CHAPTER 1

INTRODUCTION

1.1 BRIEF SURVEY

Optimization is the heart of applied mathematics. Optimization is an essential process in many businesses, management, and engineering applications. From a purely mathematical point of view optimization is interesting in itself, for it contains quite interesting theories and results. It offers us challenging open problems of theoretical nature as well as of algorithmic or numerical character.

Almost all practical optimization problems are concerned with more than a single objective function. Real life problems require the optimization of multiple objectives at the same time. For example, in planning of transportations or distribution of schedules, the total transit time is usually not the only criterion to consider. A planner may also consider the cost and the reliability of the planner schedule before it is actually executed. When the evaluation process deals with more than one objective, the problems of this nature are described as multiple objective optimizations. These objectives are often inter-conflicting. When objectives are conflicting, this implies that an objective cannot be improved without affecting the optimality of the other objectives. A possible solution to multiple criteria optimization should provide balance in objectives. These solutions may be suboptimal with respect to single objective programming problem. In fact, they are called trade-off solutions that are regarded as the best solution. Multiple criteria optimization is most often applied to deterministic problem in which the number of feasible alternatives is large. It is more useful with less controversial in business and government such as in oil refinery, scheduling, production planning, capital budgeting, forest management, determining reservoir release policy, allocation of audit staff in a firm, transportation and many others.

An optimization problem consists of an objective function that requires to be optimized subject to certain limitations which are termed as constraints. The idea is to select the best solution specified by the objective function from a set of various alternatives. Quite often more than one objective function with conflicting nature arises in many real world problems.
Such optimization problems are referred to as vector optimization problems or multiobjective optimization problems. Vector optimization problem was first considered in the legendary paper of Kuhn and Tucker (1951). Finding a solution that satisfies all the objective functions is almost an impossible task. Many solution concepts based on the principal of compromise has been proposed for vector optimization problems in literature. In 1986, a solution concept for vector optimization was proposed by Italian economist, sociologist and philosopher Vilfredo Federico Damaso Pareto, which was later referred to as Pareto solution. For introduction to vector optimization theory and applications one may refer book by Jahn (2004).

In vector optimization one investigates optimal elements of a set in a pre-ordered space. For these problems, one cannot identify a single solution that simultaneously optimizes each of the objectives. A solution of a vector optimization problem is called non-dominated, Pareto optimal, or Pareto efficient if it cannot be eliminated from consideration by replacing it with another solution which improves an objective without worsening another one.

The first and foremost idea of optimality in the setting of multiobjective programming goes back to Edgeworth in 1881 and Pareto in 1896 as well known historical references and is still the most extensively used. In (Edgeworth) Pareto-optimality every feasible alternative that is not dominated by any other in terms of the component wise partial order is considered to be optimal. Hence, each solution is considered optimal that is not definitely worse than another. In essence, multiobjective optimization does not yield a single or a set of equally good answers, but rather suggests a range of potentially very different answers.

Optimality criteria are extensively studied as a vital organic part in the theory of single or multiple optimization theory because these criteria lay the foundation of duality which is a natural and significant concept. The term duality used in our daily life means the sort of harmony of two opposite or complementary parts by which they integrate into whole. Inner beauty in natural phenomena is bound up with duality, which has always been a rich source of inspiration in human knowledge through the centuries. The theory of duality is a vast subject, significant in art and natural science. Mathematics lies in its roots. The concept of duality has proved to be valuable notion in analysis of linear and nonlinear programming. According to Dantzig (1963) the notion of duality was first introduced by Von-Neumann (1947) and was subsequently formulated in the precise form by Gale, Kuhn and Tucker (1951). The concept
of duality is to associate with each mathematical programming, called primal, (a Latin word, which means original) another mathematical programming called dual program. This idea is useful in economics where the dual problem can be stated in terms of price, in mechanics where primal and dual problems are two well-known forms of conservation principles.

Kuhn and Tucker (1951) were the first to incorporate some interesting results concerning multiobjective optimization in 1951. Since then, research in this area has made remarkable progress both theoretically and practically. Some of the earliest attempts to obtain conditions for efficiency were carried out by Kuhn and Tucker (1951), Arrow et al. (1958). Their research has been inherited by Da Cunha and Polak (1967), Neustadt (1976), Ritter (1969-1970), Smale (1973), Aubin (1979), Gulati et al. (1997-2006) and others.

Duality, which plays an important role in traditional mathematical programming, has been extended to multiobjective optimization since the late 1970’s. Isermann (1974-1978) developed multiobjective duality in linear case while results for nonlinear cases have been given by Schonfeld (1970), Tanino and Sawaragi (1979), Mazzoleni (1980), Corley (1981), Nakayama (1984) and others.

Concept of mixed type multiobjective duality seems to be quite interesting and useful from practical as well as from algorithmic point of view. The computational advantage of mixed type dual formulations involves the flexibility of the choice of constraints to be put in the Lagrange function can be exploited to develop certain efficient solution procedures for solving mathematical programming problems.

1.2 NOTATIONS, DEFINITIONS AND MAIN CONCEPTS

Notations 1.2.1

In this section, we shall incorporate major symbols which are used throughout the research work reported in this thesis.

\[ \mathbb{R}^n = \text{n-dimensional Euclidean space}, \]

\[ \mathbb{R}_+^n = \text{The non-negative orthant in } \mathbb{R}^n, \]

\[ \mathbb{A}^T = \text{Transpose of the matrix A}. \]
Let $f$ be a numerical function defined on an open set $\Gamma$ in $\mathbb{R}^n$, then $\nabla f(x)$ denotes the gradient of $f$ at $x$, that is

$$\nabla f(x) = \left[ \frac{\partial f(x)}{\partial x^1}, \ldots, \frac{\partial f(x)}{\partial x^n} \right]^T$$

Let $\phi$ be a real valued twice continuously differentiable function defined on an open set contained in $\mathbb{R}^n \times \mathbb{R}^m$. Then $\nabla_x \phi(x, y)$ and $\nabla_y \phi(x, y)$ denote the gradient (column) vector of $\phi$ with respect to $x$ and $y$ respectively i.e.,

$$\nabla_x \phi(x, y) = \left( \frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \ldots, \frac{\partial \phi}{\partial x^n} \right)^T_{(x, y)}$$

$$\nabla_y \phi(x, y) = \left( \frac{\partial \phi}{\partial y^1}, \frac{\partial \phi}{\partial y^2}, \ldots, \frac{\partial \phi}{\partial y^m} \right)^T_{(x, y)}$$

Further $\nabla^2_{xx} \phi(x, y)$ and $\nabla^2_{yy} \phi(x, y)$ denote respectively the $(n \times n)$ and matrices of second-order partial derivative i.e.,

$$\nabla^2_{xx} \phi(x, y) = \left( \frac{\partial^2 \phi}{\partial x^i \partial x^j} \right)_{(x, y)}$$

$$\nabla^2_{yy} \phi(x, y) = \left( \frac{\partial^2 \phi}{\partial y^i \partial y^j} \right)_{(x, y)}$$

The symbols $\nabla^2_{xy} \phi(x, y)$ and $\nabla^2_{yx} \phi(x, y)$ are similarly defined.

**Definitions 1.2.2:** Let $X \subseteq \mathbb{R}^n$ be an open and convex set and $f : X \to \mathbb{R}$ be differentiable. Then we define $f$ to be

1. **Convex**, if for all $x_1, x_2 \in X$,

$$f(x_1) - f(x_2) \geq (x_1 - x_2) \nabla f(x_2).$$

2. **Strictly convex**, if for all $x_1, x_2 \in X$ and $x_1 \neq x_2$,

$$f(x_1) - f(x_2) > (x_1 - x_2) \nabla f(x_2).$$
3. **Quasiconvex**, if for all \( x_1, x_2 \in X \),

\[
    f(x_1) \leq f(x_2) \Rightarrow (x_1 - x_2) \nabla f(x_2) \leq 0.
\]

4. **Pseudoconvex**, if for all \( x_1, x_2 \in X \),

\[
    (x_1 - x_2) \nabla f(x_2) \geq 0 \Rightarrow f(x_1) \geq f(x_2).
\]

5. **Strictly pseudoconvex**, if for all \( x_1, x_2 \in X \) and \( x_1 \neq x_2 \),

\[
    (x_1 - x_2) \nabla f(x_2) \geq 0 \Rightarrow f(x_1) > f(x_2).
\]

6. **Invex**, if there exists a vector function \( \eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that for all \( x_1, x_2 \in X \),

\[
    f(x_1) - f(x_2) \geq \eta(x_1, x_2)^T \nabla f(x_2).
\]

7. **Pseudoinvex**, if there exists a vector function \( \eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that for all \( x_1, x_2 \in X \),

\[
    \eta^T (x_1, x_2) \nabla f(x_2) \geq 0 \Rightarrow f(x_1) \geq f(x_2).
\]

8. **Strictly pseudoinvex**, if there exists a vector function \( \eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that for all \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \),

\[
    \eta^T (x_1, x_2) \nabla f(x_2) \geq 0 \Rightarrow f(x_1) > f(x_2).
\]

Equivalently, if

\[
    f(x_1) \leq f(x_2) \Rightarrow \eta^T (x_1, x_2) \nabla f(x_2) < 0.
\]

9. **Quasi-invex**, if there exists a vector function \( \eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that for all \( x_1, x_2 \in X \),

\[
    f(x_1) \leq f(x_2) \Rightarrow \eta^T (x_1, x_2) \nabla f(x_2) \leq 0.
\]

Clearly, a differentiable convex, pseudoconvex, quasiconvex function is invex, pseudoinvex or quasi invex respectively with \( \eta^T (x_1, x_2) = (x_1 - x_2) \). Further we define \( f(.) \) to be concave, strictly concave pseudoconcave, quasiconcave, strictly pseudo convex on \( X \).
according as \(-f(.)\) is convex, strictly convex, quasi convex, pseudoconvex, and strictly pseudoconvex.

In the following definitions we shall use \(D\) and \(D^2\) for customary symbols \(\frac{d}{dt}\) and \(\frac{d^2}{dt^2}\):

10. Invexity and generalized invexity in continuous programming

1. Invexity, if there exists vector function \(\eta(t,x,u) \in \mathbb{R}^a\) with \(\eta = 0\) and \(x(t) = u(t)\), \(t \in I = [a,b]\), a real interval, such that for a scalar function \(\phi(t,x,\dot{x})\), the functional \(\Phi(x) = \int_I \phi(t,x,\dot{x})\,dt\) satisfies

\[
\Phi(u) - \Phi(x) \geq \int_I \left( \eta \phi_x(t,x,\dot{x}) + (D\eta)^T \phi_x(t,x,\dot{x}) \right) dt,
\]

\(\Phi\) is said to be invex in \(x\) and \(\dot{x}\) on \(I\) with respect to \(\eta\).

