The space $BV^f_\sigma$ is solid follows from the following inclusion relation,

$$\{k \in N : |\phi_{m,k}(x)| \geq \epsilon\} \supseteq \{k \in N : |\alpha_k \phi_{m,k}(x)| \geq \epsilon\}.$$ 

Also a sequence space is solid implies monotone. Hence the space $BV^f_\sigma$ is monotone.

**Theorem 5.2.6.** $c_0^f \subset BV^f_\sigma \subset l^f_\infty$ and the inclusions are proper.

**Proof.** Let $x = (x_k) \in c_0^f$. Then we have

$$\{k \in N : |x_k| \geq \epsilon\} \in I.$$ 

Since $c_0 \subset BV_\sigma$, $x = (x_k) \in BV^f_\sigma$ implies

$$\{k \in N : |\phi_{m,k}(x)| \geq \epsilon\} \in I.$$ 

Now let,

$$A_1 = \{k \in N : |x_k| < \epsilon\} \in I;$$

$$A_2 = \{k \in N : |\phi_{m,k}(x)| < \epsilon\} \in I$$

be such that $A_1, A_2 \in I$. As

$$l_\infty = \{x = (x_k) : \sup_k |x_k| < \infty\},$$

taking supremum over $k$ we get $A_1^c \subset A_2^c$. Hence

$$c_0^f \subset BV^f_\sigma \subset l^f_\infty.$$ 

**Theorem 5.2.7.**

$$c^f \subset BV^f_\sigma \subset l^f_\infty$$

and the inclusions are proper.

**Proof.** Let $x = (x_k) \in c^f$. Then we have

$$\{k \in N : |x_k - L| \geq \epsilon\} \in I.$$
Since
\[ c \subset BV_{\sigma} \subset l_{\infty}, \]
\[ x = (x_k) \in BV_{\sigma}^f \]
implies
\[ \{ k \in \mathbb{N} : |\phi_{m,k}(x) - L| \geq \epsilon \} \in I. \]

Now let,
\[ B_1 = \{ k \in \mathbb{N} : |\phi_k - L| < \epsilon \} \in I \]
\[ B_2 = \{ k \in \mathbb{N} : |\phi_{m,k}(x) - L| < \epsilon \} \in I \]
be such that \( B_1, B_2 \in I. \) As
\[ l_{\infty} = \{ x = (x_k) : \sup_k |x_k| < \infty \}, \]
taking supremum over \( k \) we get \( B_1^c \subset B_2^c. \) Hence
\[ c^f \subset BV_{\sigma}^f \subset l_{\infty}^f. \]
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6.1. INTRODUCTION

The following subspaces of \( \omega \) were first introduced and discussed by Maddox [59].

\[
\ell(p) = \{ x \in \omega : \sum_k |x_k|^{p_k} < \infty \}, \quad \ell_\infty(p) = \{ x \in \omega : \sup_k |x_k|^{p_k} < \infty \},
\]

\[
e(p) := \{ x \in \omega : \lim_k |x_k - t|^p_k = 0, \text{ for some } t \in \mathcal{C} \},
\]

\[
c_0(p) := \{ x \in \omega : \lim_k |x_k|^{p_k} = 0, \}
\]

where \( p = (p_k) \) is a sequence of strictly positive real numbers.

After then Lascarides [55-56] defined the following sequence spaces

\[
\ell_\infty(p) = \{ x \in \omega : \text{there exists } r > 0 \text{ such that } \sup_k |x_k r|^{p_k} t_k < \infty \},
\]

\[
c_0(p) = \{ x \in \omega : \text{there exists } r > 0 \text{ such that } \lim_k |x_k r|^{p_k} t_k = 0, \}
\]

\[
\ell(p) = \{ x \in \omega : \text{there exists } r > 0 \text{ such that } \sum_k |x_k r|^{p_k} t_k < \infty \},
\]

Where \( t_k = p_k^{-1} \), for all \( k \in \mathbb{N} \).