2. Pseudoinvexity, \(\Phi\) is said to be pseudoinvex in \(x\) and \(\dot{x}\) with respect to \(\eta\) if

\[
\int_I \left( \eta^T \phi_x(t,x,\dot{x}) + (D\eta)^T \phi_x(t,x,\dot{x}) \right) dt \geq 0 \Rightarrow \Phi(u) \geq \Phi(x).
\]

3. Quasi-invex, \(\Phi\) is said to quasi-invex in \(x\) and \(\dot{x}\) with respect to \(\eta\) if

\[
\Phi(u) \leq \Phi(x) \Rightarrow \int_I \left( \eta^T \phi_x(t,x,\dot{x}) + (D\eta)^T \phi_x(t,x,\dot{x}) \right) dt \leq 0.
\]

4. Efficient and properly efficient solutions

We consider the following multiobjective variational problem as:

\((P):\) Minimize \(\int_I f(t,x(t),\dot{x}(t))\,dt = \left( \int_I f^1(t,x(t),\dot{x}(t))\,dt, \ldots, \int_I f^p(t,x(t),\dot{x}(t))\,dt \right)\)

subject to

\(x(a) = \alpha, x(b) = \beta,\)

\(g(t,x(t),\dot{x}(t)) \leq 0, \ t \in I.\)

Let \(K\) the set of feasible solutions for \((P)\) be given by
\[ K = \left\{ x : C(I, \mathbb{R}^n) \Big| x(a) = \alpha, x(b) = \beta, g(t, x(t), \dot{x}(t)) \leq 0, t \in I \right\}. \]

4 (i): A point \( x^* \) in \( K \) is said to be an efficient solution of (P) if for all \( x \) in \( K \),

\[
\int_I f^i(t, x^*(t), \dot{x}^*(t)) dt \geq \int_I f^i(t, x(t), \dot{x}(t)) dt, \quad \forall i \in \{1, 2, \ldots, p\},
\]

\[
\Rightarrow \int_I f^i(t, x^*(t), \dot{x}^*(t)) dt = \int_I f^i(t, x(t), \dot{x}(t)) dt, \quad \forall i \in \{1, 2, \ldots, p\}.
\]

4 (ii) (Borwein (1974)): A point \( x^* \) in \( K \) is said to be a weak minimum for (P) if there exists no other \( x \) in \( K \) for which

\[
\int_I f^i(t, x^*(t), \dot{x}^*(t)) dt > \int_I f^i(t, x(t), \dot{x}(t)) dt.
\]

From this it follows that if an \( x^* \) in \( K \) is efficient for (P) then it is also a weak minimum for the problem (P).

4 (iii) (Geoffrion (1968)): A point \( x^* \) in \( K \) is said to be properly efficient solution of (P) if there exists a scalar \( M > 0 \) such that \( \forall i \in \{1, 2, \ldots, p\} \),

\[
\int_I f^i(t, x^*(t), \dot{x}^*(t)) dt - \int_I f^i(t, x(t), \dot{x}(t)) dt \leq M \left[ \int_I f^j(t, x(t), \dot{x}(t)) dt - \int_I f^j(t, x^*(t), \dot{x}^*(t)) dt \right]
\]

for some \( j \) such that

\[
\int_I f^j(t, x(t), \dot{x}(t)) dt > \int_I f^j(t, x^*(t), \dot{x}^*(t)) dt,
\]

whenever \( x \) is in \( K \) and

\[
\int_I f^i(t, x(t), \dot{x}(t)) dt < \int_I f^i(t, x^*(t), \dot{x}^*(t)) dt.
\]

11. Constraint Qualifications

There are number of constraint qualifications Mangasarian (1969), which are required to be satisfied by the constraints in establishing the necessary optimality criteria to ensure that certain Lagrange multipliers are non zero. Here we describe only a few of them for completeness.
(i) **Slater’s Constraint Qualification:** Let $X^0$ be a convex set in $R^n$. The $m$-dimensional convex vector function $g$ on $X^0$ which defines the convex feasible region $X = \{ x : x \in X^o, g(x) \leq 0 \}$ is said to satisfy Slater’s constraint qualification on $X^0$ if there exists an $\bar{x} \in X^o$ such that $g(\bar{x}) < 0$.

(ii) **The reverse convex constraint qualification:** Let $X^0$ be an open set in $R^n$. Let $g$ be $m$-dimensional vector function defined on $X^0$ and let $X = \{ x : x \in X^o, g(x) \leq 0 \}$, $g$ is said to satisfy the reverse constraint qualification at $\bar{x} \in X$, if $g$ is differentiable at $\bar{x}$ and if for each $i \in I$ either $g_i$ is concave at $\bar{x}$ or $g_i$ is linear on $R^n$, where $I = \{ i | g_i(\bar{x}) = 0 \}$.

(iii) **Linear independence constraint qualification:** The condition that the vectors $\nabla g_i(x_0), \ldots, \nabla g_m(x_0)$ are linearly independent is often referred to as linearly independence constraint qualification.

### 1.3 REVIEW OF THE RELATED WORK

#### 1.3.1 Optimality and Duality in Multiobjective Mathematical Programming

For the study of multiobjective programming problem we shall follow the following conventions for vectors $x, y$ in $R^n$

$$x < y \iff x_i < y_i, \ i = 1, 2, \ldots, n$$

$$x \preceq y \iff x_i \preceq y_i, \ i = 1, 2, \ldots, n$$

$$x \preceq y \iff x_i \preceq y_i, \ i = 1, 2, \ldots, n \text{ but } x \neq y$$

$x \prec y$ is the negation of $x \preceq y$.

Consider the following multiobjective programming problem:

**(VP):** V- Min $F(x) = (f_1(x), f_2(x), \ldots, f_p(x))$

subject to

$$g_j(x) \leq 0, \ (j = 1, 2, \ldots, m)$$
where $X \subseteq \mathbb{R}^n$ is an open convex set, and $f_i$ and $g_j$ are differentiable functions with $f_i : X \rightarrow R, i = 1, 2, \ldots, p$ and $g_j : X \rightarrow R, j = 1, 2, \ldots, m$. Here the symbol “V-Min” stands for vector minimization and minimality is taken in terms of “efficient points” given by Koopmans (1951) and Geoffrion (1968) respectively.

**Definition 1.3.1:** A feasible point $\bar{x}$ is said to be an efficient solution of (VP), if there does not exist any feasible $x$ for (VP) such that

$$f_r(x) < f_r(\bar{x}) \text{ for some } r,$$

$$f_i(x) \leq f_i(\bar{x}) \text{ for all } i = 1, 2, \ldots, p, \ i \neq r.$$

Geoffrion (1968) considered the following single objective minimization problems for fixed $\lambda \in \mathbb{R}^p$:

$$(VP)_{\lambda}: \text{Minimize } \sum_{i=1}^{p} \lambda^i f_i(x)$$

subject to

$$g_j(x) \leq 0, \ (j = 1, 2, \ldots, m)$$

and proved the following lemma connecting (VP) and $(VP)_{\lambda}$.

**Lemma 1.3.1:**

(i) Let $\lambda^i > 0, (i = 1, 2, \ldots, p), \sum_{i=1}^{p} \lambda^i = 1$ be fixed. If $\bar{x}$ is optimal for $(VP)_{\lambda}$, then $\bar{x}$ is properly efficient for (VP).

(ii) Let $f_i$ and $g_j$ be convex functions. Then $\bar{x}$ is properly efficient for (VP) if and only if $\bar{x}$ is optimal for $(VP)_{\lambda}$ for some $\lambda > 0, \sum_{i=1}^{p} \lambda^i = 1$.

If $f_i$ and $g_j$ are differentiable convex functions then $(VP)_{\lambda}$ is a convex programming problem. Therefore in relation to $(VP)_{\lambda}$, consider the scalar maximization problem,

$$(VD)_{\lambda}: \text{Maximize } \lambda^T f(x) + y^T g(x) = \lambda^T \left( f(x) + y^T g(x)e \right)$$
subject to

$$\nabla \left( \lambda^T f(x) + y^T g(x) \right) = 0,$$

$$\lambda \in \Lambda^+, \quad y \geq 0,$$

where $$e = (1,1,\ldots,1) \in R^p$$ and $$\Lambda^+ = \{ \lambda \in R^p : \lambda > 0, \lambda^T e = 1 \}$$.

Now as $$(VD)_\lambda$$ is a dual program of $$(VP)_\lambda$$, Weir (1987) considered the following vector optimization problem in relation to $$(VP)$$ as:

(DV): Maximize $$\lambda^T f(x) + y^T g(x)$$

subject to

$$\nabla \left( \lambda^T f(x) + y^T g(x) \right) = 0,$$

$$\lambda \in \Lambda^+, \quad y \geq 0.$$

Weir (1987) termed (DV) as the dual of (VP) and proved various duality theorems between (VP) and (DV) under the assumption that $$f$$ and $$g$$ are convex functions.

Further, for the purpose of weakening the convexity requirements on objective and constraint functions, Weir (1987) introduced another dual program (DV1).

(DV1): Maximize $$f(x)$$

subject to

$$\nabla \left( \lambda^T f(x) + y^T g(x) \right) = 0,$$

$$y^T g(x) \geq 0,$$

$$\lambda \in \Lambda^+, \quad y \geq 0.$$

For these problems, various duality theorems are proved by assuming the function $$f$$ to be pseudo convex and $$y^T g$$ to be quasiconvex for their feasible solutions.

1.3.2 Symmetric duality in multiobjective programming

Mond and Weir (1991) discussed symmetric duality in multiobjective programming by considering the following pair of programs:
(PS): Minimize \( f(x, y) - (y^T \nabla_y \lambda^T f(x, y)) e \)

subject to
\[ \nabla_y \lambda^T f(x, y) \leq 0, \]
\[ x \geq 0, \lambda \in \Lambda^+. \]

(DS): Maximize \( f(x, y) - (x^T \nabla_x \lambda^T f(x, y)) e \)

subject to
\[ \nabla_x \lambda^T f(x, y) \geq 0, \]
\[ y \geq 0, \lambda \in \Lambda^+, \]

where \( f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p \) and proved the symmetric duality theorem under the convexity–concavity assumptions on \( f(x, y) \). Here the minimization is taken in the sense of proper efficiency as given by Geoffrion (1968).