Mursaleen [62] defined the sequence space \( BV_\sigma \), the space of all sequences of \( \sigma \)-bounded variation

\[
BV_\sigma = \{ x \in \ell_\infty : \sum_m |\phi_{m,k}(x)| < \infty, \text{ uniformly in } n \}.
\]

**Theorem 6.1.** [62, Theorem 2.2.] \( BV_\sigma \) is a Banach space normed by

\[
\|x\| = \sup_k \sum_{m=0}^\infty |\phi_{m,k}(x)|.
\]

"The essence of mathematics lies in its freedom" - Cantor.
Recently khan, Ebadullah and Suantai[31] defined the following sequence space :

\[ BV^{I}_\sigma = \{(x_k) \in \omega : \{ k \in \mathfrak{N} : |\phi_{m,k}(x) - L| \geq \epsilon \} \in I, \text{ for some } L \in C \}. \]

6.2. Main Results.

In this chapter we introduce the sequence space.

\[ BV^{I}_\sigma(p) = \{(x_k) \in \omega : \{ k \in \mathfrak{N} : |\phi_{m,k}(x) - L|^{p_k} \geq \epsilon \} \in I, \text{ for some } L \in C \}. \]

Theorem 6.2.1. \( BV^{I}_\sigma(p) \) is a linear space.

Proof. Let \((x_k), (y_k) \in BV^{I}_\sigma(p)\) and let \(\alpha, \beta\) be scalars. Then for a given \(\epsilon > 0\), we have

\[
\{ k \in \mathfrak{N} : |\phi_{m,k}(x) - L_1|^{p_k} \geq \epsilon, \text{ for some } L_1 \in C \} \in I
\]

\[
\{ k \in \mathfrak{N} : |\phi_{m,k}(y) - L_2|^{p_k} \geq \epsilon, \text{ for some } L_2 \in C \} \in I
\]

where

\[
M_1 = D.\max\{1, \sup_k |\alpha|^{p_k}\}
\]

\[
M_2 = D.\max\{1, \sup_k |\beta|^{p_k}\}
\]

and

\[
D = \max\{1, 2^{H-1}\} \quad \text{where} \quad H = \sup_k p_k \geq 0.
\]

Let

\[
A_1 = \{ k \in \mathfrak{N} : |\phi_{m,k}(x) - L_1|^{p_k} < \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in C \} \in I,
\]

\[
A_2 = \{ k \in \mathfrak{N} : |\phi_{m,k}(y) - L_2|^{p_k} < \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in C \} \in I
\]
be such that $A_1^*, A_2^* \in I$. Then

$$A_3 = \{k \in \mathbb{N} : |(\alpha \phi_{m,k}(x) + \beta \psi_{m,k}(y) - (\alpha L_1 + \beta L_2)|^{p_k} < \epsilon \}$$

$$\sup \{k \in \mathbb{N} : |\alpha|^{p_k} |\phi_{m,k}(x) - L_1|^{p_k} < \frac{\epsilon}{2M_1} |\alpha|^{p_k} \cdot D \}$$

$$\cap \{k \in \mathbb{N} : |\beta|^{p_k} |\psi_{m,k}(y) - L_2|^{p_k} < \frac{\epsilon}{2M_2} |\beta|^{p_k} \cdot D \}.$$  

Thus $A_3^* = A_1^* \cap A_2^* \in I$. Therefore $(\alpha \phi_{m,k}(x) + \beta \psi_{m,k}(y)) \in BV_\sigma(I^f(p))$. Hence $BV_\sigma(I^f(p))$ is a linear space.

**Theorem 6.2.2.** Let $(p_k) \in \ell_\infty$. Then $BV_\sigma(I^f(p))$ is a paranormed space, paranormed by

$$||x||_* = \sup_k |\phi_{m,k}(x)|^{\frac{p_k}{M}}$$

where

$$M = \max \{1, \sup_k p_k \}.$$

**Proof.** Let $x = (x_k), y = (y_k) \in BV_\sigma(I^f(p))$.

[i] Clearly, $||x||_* = 0$ if and only if $x = 0$.