Further, on the lines of scalar case (Mond and Weir (1981)) also considered another pair of symmetric dual programs and proved symmetric duality results under pseudoconvexity-pseudoconcavity:

(PS1): Minimize \( f(x, y) - (y^T \nabla_2 \lambda^T f(x, y)) e \)

subject to
\[ \nabla_2 \lambda^T f(x, y) \leq 0, \]
\[ y^T \nabla_2 \lambda^T f(x, y) \geq 0, \]
\[ x \geq 0, \lambda \in \Lambda^+. \]

(DS1): Maximize \( f(x, y) - (x^T \nabla_1 \lambda^T f(x, y)) e \)

subject to
\[ \nabla_1 \lambda^T f(x, y) \geq 0, \]
\[ x^T \nabla_1 \lambda^T f(x, y) \leq 0, \]
\[ y \geq 0, \lambda \in \Lambda^+. \]
Later, Chandra and Durga Prasad (1993) introduced the following pair of multiobjective programs by associating a vector valued infinite game:

**PS*: Minimize $f(x, y) - \left( y^T \nabla_y \mu^T f(x, y) \right) e$

subject to

$$\nabla_y \mu^T f(x, y) \leq 0,$$

$$x \geq 0, \mu \in \Lambda^+.$$

**DS*: Maximize $f(x, y) - \left( x^T \nabla_x \lambda^T f(x, y) \right) e$

subject to

$$\nabla_x \lambda^T f(x, y) \geq 0,$$

$$y \geq 0, \lambda \in \Lambda^+.$$

Here it may be noted that not the same $\lambda$ is appearing in (PS*) and (DS*) and this creates certain difficulties which are also discussed in Chandra and Durga Prasad (1993).

**1.3.3 Multiobjective variational problems**

We consider the following multiobjective variational problem introduced by Bector and Husain (1992) with its Wolfe type dual:

**P**: Minimize $\int f(t, x(t), \dot{x}(t)) dt = \left( \int f^1(t, x(t), \dot{x}(t)) dt, ..., \int f^p(t, x(t), \dot{x}(t)) dt \right)$

subject to

$$x(a) = \alpha, x(b) = \beta,$$

$$g(t, x(t), \dot{x}(t)) \leq 0, t \in I.$$

**WD**: Maximize $\int \left( \int \left( f^1(t, u(t), \dot{u}(t)) + y(t)^T g(t, u(t), \dot{u}(t)) \right) dt, ..., \int \left( f^p(t, u(t), \dot{u}(t)) + y(t)^T g(t, u(t), \dot{u}(t)) \right) dt \right)$

subject to

$$u(a) = \alpha, u(b) = \beta,$$
\[ \lambda^T f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t)) \]
\[ = D\left(\lambda^T f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t)) + z(t)^T h_u\right), \quad t \in I, \]
\[ y(t) \geq 0, \quad t \in I, \]
\[ \lambda \in \Lambda^+. \]

Bector and Husain (1992) proved the following duality theorems for the pair of problems (P) and (WD):

**Theorem 1.3.3.1 (Weak duality):** Let \( x(t) \in K \) and \((u(t), \lambda, y(t)) \in G\). Let \( f \) and \( g \) be convex at \((u, \dot{u})\) over K. Then the following cannot hold:

\[ \int_I f^i(t,x(t),\dot{x}(t))dt \leq \int_I \left(f^i(t,u(t),\dot{u}(t)) + y(t)^T g(t,u(t),\dot{u}(t))\right)dt, \quad \text{for some } i \in \{1, 2, ..., p\} \]

and

\[ \int_I f^j(t,x(t),\dot{x}(t))dt < \int_I \left(f^j(t,u(t),\dot{u}(t)) + y(t)^T g(t,u(t),\dot{u}(t))\right)dt, \text{for at least one } j, \]

where \( G \) designates the set of feasible solutions of the dual problem (WD).

**Theorem 1.3.3.2 (Strong duality):** Let \( f \) and \( g \) be convex at \((u, \dot{u})\) over K. Let \( x^* \) be normal and a properly efficient solution for (P). Then for some \( \lambda \in \Lambda^+ \), there exists a piecewise smooth function \( y^*: I \rightarrow \mathbb{R}^m \) such that \((x^*, \lambda, y^*)\) is a properly efficient solution for (WD) and

\[ \int_I f(t,x^*(t),\dot{x}^*(t))dt = \int_I \left(f(t,x^*(t),\dot{x}^*(t)) + y(t)^T g(t,x^*(t),\dot{x}^*(t))\right)dt. \]

In the following theorem, \( \theta = \theta(t,u(t),\dot{u}(t),\ddot{u}(t),\lambda, y(t), \dot{y}(t)) \)

\[ = \lambda^T f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t)) \]
\[ - D\left(\lambda^T f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t)) + z(t)^T h_u\right), \quad t \in I \]

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with $\ddot{u}(t) = D^2 u(t)$, $X_2$ denotes the space of piecewise differentiable function $x : I \to \mathbb{R}^n$ for which $x(a) = \alpha$, $x(b) = \beta$ equipped with the norm $\|x\| = \|x\|_{\infty} + \|Dx\|_{\infty} + \|D^2 x\|_{\infty}$, defining $D$ as before and $\theta$ is considered as defining a map $\psi : X_2 \times Y \times \Lambda^+ \to A$, where $Y$ is the space of piecewise differentiable function $y : I \to \mathbb{R}^m$ and $A$ is a Banach space.

**Theorem 1.3.3.3 (Converse duality):** Let $f$ and $g$ be convex at $(u, \dot{u})$ over $K$. Let $(u^*, \lambda^*, y^*)$ with $u^* \in X_2$, $\lambda \in \Lambda^+$, and $y^* \in Y$ be a properly efficient solution of (WD). Let

(i) $\psi'$ have a (weak*) closed range,

(ii) $f$ and $g$ be twice continuously differentiable,

(iii) $f^i_x - Df^i_x$, $i = 1, 2, \ldots, p$, be linearly independent, and

(iv) $\left(\beta(t)^T \theta_x - D\beta(t)^T \theta_x + D^2 \beta(t)^T \theta_x\right)\beta(t) = 0 \Rightarrow \beta(t) = 0, t \in I.$

Then the objective functions of (P) and (WD) are equal and $u^*$ is a properly efficient solution of (P).

The Mond-Weir type dual formulated by Bector and Husain (1992) is as under,

**(MD):** Maximize $\int_I f(t, u(t), \dot{u}(t))dt = \left(\int_I f^1(t, u(t), \dot{u}(t))dt, \ldots, \int_I f^p(t, u(t), \dot{u}(t))dt\right)$

subject to

\[
\begin{align*}
\lambda^T f_a(t, u(t), \ddot{u}(t)) + y(t)^T g_a(t, u(t), \dot{u}(t)) \\
= D\left(\lambda^T f_a(t, u(t), \ddot{u}(t)) + y(t)^T g_a(t, u(t), \dot{u}(t)) + z(t)^T h_u\right), t \in I,
\end{align*}
\]

$y(t) \geq 0, t \in I,$

$\lambda \in \Lambda^+.$

Bector and Husain (1992) proved the following duality theorems for the pair of problems (P) and (WD):
**Theorem 1.3.3.4 (Weak duality):** Let \( x(t) \in K \) and \( (u(t), \lambda, y(t)) \in H \). Let \( f \) and \( g \) be convex at \( (u, \dot{u}) \) over \( K \). Then the following cannot hold:

\[
\int f^i \left( t, x(t), \dot{x}(t) \right) dt \leq \int f^i \left( t, u(t), \dot{u}(t) \right) dt, \text{ for some } i \in \{1, 2, \ldots, p\}
\]

and

\[
\int f^j \left( t, x(t), \dot{x}(t) \right) dt < \int f^j \left( t, u(t), \dot{u}(t) \right) dt, \text{ for at least one } j,
\]

where \( H \) designates the set of feasible solutions of the dual problem (MD).

**Theorem 1.3.3.5 (Strong duality):** Let \( f \) and \( g \) be convex at \( (u, \dot{u}) \) over \( K \). Let \( x^* \) be normal and a properly efficient solution for (P). Then for some \( \lambda \in \Lambda^+ \), there exists a piecewise smooth function \( y^* : I \rightarrow R^m \) such that \( (x^*, \lambda, y^*) \) is a properly efficient solution for (MD) and

\[
\int f \left( t, x^* (t), \dot{x}^* (t) \right) dt = \int f \left( t, x^* (t), \dot{x}^* (t) \right) dt.
\]

**Theorem 1.3.3.6 (Converse duality):** Let \( f \) and \( g \) be convex at \( (u, \dot{u}) \) over \( K \). Let \( (u^*, \lambda^*, y^*) \) with \( u^* \in X_2, \lambda \in \Lambda^+ \) and \( y^* \in Y \) be a properly efficient solution of (MD). Let

(i) \( \psi^r \) have a (weak*) closed range,

(ii) \( f \) and \( g \) be twice continuously differentiable,

(iii) \( f_i^I - Df_i, i = 1, 2, \ldots, p \), be linearly independent and

(iv) \( \left( \beta(t)^T \partial_x - D\beta(t)^T \partial^2_x + D^2\beta(t)^T \partial^3_x \right) \beta(t) = 0 \Rightarrow \beta(t) = 0, t \in I. \)

Then the objective functions of (P) and (MD) are equal and \( u^* \) is a properly efficient solution of (P).