[ii] $||x||_* = || - x||_*$ is obvious.

[iii] Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality we have

$$\sup_k |\phi_{m,k}(x) + \phi_{m,k}(y)|^{\frac{p_k}{M}} \leq \sup_k |\phi_{m,k}(x)|^{\frac{p_k}{M}} + \sup_k |\phi_{m,k}(y)|^{\frac{p_k}{M}}.$$  

[iv] Now for any complex $\lambda$ we have $(\lambda_k)$ such that $\lambda_k \to \lambda$, $(k \to \infty)$. Let $(x_k) \in BV_\sigma(I^f(p))$ such that

$$|\phi_{m,k}(x) - L|^{p_k} \geq \epsilon.$$  

Therefore,

$$||\phi_{m,k}(x) - L||_* = \sup_k |\phi_{m,k}(x) - L|^{\frac{p_k}{M}} \leq \sup_k |\phi_{m,k}(x)|^{\frac{p_k}{M}} + \sup_k |L|^{\frac{p_k}{M}}.$$
Hence
\[ ||\lambda_n \phi_{m,k}(x) - \lambda L|| \leq ||\lambda_n \phi_{m,k}(x)|| + ||\lambda L||.\]
\[ = \lambda_n ||\phi_{m,k}(x)|| + \lambda ||L|| \quad \text{as} \quad (k \to \infty).\]

Hence \(BV^I_\sigma(p)\) is a paranormed space.

**Theorem 6.2.3.** \(BV^I_\sigma(p)\) is a closed subspace of \(\ell_\infty(p)\).

**Proof.** Let \((x^{(n)}_k)\) be a cauchy sequence in \(BV^I_\sigma(p)\) such that \(x^{(n)} \to x\). We show that \(x \in BV^I_\sigma(p)\).

Since \((x^{(n)}_k) \in BV^I_\sigma(p)\), then there exists \(a_n\) such that
\[\{k \in \mathcal{K} : ||\phi_{m,k}(x^{(n)}_k) - a_n|| < \epsilon\} \in I.\]

We need to show that
[i] \((a_n)\) converges to \(a\).

[ii] If \(U = \{k \in \mathcal{K} : |x_k - a|^p < \epsilon\}\), then \(U^c \in I\).

[i] Since \((x^{(n)}_k)\) is a cauchy sequence in \(BV^I_\sigma(p)\) then for a given \(\epsilon > 0\), there exists \(k_0 \in \mathcal{K}\) such that
\[\sup_k |\phi_{m,k}(x^{(n)}_k) - \phi_{m,k}(x^{(i)}_k)| < \frac{\epsilon}{3}, \text{ for all } n, i \geq k_0.\]

For a given \(\epsilon > 0\), we have
\[B_{n \ell} = \{k \in \mathcal{K} : |\phi_{m,k}(x^{(n)}_k) - \phi_{m,k}(x^{(i)}_k)|^p < \left(\frac{\epsilon}{3}\right)^M\}\]
\[B_{i} = \{k \in \mathcal{K} : |\phi_{m,k}(x^{(i)}_k) - a_i|^p < \left(\frac{\epsilon}{3}\right)^M\}\]
\[B_{n} = \{k \in \mathcal{K} : |\phi_{m,k}(x^{(n)}_k) - a_n|^p < \left(\frac{\epsilon}{3}\right)^M\}\]

Then \(B_{n \ell}^c, B_i^c, B_n^c \in I\). Let
\[B^c = B_{n \ell}^c \cap B_i^c \cap B_n^c,\]

where
\[B = \{k \in \mathcal{K} : |a_i - a_n|^p < \epsilon\}.\]
Then $B^c \in I$.

We choose $k_0 \in B^c$, then for each $n, i \geq k_0$, we have
\[
\{k \in \mathcal{N} : |a_i - a_n|^{p_k} < \epsilon\} \supseteq \{k \in \mathcal{N} : |\phi_{m,k}(x_k^{(i)}) - a_i|^{p_k} < \left(\frac{\epsilon}{3}\right)^M\}
\cap \{k \in \mathcal{N} : |\phi_{m,k}(x_k^{(n)}) - \phi_{m,k}(x_k^{(i)})|^{p_k} < \left(\frac{\epsilon}{3}\right)^M\}
\cap \{k \in \mathcal{N} : |\phi_{m,k}(x_k^{(n)}) - a_n|^{p_k} < \left(\frac{\epsilon}{3}\right)^M\}.
\]

Then $(a_n)$ is a cauchy sequence of scalars in $\mathcal{C}$ so there exists a scalar $a \in \mathcal{C}$ such that $(a_n) \to a$, as $n \to \infty$.

[ii] Let $0 < \delta < 1$ be given. Then we show that if
\[U = \{k \in \mathcal{N} : |\phi_{m,k}(x) - a|^{p_k} < \delta\},\]
then $U^c \in I$. Since $\phi_{m,k}(x^{(n)}) \to \phi_{m,k}(x)$, then there exists $q_0 \in \mathcal{N}$ such that
\[
P = \{k \in \mathcal{N} : |\phi_{m,k}(x^{(q_0)}) - \phi_{m,k}(x)|^{p_k} < \left(\frac{\delta}{3D}\right)^M\} \quad [6.1].
\]
Which implies that $P^c \in I$.

The number $q_0$ can be so choosen that together with [6.1], we have
\[
Q = \{k \in \mathcal{N} : |a_{q_0} - a|^{p_k} < \frac{\delta}{3D}\},
\]
such that $Q^c \in I$.

Since
\[
\{k \in \mathcal{N} : |\phi_{m,k}(x_k^{(q_0)}) - a_{q_0}|^{p_k} \geq \delta\} \in I.
\]
Then we have a subset $S$ of $\mathcal{N}$ such that $S^c \in I$, where
\[
S = \{k \in \mathcal{N} : |\phi_{m,k}(x_k^{(q_0)}) - a_{q_0}|^{p_k} < \left(\frac{\delta}{3D}\right)^M\}.
\]
Let
\[ U^c = P^c \cap Q^c \cap S^c, \]
where
\[ U = \{ k \in \mathbb{N} : |\phi_{m,k}(x) - a|^p_k < \delta \}. \]

Therefore for each \( k \in U^c \), we have
\[ \{ k \in \mathbb{N} : |\phi_{m,k}(x) - a|^p_k < \delta \} \supseteq \{ k \in \mathbb{N} : |\phi_{m,k}(x^{(q_0)}) - \phi_{m,k}(x)|^p_k < \left( \frac{\delta}{3D} \right)^M \}, \]
\[ \cap \{ k \in \mathbb{N} : |\phi_{m,k}(x^{(q_0)}) - a_{q_0}|^p_k < \left( \frac{\delta}{3D} \right)^M \}, \]
\[ \cap \{ k \in \mathbb{N} : |a_{q_0} - a|^p_k < \left( \frac{\delta}{3} \right)^M \}. \]

Then the result follows.

Since the inclusion \( BV^l(p) \subset \ell^l(p) \) is strict so in view of Theorem 6.2.3 we have the following result.

**Theorem 6.2.4.** The space \( BV^l(p) \) is nowhere dense subset of \( \ell^l(p) \).

**Theorem 6.2.5.** The space \( BV^l(p) \) is not separable.

**Proof.** Let \( M = \{ m_1 < m_2 < m_3 < \ldots \} \) be an infinite subset of \( \mathbb{N} \) such that \( M \in I \).