1.3.4 Symmetric duality for multiobjective variational problems

Gulati, Husain and Ahmad (1997) presented the following pair of symmetric dual multiobjective variational problems:
(SVP): Minimize \( \int_a^b f^1(t,x(t),\dot{x}(t),y(t),\dot{y}(t))dt, \ldots, \int_a^b f^p(t,x(t),\dot{x}(t),y(t),\dot{y}(t))dt \) subject to
\[
x(a) = 0 = x(b), \ y(a) = 0 = y(b),
\dot{x}(a) = 0 = \dot{x}(b), \ \dot{y}(a) = 0 = \dot{y}(b),
\]
\[
\left( \lambda^T f \right)_y (t,x(t),\dot{x}(t),y(t),\dot{y}(t)) \leq \left( \lambda^T f \right)_y (t,x(t),\dot{x}(t),y(t),\dot{y}(t)), t \in I,
\]
\[
y(t)^T \left( \lambda^T f \right)_y (t,x(t),\dot{x}(t),y(t),\dot{y}(t)) \geq y(t)^T \left( \lambda^T f \right)_y (t,x(t),\dot{x}(t),y(t),\dot{y}(t)), t \in I,
\]
\[
x(t) \geq 0, \ t \in I, \\
\lambda > 0.
\]

(SVD): Maximize \( \int_a^b f^1(t,u(t),\dot{u}(t),v(t),\dot{v}(t))dt, \ldots, \int_a^b f^p(t,u(t),\dot{u}(t),v(t),\dot{v}(t))dt \) subject to
\[
u(a) = 0 = u(b), \ v(a) = 0 = v(b),
\dot{u}(a) = 0 = \dot{u}(b), \ \dot{v}(a) = 0 = \dot{v}(b),
\]
\[
\left( \lambda^T f \right)_v (t,u(t),\dot{u}(t),v(t),\dot{v}(t)) \geq \left( \lambda^T f \right)_v (t,u(t),\dot{u}(t),v(t),\dot{v}(t)), t \in I,
\]
\[
u(t) \geq 0, \ t \in I, \\
\lambda > 0.
\]

The following are the duality theorems for the problems (SVP) and (SVD) validated by Gulati et al. (1997):

**Theorem 1.3.4.1 (Weak Duality):** Let

(A1): \( (x(t), y(t), \lambda) \) be feasible for (SVP) and \( (u(t), v(t), \lambda) \), feasible for (SVD)

(A2): \( \int_a^b \lambda^T f(t,\ldots,y(t),\dot{y}(t))dt \) be pseudoconvex in \( x \) for fixed \( y \) and
(A3): $\int_{a}^{b} \lambda^T f(t, x(t), \dot{x}(t), \ldots) dt$ be pseudoconcave in $y$ for fixed $x$.

Then

$$\int_{a}^{b} f(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) dt \leq \int_{a}^{b} f(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) dt.$$ 

**Theorem 1.3.4.2 (Strong duality):** Assume that the pseudoconvexity-pseudoconcavity conditions of Theorem 1.3.4.1 are satisfied. Let

\( (B_1): \left( x^0, y^0, \lambda^0 \right) \) be a properly efficient for \((VP)_\lambda\),
\( (B_2)\): the system

\[
\begin{bmatrix}
\phi(t)^T 
\end{bmatrix}
\begin{bmatrix}
\left( \lambda^{0T} f \right)_{yy} - D\left( \lambda^{0T} f \right)_{yy}
\end{bmatrix}
-D\left( \phi(t)^T \left( D\left( \lambda^{0T} f \right)_{yx} - D\left( \lambda^{0T} f \right)_{yx} \right) \right)
+D^2 \left( -\phi(t)^T \left( \lambda^{0T} f \right)_{yy} \right) \phi(t) = 0 \Rightarrow \phi(t) = 0, t \in I
\] 

and

\( (B_3): \) the set \( \left\{ \left( f_{y}^1 - Df_{y}^1 \right), \ldots, \left( f_{y}^p - Df_{y}^p \right) \right\} \) be linearly independent.

Then \( \left( x^0, y^0 \right) \) is a properly efficient solution of \((VD)\) with \( \lambda = \lambda^0 \).

**Theorem 1.3.4.3 (Converse duality):** Assume that the hypotheses of Theorem 1.3.4.1 are satisfied. Let

\( (C_1): \left( u^0, v^0, \lambda^0 \right) \) be a properly efficient solution of \((VD)\),
\( (C_2): \) the system

\[
\begin{bmatrix}
\phi(t)^T 
\end{bmatrix}
\begin{bmatrix}
\left( \lambda^{0T} f \right)_{xx} - D\left( \lambda^{0T} f \right)_{xx}
\end{bmatrix}
-D\left( \phi(t)^T \left( D\left( \lambda^{0T} f \right)_{xx} - D\left( \lambda^{0T} f \right)_{xx} \right) \right)
+D^2 \left( -\phi(t)^T \left( \lambda^{0T} f \right)_{xx} \right) \phi(t) = 0 \Rightarrow \phi(t) = 0, t \in I
\] 

and

\( (C_3): \) the set \( \left\{ \left( f_{x}^1 - Df_{x}^1 \right), \ldots, \left( f_{x}^p - Df_{x}^p \right) \right\} \) be linearly independent.

Then \( \left( x^0, y^0 \right) \) is a properly efficient solution of \((VP)\) with \( \lambda = \lambda^0 \).
1.3.5 Multiobjective control problems

Bhatia and Kumar (1995) extended the work of Mond and Hanson (1968) to the content of multiobjective control problems and established duality results for Wolfe as well as Mond-Weir type duals under $\rho$-invexity assumptions and their generalizations. They introduced the following multiobjective control problem and formulated its Wolfe and Mond-Weir type duals:

(VCP): Minimize \[
\int_{I}^{f^1(t,x,y)} dt, \ldots, \int_{I}^{f^p(t,x,y)} dt
\]
subject to
\[
x(a) = \alpha, \ x(b) = \beta,
\]
\[
\dot{x} = h(t,x,y), \ t \in I,
\]
\[
g(t,x,y) \leq 0, \ t \in I.
\]

Here $R^n$ denotes an n-dimensional Euclidean space and $I = [a,b]$ is a real interval. Each $f^i : I \times R^n \times R^m \to R$, $i = 1,2,\ldots, p$, $g : I \times R^n \times R^m \to R^p$ and $h : I \times R^n \times R^m \to R^n$ is a continuously differentiable functions.

Let $x : I \to R^n$ be differentiable with its derivative $\dot{x}$, and let $y : I \to R^m$ be a smooth function. Denote the partial derivatives of $f^i$ with respect to $t$, $x$ and $y$ respectively, by $f^i_t$, $f^i_x$ and $f^i_y$ such that

\[f^i_x = \left( \frac{\partial f^i}{\partial x^1}, \frac{\partial f^i}{\partial x^2}, \ldots, \frac{\partial f^i}{\partial x^n} \right), \ f^i_u = \left( \frac{\partial f^i}{\partial y^1}, \frac{\partial f^i}{\partial y^2}, \ldots, \frac{\partial f^i}{\partial y^m} \right),\]

where superscripts denote the vector components.

The following are the Wolfe and Mond-Weir type duals to (VCP) constructed by Bhatia and Kumar (1995):

(WVCD): Maximize \[
\int_{I}^{f^1(t,u,v)+w(t)^T g(t,u,v)} dt, \ldots, \int_{I}^{f^p(t,u,v)+w(t)^T g(t,u,v)} dt
\]
subject to
\[
u(a) = \alpha, \ u(b) = \beta.
\]
\[
\begin{align*}
\sum_{i=1}^{p} \lambda^i f_u^i (t,u,v) + w(t)^T g_u (t,u,v) + z(t)^T h_u (t,u,v) + \dot{z}(t) &= 0, \quad t \in I, \\
\sum_{i=1}^{p} \lambda^i f_v^i (t,u,v) + w(t)^T g_v (t,u,v) + z(t)^T h_v (t,u,v) &= 0, \quad t \in I, \\
\int_I z(t)^T [h(t,u,v) - \dot{u}(t)] dt &\geq 0, \\
w(t) &\geq 0, \quad t \in I \\
\lambda^i &\geq 0, \quad i = 1, 2, \ldots, p, \quad \sum_{i=1}^{p} \lambda^i = 1.
\end{align*}
\]

(MVCD): Maximize \[
\left\{ \int_I f^1 (t,u,v) dt, \ldots, \int_I f^p (t,u,v) dt \right\}
\]
subject to
\[
x(a) = \alpha, \quad x(b) = \beta,
\]
\[
\begin{align*}
\sum_{i=1}^{p} \lambda^i f_u^i (t,u,v) + w(t)^T g_u (t,u,v) + z(t)^T h_u (t,u,v) + \dot{z}(t) &= 0, \quad t \in I, \\
\sum_{i=1}^{p} \lambda^i f_v^i (t,u,v) + w(t)^T g_v (t,u,v) + z(t)^T h_v (t,u,v) &= 0, \quad t \in I, \\
\int_I z(t)^T [h(t,u,v) - \dot{u}(t)] dt &\geq 0, \\
\int_I w(t)^T g(t,u,v) dt &\geq 0, \\
w(t) &\geq 0, \quad t \in I, \\
\lambda^i &\geq 0, \quad i = 1, 2, \ldots, p, \quad \sum_{i=1}^{p} \lambda^i = 1.
\end{align*}
\]

The duality relationship between the vector control problems (VCP) and (WVCD) is stated in the form of the following theorems:

**Theorem 1.3.5.1 (Weak duality):** Assume that for all feasible \((x,y)\) for (VCP) and all \((u,v,w,z,\lambda)\) for (WVCD)
(i): \( f^i(t,x,y) + w(t)^T g(t,x,y) \) is \( \rho_i \)-PIX (\( \rho_i \)-pseudoinvex) with respect to the functions \( \eta_i, \zeta_i \) and \( \rho_i' \)-SPIY (\( \rho_i' \)-strict pseudoincave) with respect to the functions \( \eta_2, \zeta_2 \) (or \( \rho_i' \)-SPIX (\( \rho_i' \)-strict pseudoinvex) with respect to the functions \( \eta_i, \zeta_i \) and \( \rho_i' \)-PIY (\( \rho_i' \)-pseudoincave) with respect to the functions \( \eta_2, \zeta_2 \)) for all \( i \in \{1,2,\ldots,p\} \).

(ii): \( z(t)^T \left[ (h(t,x,y) - \dot{x}(t)) \right] \) is \( \sigma \)-QIX (\( \sigma \)-Quasi-invex) with respect to the same functions \( \eta_i, \zeta_i \) and \( \sigma' \)-QIY (\( \sigma' \)-Quasi-incave) with respect to the same functions \( \eta_2, \zeta_2 \).

(iii): \( \sum_{i=1}^p \lambda_i^i \rho_i + \sigma \geq 0 \) and \( \sum_{i=1}^p \lambda_i^i \rho_i + \sigma' \geq 0 \).

Then the following cannot hold:

\[
\int_I f^i(t,x,y) \, dt < \int_I \left( f^i(t,u,v) + w(t)^T g(t,u,v) \right) \, dt, \quad \text{for some } i \in \{1,2,\ldots,p\},
\]

\[
\int_I f^j(t,x,y) \, dt \leq \int_I \left( f^j(t,u,v) + w(t)^T g(t,u,v) \right) \, dt, \quad \text{for all } j \in \{1,2,\ldots,p\}.
\]

**Theorem 1.3.5.2 (Strong duality):** Let \( (u^0,v^0) \) be efficient for (VCP) and assume that \( (u^0,v^0) \) satisfies the constraints of Theorem (Weak Duality) for \( P_k(u^0,v^0) \) for at least one \( k \in \{1,2,\ldots,p\} \). Then there exists \( \lambda^0 \in \mathbb{R}^k \), and piecewise smooth functions \( w^0 : I \to \mathbb{R}^m \) and \( z^0 : I \to \mathbb{R}^n \) such that \( (u^0,v^0,w^0,z^0,\lambda^0) \) is feasible for (WVCD) and \( w^0(t)^T g(t,u^0,v^0) = 0 \).