Let
\[ p_k = \begin{cases} 1, & \text{if } k \in M, \\ 2, & \text{otherwise}. \end{cases} \]

Let
\[ P_0 = \{ (x_k) : \phi_{m,k}(x) = 0 \text{ or } 1, \text{ for } k = m_j, j \in \mathbb{N} \text{ and } \phi_{m,k}(x) = 0, \text{otherwise} \}. \]

Since \( M \) is infinite, so \( P_0 \) is uncountable. Consider the class of open balls
\[ B_1 = \{ B(z, \frac{1}{2}) : z \in P_0 \}. \]
Let $C_1$ be an open cover of $BV_0'(p)$ containing $B_1$. Since $B_1$ is uncountable, so $C_1$ cannot be reduced to a countable subcover for $BV_0'(p)$. Thus $BV_0'(p)$ is not separable.

**Theorem 6.2.6.** The function $h : BV_0'(p) \to$ is the Lipschitz function and is uniformly continuous.

**Proof.** Let $x, y \in BV_0'(p)$ and $x \neq y$. Then the sets

$$A_x = \{ k \in \mathbb{N} : |\phi_{m,k}(x) - h(x)|^{p_k} \geq ||x - y||_* \} \in I,$$

$$A_y = \{ k \in \mathbb{N} : |\phi_{m,k}(y) - h(y)|^{p_k} \geq ||x - y||_* \} \in I.$$

Thus the sets,

$$B_x = \{ k \in \mathbb{N} : |\phi_{m,k}(x) - h(x)|^{p_k} < ||x - y||_* \} \in BV_0'(p),$$

$$B_y = \{ k \in \mathbb{N} : |\phi_{m,k}(y) - h(y)|^{p_k} < ||x - y||_* \} \in BV_0'(p).$$

Hence also $B = B_x \cap B_y \in BV_0'(p)$, so that $B \neq \emptyset$.

Now taking $k \in B$,

$$|h(x) - h(y)|^{p_k} \leq |h(x) - \phi_{m,k}(x)|^{p_k} + |\phi_{m,k}(x) - \phi_{m,k}(y)|^{p_k} + |\phi_{m,k}(y) - h(y)|^{p_k},$$

$$\leq 3||x - y||_*.$$

Thus $h$ is a Lipschitz function.

**Theorem 6.2.7.** $c_0'(p) \subset BV_0'(p) \subset \ell_\infty'(p)$.

**Proof.** Let $(x_k) \in c_0'(p)$. Then we have

$$\{ k \in \mathbb{N} : |x_k|^{p_k} \geq \epsilon \} \in I,$$

Since $c_0 \subset BV_0$, $(x_k) \in BV_0'(p)$ implies

$$\{ k \in \mathbb{N} : |\phi_{m,k}(x)|^{p_k} \geq \epsilon \} \in I.$$
Now let
\[ A_1 = \{ k \in \mathcal{N} : |x_k|^p < \varepsilon \} \in I. \]
\[ A_2 = \{ k \in \mathcal{N} : |\phi_{m,k}(x)|^p < \varepsilon \} \in I. \]
be such that \( A_1^c, A_2^c \in I \). As
\[ \ell_\infty(p) = \{ x = (x_k) : \sup_k |x_k|^p < \infty \}, \]
taking supremum over \( k \) we get \( A_1^c \subset A_2^c \). Hence
\[ c_0^f(p) \subset BV_\sigma^f(p) \subset \ell_\infty(p). \]

**Theorem 6.2.8.** \( c^f(p) \subset BV_\sigma^f(p) \subset \ell_\infty(p). \)

**Proof.** Let \((x_k) \in c^f(p)\). Then we have
\[ \{ k \in \mathcal{N} : |x_k - L|^p \geq \varepsilon \} \in I. \]
Since
\[ c \subset BV_\sigma \subset \ell_\infty. \]
\[ (x_k) \in BV^f_\sigma(p) \]
implies
\[ \{ k \in \mathcal{N} : |\phi_{m,k}(x) - L|^p \geq \varepsilon \} \in I. \]
Now let
\[ B_1 = \{ k \in \mathcal{N} : |\phi_k - L|^p < \varepsilon \} \in I, \]
\[ B_2 = \{ k \in \mathcal{N} : |\phi_{m,k}(x) - L|^p < \varepsilon \} \in I. \]
be such that \( B_1^c, B_2^c \in I \).
As
\[ \ell_\infty(p) = \{ x = (x_k) : \sup_k |x_k|^p < \infty \}, \]
taking supremum over $k$ we get $B_1^c \subset B_2^c$. Hence