If weak duality also holds between (VCP) and (WVCD), then \( (u^0,v^0,w^0,z^0,\lambda^0) \) is efficient for (WVCD).

The duality relationship between the vector control problems (VCP) and (MVCD) is stated in the form of the following theorems:

**Theorem 1.3.5.3 (Weak duality):** Assume that for all feasible \( (x,y) \) for (VCP) and all \( (u,v,w,z,\lambda) \) for (MVCD) if,

(i): \( f^i(t,x,y) \) is \( \rho_i \)-QIX (\( \rho_i \)-Quasi-invex) with respect to the functions \( \eta_i, \zeta_i \) and \( \rho_i' \)-QIY (\( \rho_i' \)-Quasi-incave) with respect to the functions \( \eta_2, \zeta_2 \) for all \( i \in \{1,2,\ldots,p\} \).
(ii): $w(t)^Tg(t,x,y)$ is $\sigma$-QIX ($\sigma$-Quasi-invex) with respect to the same functions $\eta_1, \zeta_1$ and $\sigma'$-QIY ($\sigma'$-Quasi-incave) with respect to the same functions $\eta_2, \zeta_2$.

(iii): $z(t)^T\left(h(t,x,y) - \dot{x}(t)\right)$ is $\mu$-SQIX ($\mu$-strict Quasi-invex) with respect to the same functions $\eta_1, \zeta_1$ and $\mu'$-QIY ($\mu'$-Quasi-incave) with respect to the same functions $\eta_2, \zeta_2$ (or $\mu$-QIX ($\mu$-Quasi-invex) with respect to the same functions $\eta_1, \zeta_1$ and $\mu'$-SQIY ($\mu'$-strict Quasi-incave) with respect to the same functions $\eta_2, \zeta_2$ respectively), and

(iv): $\sum_{i=1}^{p} \lambda_i^j \rho_i + \sigma + \mu \geq 0$ and $\sum_{i=1}^{p} \lambda_i^j \rho_i + \sigma + \mu' \geq 0$.

Then the following cannot hold:

\[ \int f^i(t,x,y) dt < \int f^i(t,u,v) dt, \quad \text{for some } i \in \{1,2,\ldots,p\}, \]

\[ \int f^j(t,x,y) dt \leq \int f^j(t,u,v) dt, \quad \text{for all } j \in \{1,2,\ldots,p\}. \]

**Theorem 1.3.5.4 (Strong duality):** Let \( (u^0, v^0) \) is efficient for (VCP) and assume that \( (u^0, v^0) \) satisfies the constraints of Theorem (Weak Duality) for $P_k(u^0, v^0)$ for at least one $k \in \{1, 2, \ldots, p\}$. Then there exists $\lambda^0 \in R^k$ and piecewise smooth functions $w^0: I \rightarrow R^m$ and $z^0: I \rightarrow R^n$ such that $(u^0, v^0, w^0, z^0 \lambda^0)$ is feasible for (MVCD) and $w^0(t)^Tg(t,u^0,v^0)=0$. If weak duality also holds between (VCP) and (MVCD), then $(u^0, v^0, w^0, z^0 \lambda^0)$ is efficient for (MVCD).

1.4 A BRIEF ACCOUNT OF GAMES

The theory of games started in 20th century but the mathematical treatment of games took fire when John Von Neumann and Morgenstern (1944) published their well known book, “Theory of Games and economic behaviour” the Neumann’s approach uses the minimax principle which involves the fundamental idea of the minimization of the maximum loss. Many of the
competitive problems can be handled by this game theory. However, not all the competitive problems can be analyzed with the help of game theory.

The competition between firms, the conflict between management and labour, the fight to get bills through Congress, the power of judiciary, war and peace negotiations between countries, and so on, all provide examples of games in action. There are also psychological games played on a personal level, where the weapons are words, and the pay-offs are good or bad feelings. There are biological games, the competition between is species, where natural selection can be modelled as a game played between genes. There is a connection between game theory and mathematical areas of logic and computer science. One may view theoretical statistics as a two person game in which nature takes the roll of one of the players.

We denote the strategy set or action space of player \( i \) by \( A_i \), \( i = 1, 2, \ldots, n \). Suppose the player \( I \) chooses \( a_1 \in A_1 \). Player two chooses \( a_2 \in A_2 \) etc. and player \( n \) chooses \( a_n \in A_n \). Then we denote the payoff to the player \( j \) for \( j = 1, 2, \ldots, n \) by \( f_j \left( a_1, a_2, \ldots, a_n \right) \) and call it payoff function for the player \( j \). The strategic form of a game is defined then by three objects.

(i) The set \( N = \{1, 2, \ldots, n\} \), of players,
(ii) The sequence \( A_1, A_2, \ldots, A_n \) of strategy sets of the players, and
(iii) The sequence \( f_1 \left( a_1, a_2, \ldots, a_n \right), \ldots, f_n \left( a_1, a_2, \ldots, a_n \right) \) of real-valued payoff functions of the players.

A game in strategic form is said to be zero-sum if the sum of the payoffs to the players is zero no matter what actions are chosen by the players. That is, the game is zero-sum if

\[
\sum_{j=1}^{n} f_j \left( a_1, a_2, \ldots, a_n \right) = 0, \text{ for all } a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n.
\]

A two person zero-sum game can also be solved by linear programming approach. The major advantage of using linear programming technique is that it solves mixed strategy of any size.

In game theory, it is usually assumed that each player has only one payoff function and the strategy set of the game is composed of the product of individual player’s strategy sets. However in reality, player’s strategy sets may be interactive and each player may have more than one payoff function. Such games are called multicriteria games or games with vector
payoffs, and are used in modelling various real-life situations where several objectives have to be taken into account such as in politics, management decisions, problems in system design and control problems specially in situations in which the agents do not have an a priori opinion on the relative importance of the components of their payoff vectors.

Dynamic game theory is related to the modelling of large scale systems which have independent decision makers with individual payoff functions. Applications of these games dealt with environment problems, resource problems, aerospace problems and energy management.

1.5 SUMMARY OF THE THESIS

The thesis comprises seven chapters and the gist of each chapter is given below:

Chapter 1: Introduction

This chapter is an introductory one and contains a brief survey of related literatures and summary of the research work presented in the thesis.

Chapter 2: Nondifferentiable Multiobjective Programming with Equality and Inequality Constraints

This chapter has six sections. The section 2.1 gives some introductory remarks and the section 2.6 reflects conclusion.

In section 2.2, we consider the following multiobjective programming problem containing square root of a certain quadratic form in each component of the objective:

\[
\text{(VEP): Minimize } \left( f_1(x) + \left( x^T B_1 x \right)^{1/2}, \ldots, f_k(x) + \left( x^T B_k x \right)^{1/2} \right) \\
\text{subject to } \\
g(x) \leq 0, \\
h(x) = 0.
\]

where (i) \( f : R^n \to R, g : R^m \to R^m \) and \( h : R^n \to R^n \) are continuously differentiable.

(ii) \( B \) is a \( n \times n \) symmetric positive semi definite matrix.
In section 2.3, the optimality conditions for the problem (VEP) are obtained. In section 2.4, we formulate the following differentiable multiobjective dual nonlinear problem for (VEP):

**(M-WED):** Maximize \( f_1(u) + u^T B_1 w_1, ..., f_k(u) + u^T B_k w_k \)

subject to

\[
\sum_{i=1}^{k} \lambda_i \left( \nabla f_i(u) + B_i w_i \right) + \nabla y^T g(u) + \nabla z^T h(u) = 0,
\]

\[
y^T g(u) \geq 0,
\]

\[
z^T h(u) \geq 0,
\]

\[
w_i^T B_i w_i \leq 1, \ i = 1, 2, ..., k,
\]

\[
\lambda > 0,
\]

\[
y > 0.
\]

In the following, we shall use \( \Omega \) and \( \Gamma \) for the sets of feasible solutions of (VEP) and (M-WED) respectively. The following duality theorems between the problems (VEP) and (M-WED):

**Theorem 2.4.1 (Weak Duality):** Let \( x \in \Omega \) and \( (u, \lambda, y, z) \in \Gamma \) such that with respect to the same \( \eta \),

\( (A_1): \sum_{i=1}^{k} \lambda_i \left( f_i(.) + (.)^T B_i w_i \right) \) is pseudoinvex,

\( (A_2): y^T g(.) \) is quasi-invex and

\( (A_3): z^T h(.) \) is quasi-invex.

Then

\[
f_r(x) + \left( x^T B_r x \right)^{\frac{1}{2}} < f_r(u) + u^T B_r w_r, \text{ for some } r \in K,
\]

\[
f_i(x) + \left( x^T B_i x \right)^{\frac{1}{2}} \leq f_i(u) + u^T B_i w_i, \ i \in K_r = K - \{r\},
\]

cannot hold.
Theorem 2.4.2 (Strong Duality): Let $\bar{x}$ satisfy MFCQ and be an efficient solution (VEP). Then there exist $\lambda \in \mathbb{R}^k$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^p$ and $w \in \mathbb{R}^n$ such that $(\bar{x}, y, z, \lambda, w)$ is feasible for (M-WED) and the two objective functions are equal. Furthermore, if the weak duality holds for all feasible solution of (VEP) and (M-WED), then $(\bar{x}, y, z, \lambda, w)$ is an efficient solution of the (M-WED).

Theorem 2.4.3 (Strict converse duality): Let $\bar{x}$ and $(\bar{u}, \lambda, y, z, w)$ be an efficient solution of (VEP) and (M-WED), such that

$$
\sum_{i=1}^{k} \lambda_i \left( f_i(\bar{x}) + \bar{x}^T B_i w_i \right) = \sum_{i=1}^{k} \lambda_i \left( f_i(\bar{u}) + \bar{u}^T B_i w_i \right).
$$

If with respect to the same $\eta$,

(B1): $\sum_{i=1}^{k} \lambda_i \left( f_i(\cdot) + \cdot^T B_i w_i \right)$ is strictly pseudoinvex,

(B2): $y^T g(\cdot)$ is quasi-invex and

(B3): $z^T h(\cdot)$ is quasi-invex

then

$$
\bar{x} = \bar{u}.
$$

i.e. $\bar{u}$ is an optimal solution of (VEP).