$$c^I(p) \subset BV^I_\sigma(p) \subset \ell^I_\infty(p).$$

**Theorem 6.2.9.** If $H = \sup_k p_k < \infty$, then we have

$$\ell^I_\infty \subset M(BV^I_\sigma(p)),$$

where the inclusion may be proper.

**Proof.** Let $a \in \ell^I_\infty$. This implies that $\sup_k |a_k| < 1 + K$ for some $K > 0$ and all $k$. Therefore $x \in BV^I_\sigma(p)$ implies

$$\sup_k(|a_k \phi_{m,k}(x)|^{p_k}) \leq (1 + K)^H \sup_k(|\phi_{m,k}(x)|^{p_k}) < \infty.$$

Which gives

$$\ell^I_\infty \subset M(BV^I_\sigma(p)).$$

To show that the inclusion may be proper, consider the case when $p_k = \frac{1}{k}$ for all $k$. Take $a_k = k$ for all $k$. Therefore $x \in BV^I_\sigma(p)$ implies

$$\sup_k(|a_k \phi_{m,k}(x)|^{p_k}) \leq \sup_k(|k|^{\frac{1}{k}}) \sup_k(|\phi_{m,k}(x)|^{p_k}) < \infty.$$

Thus in this case

$$a = \{a_k\} \in M(BV^I_\sigma(p),$$

while $a \notin \ell^I_\infty$.

**Theorem 6.2.10.** Let $(p_k)$ and $(q_k)$ be two sequences of positive real numbers. Then $BV^I_\sigma(p) \supseteq BV^I_\sigma(q)$ if and only if $\liminf_{k \in I} \frac{p_k}{q_k} > 0$, where $K^c \subseteq \mathcal{N}$ such that $K \in I$.

**Proof.** Let $\liminf_{k \in K} \frac{p_k}{q_k} > 0$ and $(x_k) \in BV^I_\sigma(q)$. Then there exists $\beta > 0$ such that

$$p_k > \beta q_k,$$

for all sufficiently large $k \in K$. Since $(x_k) \in BV^I_\sigma(q)$ for a given $\epsilon > 0$, we have

$$B_0 = \{k \in \mathcal{N} : |\phi_{m,k}(x)| - L|^{q_k} \geq \epsilon \} \in I.$$
Let $G_0 = K^c \cup B_0$ Then $G_0 \in I$. Then for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : |\phi_{m,k}(x) - L|^{p_k} \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : |\phi_{m,k}(x) - L|^{q_k} \geq \varepsilon\} \in I.$$ 

Therefore $(x_k) \in BV^I_\sigma(p)$. The converse part of the result follows obviously.

**Theorem 6.2.11.** Let $(p_k)$ and $(q_k)$ be two sequences of positive real numbers. Then

$$BV^I_\sigma(q) \supseteq BV^I_\sigma(p),$$

if and only if

$$\liminf_{k \in K} \frac{q_k}{p_k} > 0,$$

where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

**Proof.** The proof follows similarly as the proof of Theorem 6.2.8.

**Theorem 6.2.12.** Let $(p_k)$ and $(q_k)$ be two sequences of positive real numbers. Then

$$BV^I_\sigma(p) = BV^I_\sigma(q),$$

if and only if

$$\liminf_{k \in K} \frac{p_k}{q_k} > 0,$$

and

$$\liminf_{k \in K} \frac{q_k}{p_k} > 0,$$

where, $K \subseteq \mathbb{N}$ such that $K^c \in I$.

**Proof.** On combining Theorem 6.2.10 and 6.2.11 we get the required result.