Theorem 2.4.4 (Converse duality): Let $(\bar{x}, y, \lambda, z, w_1, ..., w_k)$ be an efficient solution of (M-WED) at which the matrix $\nabla^2 \left( \lambda^T f(\bar{x}) + y^T g(\bar{x}) + z^T h(\bar{x}) \right)$ is positive or negative definite and the vectors $\nabla_y^T g(\bar{x})$ and $\nabla_z^T h(\bar{x})$ are linearly independent. If, for all feasible $(\lambda, \bar{x}, u, y, z, w_1, ..., w_k)$,

$$
\sum_{i=1}^{k} \lambda_i \left( f_i(\cdot) + \cdot^T B_i w_i \right)
$$

is pseudoinvex, $y^T g(\cdot)$ is quasi-invex and $z^T h(\cdot)$ is quasi-invex with respect to the same $\eta$, then $\bar{x}$ is an efficient solution (EP).

In section 2.5, a special case is described.

CHAPTER 3: Optimality Conditions and Second-Order Duality for Non-differentiable Multiobjective Continuous Programming Problems

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This chapter has comprises six sections. The section 3.1 gives some introductory remarks and the section 3.7 reflects conclusion.

In section 3.2, some definitions and related pre-requisites are incorporated. In section 3.3, we consider the following nondifferentiable Multiobjective variational problem:

\[(VCP): \text{Minimize } \left\{ \int_{I} \left[ f^1(t,x(t),\dot{x}(t)) + \left(x(t)^T B^1(t)x(t)\right)^{\frac{1}{2}} \right] \right\} dt, \ldots, \]
\[
\int_{I} \left[ f^p(t,x(t),\dot{x}(t)) + \left(x(t)^T B^p(t)x(t)\right)^{\frac{1}{2}} \right] dt \right]\]

subject to

\[x(a) = \alpha, \ x(b) = \beta,\]
\[g(t,x(t),\dot{x}(t)) \leq 0, \ t \in I\]

where (i) \(C(I,R^n)\) denote the space of piecewise smooth functions \(x\) with norm \(\|x\| = \|x\|_\infty + \|Dx\|_\infty\) , where \(I = [a,b]\) and \(D\) is a differentiation operator.

(ii) \(f^i : I \times R^n \times R^n \to R, \ i \in K = \{1,2,\ldots,p\}, \ g : I \times R^n \times R^n \to R^n\) are assumed to be continuously differentiable functions, and

(iii) for each \(t \in I, \ i \in K = \{1,2,\ldots,p\}\), \(B^i(t)\) is an \(n\times n\) positive semi definite (symmetric) matrix, with \(B(.)\) continuous on \(I\).

In section 3.4, we present the following Mond-Weir type second–order dual to (VCP) and validate various duality theorems:

\[(M\text{-}WCD): \text{Maximize } \left\{ \int_{I} \left[ f^1(t,x,\dot{x}) + x(t)^T B^1(t)z(t) - \frac{1}{2} \beta(t)^T H^1\beta(t) \right] dt, \ldots, \]
\[
\int_{I} \left[ f^p(t,x,\dot{x}) + x(t)^T B^p(t)z(t) - \frac{1}{2} \beta(t)^T H^p\beta(t) \right] dt \right]\]

subject to

\[x(a) = \alpha, \ x(b) = \beta,\]
\[
\sum_{i=1}^{n} \lambda^i \left( f^i_x + B^i(t)z^i(t) - Df^i_x + H^i\beta(t) \right) + \gamma(t)^T g_x - D\gamma(t)^T g_x + G\beta(t) = 0, \ t \in I,\]

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\[
\int_t \left(y(t)^T g(t,x,\dot{x}) - \frac{1}{2} \beta(t)^T G \beta(t)\right) dt \geq 0,
\]
\[
\lambda > 0, \ y(t) \geq 0,
\]
\[
z^i(t)^T B^i(t) z^i(t) \leq 1, \ i \in K, \ t \in I,
\]

where
\[
G = \left(y^T g_x\right)_x - 2D \left(y^T g_x\right)_x + D^2 \left(y^T g_x\right)_x - D^3 \left(y^T g_x\right)_x
\]

and
\[
H^i = f^i_x - 2Df^i_x + D^2 \lambda^T f_{xx} - D^3 f_{xx}, \ t \in I, \ i \in K = \{1,...,p\}.
\]

We denote by \(C_p\) and \(C_D\) the sets of feasible solutions to (VCP) and (M-WCD) respectively.

The following theorems giving duality relationship between (VCP) and (M-WCD) are validated under appropriate generalized invexity assumptions of functionals:

**Theorem 3.4.1 (Weak Duality):** Assume that

(A1): \(\bar{x}(t) \in C_p\) and \((x(t), y(t), z^1(t),...,z^p(t), \beta(t)) \in C_D\).

(A2): \(\sum_{i=1}^{k} \lambda^i \int_I \left[f^i(t,..,.) + (.)^T B^i(t) z^i(t)\right] dt\) is second-order pseudoinvex and

(A3): \(\int_I y(t)^T g(t,..,.) dt\) is second-order quasi-invex.

Then
\[
\int_I \left[f^r(t,\bar{x},\dot{x}) + (\bar{x}(t)^T B^r(t) \bar{x}(t))^{\frac{1}{2}}\right] dt < \int_I \left[f^r(t,\bar{x},\dot{x}) + x(t)^T B^r(t) z^r(t) - \frac{1}{2} \beta(t)^T H^r \beta(t)\right] dt,
\]
for some \(r \in K\)

and
\[
\int_I \left[f^i(t,\bar{x},\dot{x}) + (\bar{x}(t)^T B^i(t) \bar{x}(t))^{\frac{1}{2}}\right] dt \leq \int_I \left[f^i(t,\bar{x},\dot{x}) + x(t)^T B^i(t) z^i(t) - \frac{1}{2} \beta(t)^T H^i \beta(t)\right] dt,
\]
\(i \in K_r,\)

cannot hold.
Theorem 3.4.2 (Strong duality): Let $\bar{x}(t)$ be normal and is an efficient solution of $(V_E P)$. Then there exist $\lambda \in \mathbb{R}^p$ and piecewise smooth functions $y : I \to \mathbb{R}^m$ and $z^i : I \to \mathbb{R}^n$, $i = 1, 2, \ldots, n$ such that $(\bar{x}(t), y(t), \lambda, z^1(t), \ldots, z^p(t), \beta(t) = 0)$ is feasible for (M-WCD) and the two objective functions are equal. Furthermore, if the hypotheses of Theorem 3.4.1 hold for all feasible solutions of (VCP) and (M-WCD), then $(\bar{x}(t), y(t), \lambda, z^1(t), \ldots, z^p(t), \beta(t))$ is an efficient solution of (M-WCD).

Theorem 3.4.3 (Converse duality): Assume that

(B_1): $\bar{x}, \lambda, y, z^1, \ldots, z^n, \beta \bar{x}$ is an efficient solution of (M-WCD),

(B_2): the vectors $\{H^i, G_j, t \in I, i \in K, j = 1, 2, \ldots, n\}$ are linearly independent where $H^i$ the $j^{th}$ row of is $H^i$ and $G_j$ is the $j^{th}$ row of $G$,

(B_3): $f^i(t, \bar{x}, \bar{x}) + B^i(t) z^i(t) - D_{f^i}(t, \bar{x}, \bar{x}) + H^i \beta(t)$, $t \in I, i \in K$ are linearly independent and

(B_4): for $t \in I$ either

(a) $\int_I \beta(t)^T \left( G + \left( y(t)^T g_x \right) \right) \beta(t) > 0$ and $\int_I \beta(t)^T \left( y(t)^T g_x \right) > 0$,

(b) $\int_I \beta(t)^T \left( G + \left( y(t)^T g_x \right) \right) \beta(t) < 0$ and $\int_I \beta(t)^T \left( y(t)^T g_x \right) < 0$.

Then $\bar{x}(t)$ is feasible for (VCP) and the two objective functionals have the same value. Also, if Theorem 3.4.1 holds for all feasible solutions of (VCP) and (M-WCD), the $\bar{x}(t)$ is an efficient solution of (VCP).

Theorem 3.4.4 (Strict converse duality): Assume that $\sum_{i=1}^n \lambda^i \int_I \left( f^i(t, \ldots) + (t)^T B^i z^i(t) \right) dt$ is second-order pseudoinvex and $\int_I y^T(t) g(t, \ldots) dt$ is quasi-invex with respect to the same. Further assume that (VCP) has an optimal solution $\bar{x}(t)$ which is normal. If
(\hat{\lambda}, \hat{u}(t), \hat{y}(t), \hat{z}^1(t), \ldots, \hat{z}^p(t), \beta(t)) is an optimal solution of (M-WCD), then \( \hat{u}(t) \) is an efficient solution of (VCP) with \( \bar{x}(t) = \hat{u}(t), t \in I \).

In section 3.5, we formulate a pair of nondifferentiable Mond-Weir type dual variational problems with natural boundary values rather than fixed end points. In section 3.6, if the time dependency of the problems with natural boundary conditions is ignored, then these problems reduce to the nondifferentiable second-order nonlinear problems already studied in the literature.

CHAPTER 4: On Multiobjective Variational Problems with Equality and Inequality Constraints

This chapter involves four sections. The section 4.1 gives some introductory remarks and the section 4.4 gives conclusion. Each of the sections 4.2 and 4.3 has four subsections.

In section 4.2, we consider the following constrained multiobjective variational problem:

(VEP): Minimize \( \int_I f^1(t,x,\dot{x})\,dt, \ldots, \int_I f^p(t,x,\dot{x})\,dt \)

subject to

\( x(a) = \alpha, x(b) = \beta, \)
\( g(t,x,\dot{x}) \leq 0, t \in I, \)
\( h(t,x,\dot{x}) = 0, t \in I, \)

where \( f^i : I \times R^n \times R^n \rightarrow R, i \in K = \{1,2,\ldots,p\}, g : I \times R^n \times R^n \rightarrow R^n, h : I \times R^n \times R^n \rightarrow R^l \) are continuously differentiable functions.

In subsection 4.2.1, we have obtained necessary optimality conditions for the above problem. In subsection 4.2.2, we present the following Wolfe type second-order dual to the problem (VEP):

(WVED): Maximize \( \int_I \left( f^1(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x}) - \frac{1}{2} \beta(t)^T H^1 \beta(t) \right)\,dt, \)
\( \ldots, \int_I \left( f^p(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x}) - \frac{1}{2} \beta(t)^T H^p \beta(t) \right)\,dt \)
subject to
\[ x(a) = \alpha, x(b) = \beta, \]
\[ \lambda^T f_x + y(t)^T g_x + z(t)^T h_x - D\left(\lambda^T f_x + y(t)^T g_x + z(t)^T h_x\right) + H \beta(t) = 0, \]
\[ y(t) \geq 0, \]
\[ \lambda > 0, \lambda^T e = 1, \text{ where } e = (1, 1, \ldots, 1) \in \mathbb{R}^n \]

where
\[ H^i = f^i_{xx} - 2Df^i_{x\lambda} + D^2 f^i_{\lambda \lambda} - D^3 f^i_{x\lambda} + \left( y(t)^T g_x + z(t)^T h_x \right) - 2D \left( y(t)^T g_x + z(t)^T h_x \right)_{x} \]
\[ + D^2 \left( y(t)^T g_x + z(t)^T h_x \right)_{x} - D^3 \left( y(t)^T g_x + z(t)^T h_x \right)_{x} , \]
\[ H = \lambda^T f_x - 2D\lambda^T f_{x\lambda} + D^2 \lambda^T f_{\lambda \lambda} - D^3 \lambda^T f_{x\lambda} + \left( y(t)^T g_x + z(t)^T h_x \right) \]
\[ - 2D \left( y(t)^T g_x + z(t)^T h_x \right) + D^2 \left( y(t)^T g_x + z(t)^T h_x \right) - D^3 \left( y(t)^T g_x + z(t)^T h_x \right)_{x} \]
and \[ A^i = f^i_{xx} - 2Df^i_{x\lambda} + D^2 f^i_{\lambda \lambda} - D^3 f^i_{x\lambda}. \]

The following are the duality theorems validated under suitable generalized second-order invexity conditions:

**Theorem 4.2.2.1 (Weak Duality):** Assume that for all feasible \( x \) for (VEP) and all feasible \((u, y, z, \lambda, \beta)\) for (WVED).
\[ \int_{I} \left\{ \left[ f^i(t, x, \dot{x}) + y(t)^T g(t, x, \dot{x}) + z(t)^T h(t, x, \dot{x}) \right] \right\} dt \]
is second-order pseudoinvex with respect to \( \eta \). Then the following cannot hold:
\[ \int_{I} f^i(t, x, \dot{x}) dt < \int_{I} \left\{ f^i(t, u, \dot{u}) + y(t)^T g(t, u, \dot{u}) + z(t)^T h(t, u, \dot{u}) - \frac{1}{2} \beta(t)^T H^i \beta(t) \right\} dt, \]
for some \( i \in \{1, 2, \ldots, p\} \),
\[ \int_{I} f^i(t, x, \dot{x}) dt \leq \int_{I} \left\{ f^j(t, u, \dot{u}) + y(t)^T g(t, u, \dot{u}) + z(t)^T h(t, u, \dot{u}) - \frac{1}{2} \beta(t)^T H^i \beta(t) \right\} dt, \]
for all \( j \in \{1, 2, \ldots, p\} \).
In the following theorem, the $P_k(\bar{x})$ is defined as:

$$P_k(\bar{x}): \text{Minimize } \int_I f^k(t,x,\dot{x})dt$$

subject to

$$x(a) = \alpha, \ x(b) = \beta,$$

$$g(t,x,\dot{x}) \leq 0, \ t \in I,$$

$$h(t,x,\dot{x}) = 0, \ t \in I.$$

**Theorem 4.2.2.2 (Strong Duality):** Let $\bar{x}$ is efficient solution of (VEP) and satisfies the constraint qualification for $P_k(\bar{x})$, for at least one $k \in \{1,2,...,p\}$. Then there exist $\lambda' \in R^p$ and piecewise smooth functions $y: I \to R^m$ and $z: I \to R^l$ such that $(\bar{x},\bar{\lambda},\bar{y},\bar{z},\beta = 0)$ is feasible for (WVED) and $\bar{y}(t)^Tg(t,\bar{x},\dot{\bar{x}}) = 0, \ t \in I$. If the weak duality also holds between (VEP) and (WVED) then $(\bar{x},\bar{\lambda},\bar{y},\bar{z},\beta = 0)$ is efficient for (WVED).

**Theorem 4.2.2.3 (Strict converse Duality):** Let $\bar{x}$ be normal and efficient solution of (VEP) and $(u,y,z,\lambda,\beta)$ be efficient for (WVED) such that

$$\int \sum_{i=1}^{p} \lambda_i f^i(t,\bar{x},\dot{\bar{x}})dt = \int \left\{ \lambda_i f^i(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)^Tg(t,\bar{u},\dot{\bar{u}}) + \bar{z}(t)^Th(t,\bar{u},\dot{\bar{u}}) - \frac{1}{2} \beta(t)^TH\beta(t) \right\}dt.$$

If $\int \left( \lambda^Tf_x + y(t)^Tg + z(t)^Th \right)dt$ is second-order strictly pseudoinvex with respect to $\eta$, then

$$\bar{x}(t) = \bar{u}(t), \ t \in I.$$

i.e. $\bar{u}(t)$ is an efficient solution of the problem (VEP).

**Theorem 4.2.2.4 (Converse duality):** Let $(\bar{x}(t),y(t),z(t),\lambda(t),\beta(t))$ be an efficient solution of (WVED) for which

$$(C_1): \text{H is non singular}$$
\[(C_2)\]:

\[
\begin{bmatrix}
\left(\sigma(t)^T H^i \sigma(t)\right)_x - D\left(\sigma(t)^T H^i \sigma(t)\right)_x \\
+ D^2\left(\sigma(t)^T H^i \sigma(t)\right)_x - D^3\left(\sigma(t)^T H^i \sigma(t)\right)_x + D^4\left(\sigma(t)^T H^i \sigma(t)\right)_x \\
\end{bmatrix}
\]

\[= 0 \Rightarrow \sigma(t) = 0,\]

where \(\sigma(t)\) is a vector function.

Then \(x(t)\) is feasible for (VEP) and the two objectives functional have the same value. Also, if the weak duality theorem (Theorem 4.2.2.1) holds for all feasible of (VEP) and (WVED), then \(x(t)\) is efficient.

In subsection 4.2.3, a pair of Wolfe type second-order Multiobjective variational problems with natural values is formulated. In subsection 4.2.4, if the problems (VEP\(_0\)) and (WVED\(_0\)) are independent of \(t\), that is, if \(f, g\) and \(h\) do not depend explicitly on \(t\), then these problems essentially reduce to the static cases of nonlinear programming studied by Mond and J. Zhang (1985).

In section 4.3, we consider the following nondifferentiable multiobjective variational problems containing terms of square root functions:

(\textbf{NWVEP}): Minimize

\[
\int \left\{ f^1 (t,x(t),\dot{x}(t)) + \left( x(t)^T B^1 (t)x(t) \right)^{1/2} \right\} dt, 
\]

subject to

\[
x(a) = 0, \ x(b) = 0,
\]

\[
g(t,x(t),\dot{x}(t)) \leq 0, \ t \in I,
\]

\[
h(t,x(t),\dot{x}(t)) = 0, \ t \in I,
\]

where (i) \(f^i, g\) and \(h\) are same as in the previous section and
(ii) for each \( t \in I, \ i \in K = \{1, 2, \ldots, p\} \), \( B^i(t) \) is an \( n \times n \) positive semi definite (symmetric) matrix, with \( B^i(.) \) continuous on \( I \).

In **subsection 4.3.1**, necessary optimality conditions are derived. In **subsection 4.3.2**, we construct the following problem (NWVED) as the dual to the problem (NWVEP) and validated various duality theorems for the pair of problems (NWVEP) and (NWVED):

\[
\text{(NWVED): Maximize} \left[ \int f^1(t,u(t),\hat{u}(t)) + u(t)^T B^1(t) w^1(t) + y(t)^T g(t,u(t),\hat{u}(t)) + z(t)^T h(t,u(t),\hat{u}(t)) - \frac{1}{2} \beta(t)^T H^1 \beta(t) \right] dt, \ldots
\]

\[
\int f^p(t,u(t),\hat{u}(t)) + u(t)^T B^p(t) w^p(t) + y(t)^T g(t,u(t),\hat{u}(t)) + z(t)^T h(t,u(t),\hat{u}(t)) - \frac{1}{2} \beta(t)^T H^p \beta(t) \right] dt
\]

subject to

\[
u(a) = 0 = u(b),
\]

\[
\sum_{i=1}^{p} \lambda^i \left( f^i_a(t,u(t),\hat{u}(t)) + B^i(t) w^i(t) - D f^i_a(t,u(t),\hat{u}(t)) \right) + y(t)^T g_a(t,u(t),\hat{u}(t)) + z(t)^T h_a(t,u(t),\hat{u}(t)) + H \beta(t) = 0, t \in I,
\]

\[
w^i(t)^T B^i(t) w^i(t) \leq 1, \ i \in K,
\]

\[
y(t) \geq 0, \ t \in I,
\]

\[
\lambda > 0, \sum_{i=1}^{p} \lambda^i = 1.
\]

In **subsection 4.3.3** gives a pair of Wolfe type second-order nondifferentiable multiobjective variational problems with natural boundary values. In **subsection 4.3.4**, the relationship between our second-order duality results and those of second-order nondifferentiable multiobjective mathematical programming problems is indicated.
CHAPTER 5: A Class of Nondifferentiable Multiobjective Control Problems

This chapter consists of six sections. The section 5.1 gives some introductory remarks and the section 5.6 reflects conclusion.

In section 5.2, we consider the following multiobjective control problem:

(VCP): Minimize \( \left[ \int f^1(t,x,u) + \left( u(t)^T B^1(t)u(t) \right)^{1/2} \right] dt, \ldots, \int \left[ f^p(t,x,u) + \left( u(t)^T B^p(t)u(t) \right)^{1/2} \right] dt \)

subject to

\[ x(a) = \alpha, \quad x(b) = \beta, \]
\[ \dot{x} = h(t,x,u), \quad t \in I, \]
\[ g(t,x,u) \leq 0, \quad t \in I. \]

In section 5.3, we have obtained necessary optimality conditions for the nondifferentiable multiobjective control problems (VCP). In section 5.4, we propose the following Mond-Weir type dual to (VCP) and establish various duality results under suitable generalized invexity:

(VCD): Maximize \( \left[ \int f^1(t,x,u) + \left( u(t)^T B^1(t)u(t) \right) \right] dt, \ldots, \int \left[ f^p(t,x,u) + \left( u(t)^T B^p(t)u(t) \right) \right] dt \)

subject to

\[ x(a) = \alpha, \quad x(b) = \beta, \]
\[ \sum_{i=1}^{p} \lambda^i \left( f^i_x(t,x,u) + y(t)^T g^i_x(t,x,u) + z(t)^T h^i_x(t,x,u) + \dot{z}(t) = 0, \quad t \in I, \right. \]
\[ \sum_{i=1}^{p} \lambda^i \left( f^i_u(t,x,u) + B^i(t)w^i(t) \right) + y(t)^T g^i_u(t,x,u) + z(t)^T h^i_u(t,x,u) = 0, \quad t \in I, \]
\[ \int y(t)^T g(t,x,u) dt \geq 0, \]
\[ \int z(t)^T \left( h(t,x,u) - \dot{x}(t) \right) dt \geq 0, \]
\[ y(t) \geq 0, \quad t \in I, \]
\[ w^i(t)^T B^i(t)w^i(t) \leq 1, \quad t \in I, \quad i \in K, \]
\( \lambda > 0 \).

In section 5.5, we have discussed related problems.

CHAPTER 6: On Vector-Valued Variational Problems

This chapter has seven sections. The section 6.1 gives some introductory remarks and the section 6.7 mention conclusion.

In section 6.2, we consider the following constrained multiobjective variational problem:

(VEP): Minimize \( \left[ \int f^1(t,x,x,t)dt,...,\int f^p(t,x,x,t)dt \right] \)

subject to
\[
\begin{align*}
x(a) &= \alpha, \ x(b) = \beta, \\
g(t,x,x) &\leq 0, \ t \in I, \\
h(t,x,x) &= 0, \ t \in I,
\end{align*}
\]

where \( f^i \), \( g \) and \( h \) are same as in the previous section.

In section 6.3, we propose the following Mond-Weir type second-order to the problem (VEP) and derive various duality results under suitable generalized invexity:

(M-WED): Maximize \( \left[ \int \left( f^1(t,u,u,t) - \frac{1}{2} \beta(t)^T F^1 \beta(t) \right) dt,...,\int \left( f^p(t,u,u,t) - \frac{1}{2} \beta(t)^T F^p \beta(t) \right) dt \right] \)

subject to
\[
\begin{align*}
u(a) &= \alpha, \ u(b) = \beta, \\
\lambda^T f_u + y(t)^T g_u + z(t)^T h_u - D\left( \lambda^T f_\alpha + y(t)^T g_\alpha + z(t)^T h_\alpha \right) + B\beta(t) &= 0, \ t \in I, \\
\int \left( y(t)^T g(t,u,u) - \frac{1}{2} \beta(t)^T G \beta(t) \right) dt &\geq 0, \\
\int \left( z(t)^T h(t,u,u) - \frac{1}{2} \beta(t)^T H \beta(t) \right) dt &\geq 0, \\
y(t) &\geq 0, \ t \in I,
\end{align*}
\]

\( \lambda > 0 \).
where
\[
B = \lambda^T F + G + H \quad \text{with} \quad F^i = f^i_{uu} - 2Df^i_{u\bar{u}} + D^2 f^i_{\bar{u}\bar{u}} - D^3 f^i_{\bar{u}\bar{u}\bar{u}}, \quad t \in I, \ i \in K.
\]
\[
G = \left( y(t)^T g_a \right)_u - 2D \left( y(t)^T g_a \right)_{\bar{u}} + D^2 \left( y(t)^T g_a \right)_{\bar{u}\bar{u}} - D^3 \left( y(t)^T g_a \right)_{\bar{u}\bar{u}\bar{u}}, \quad t \in I
\]
and
\[
H = \left( z(t)^T h_u \right)_u - 2D \left( z(t)^T h_u \right)_{\bar{u}} + D^2 \left( z(t)^T h_u \right)_{\bar{u}\bar{u}} - D^3 \left( z(t)^T h_u \right)_{\bar{u}\bar{u}\bar{u}}, \quad t \in I
\]
are \( n \times n \) symmetric matrices.

In section 6.4. In relation to the problem (VEP), we have considered the following mixed type dual multiobjective variational problem:

(MVED): Maximize \[
\int_I \left( f^1(t,u,\dot{u}) + \sum_{j \in I_0} y^j(t) g^j(t,u,\dot{u}) + \sum_{k \in I_0} z^k(t) h^k(t,u,\dot{u}) - \frac{1}{2} \beta(t)^T A^1 \beta(t) \right) dt,
\]
...\[
\int_I \left( f^p(t,u,\dot{u}) + \sum_{j \in I_0} y^j(t) g^j(t,u,\dot{u}) + \sum_{k \in I_0} z^k(t) h^k(t,u,\dot{u}) - \frac{1}{2} \beta(t)^T A^p \beta(t) \right) dt
\]
subject to
\[
u(a) = \alpha, \quad u(b) = \beta,
\]
\[
\lambda^T f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) + z(t)^T h_u(t,u,\dot{u})
\]
\[-D \left( \lambda^T f_{\bar{u}}(t,u,\dot{u}) + y(t)^T g_{\bar{u}}(t,u,\dot{u}) + z(t)^T h_{\bar{u}}(t,u,\dot{u}) \right) + BA \beta(t) = 0, \quad t \in I,
\]
\[
\sum_{j \in I_0} \int_I \left( y^j(t) g^j(t,u,\dot{u}) - \frac{1}{2} \beta(t)^T G^j \beta(t) \right) dt \geq 0, \quad \alpha = 1, 2, ... , r,
\]
\[
\sum_{k \in I_0} \int_I \left( z^k(t) h^k(t,u,\dot{u}) - \frac{1}{2} \beta(t)^T H^k \beta(t) \right) dt \geq 0, \quad \alpha = 1, 2, ... , r,
\]
\[
y(t) \geq 0, \quad t \in I,
\]
\[
\lambda > 0, \quad \sum_{i=1}^p \lambda^i = 1, \quad \alpha = 1, 2, ... , r.
\]
where

\[
A^j = f^j_{uu} - 2Df^j_{uu} + D^2f^j_{uu} - D^3f^j_{uu} + \sum_{j \in I} \left( y^j(t) g^j_u(t, u, \dot{u}) \right)_u - 2D \sum_{j \in I} \left( y^j(t) g^j_u(t, u, \dot{u}) \right)_{\dot{u}} + D^2 \sum_{j \in I} \left( y^j(t) g^j_u(t, u, \dot{u}) \right)_{\ddot{u}} - D^3 \sum_{j \in I} \left( y^j(t) g^j_u(t, u, \dot{u}) \right)_{\dddot{u}}
\]

and the matrix B is the same as defined in the previous section.

\[
G^j = \left( y^j(t) g^j_u(t, u, \dot{u}) \right)_{\dot{u}} - 2D \left( y^j(t) g^j_u(t, u, \dot{u}) \right)_{\ddot{u}} + D^2 \left( y^j(t) g^j_u(t, u, \dot{u}) \right)_{\dddot{u}} - D^3 \left( y^j(t) g^j_u(t, u, \dot{u}) \right)_{\ddddot{u}},
\]

\[
H^k = \left( z^k(t) h^k_u(t, u, \dot{u}) \right)_{\dot{u}} - 2D \left( z^k(t) h^k_u(t, u, \dot{u}) \right)_{\ddot{u}} + D^2 \left( z^k(t) h^k_u(t, u, \dot{u}) \right)_{\dddot{u}} - D^3 \left( z^k(t) h^k_u(t, u, \dot{u}) \right)_{\dddddot{u}},
\]

\[
M = \{1, 2, \ldots, m\}, L = \{1, 2, \ldots, l\}, I_\alpha \subseteq M, \alpha = 0, 1, 2, \ldots, r \text{ with } I_\alpha \cap I_\beta = \phi, \alpha \neq \beta, \bigcup_{\alpha=1}^{r} I_\alpha = M
\]

and \( J_\alpha \subseteq L, \alpha = 0, 1, 2, \ldots, r \) with \( J_\alpha \cap J_\beta = \phi, \alpha \neq \beta, \bigcup_{\alpha=1}^{r} J_\alpha = L \).

In section 6.5, it is discussed that the duality theorems validated in the previous two sections can be extended to the corresponding multiobjective variational problem with natural boundary values rather than fixed end points. In section 6.6, we have indicated the linkage of our results to those of multiobjective nonlinear programming problems.

CHAPTER 7: Constrained Vector-Valued Dynamic Game and Symmetric Duality for Multiobjective Variational Problems

This chapter involves seven sections. The section 7.1 gives some introductory remarks and the section 7.7 provides concluding remarks.

In section 7.2, in order to establish the equivalence of a constrained vector-valued dynamic game to a pair of symmetric variational problems, we associate the vector-valued game,

\[
G = \left( X, Y, F(x, y) \right)
\]

to the game \( G' = \left( X, Y, \lambda^T F(x, y) \right) \), where

(i) \( X = \{ x: I \to \mathbb{R}^n \mid x(a) = 0, x(b) = 0, p_i(t, x, \dot{x}) \geq 0, t \in I, i = 1, 2, \ldots, k \text{ and } I = [a, b] \subseteq \mathbb{R} \}, \)
(ii) \( Y = \{ y : I \rightarrow \mathbb{R}^n \mid y(a) = 0, y(b) = 0, q_j(t, y, y') \geq 0, t \in I, j = 1,2,\ldots,l \} \)

with \( p_i : I \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, i = 1,\ldots,k \), \( q_j : I \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}, j = 1,2,\ldots,l \) have continuous derivative up to and including second-order with respect to each of their arguments \( x \) and \( y \) and \( x : I \rightarrow \mathbb{R}^n \), \( y : I \rightarrow \mathbb{R}^m \) with derivatives \( \dot{x} \) and \( \dot{y} \) with respect to \( t \).

(iii) \( F : X \times Y \rightarrow \mathbb{R}^p \), defined by

\[
F(x, y) = \left( \int_a^b f^1(t, x, \dot{x}, y, \dot{y}) dt, \ldots, \int_a^b f^p(t, x, \dot{x}, y, \dot{y}) dt \right),
\]

where \( f^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R} \) is assumed to be continuously differentiable function \( f^i, i = 1,2,\ldots,p \) and

(iv) \( X \) and \( Y \) represent the strategy spaces for the players I and II respectively and \( \lambda^T F(x, y) \) with \( \lambda = (\lambda^1,\ldots,\lambda^p) \in \mathbb{R}^p \) represents the payoff to the player II from the player I. The player I is assumed to be a minimizing player and player II a maximizing player. Thus the player I wishes to solve \( \min_{x \in X} \max_{y \in Y} \lambda^T F(x, y) \) and the player II wishes to solve \( \max_{y \in Y} \min_{x \in X} \lambda^T F(x, y) \).

In this section, we have shown that our dynamic game is equivalent to the following multiobjective variational problems:

\[\text{(VP): Minimize } \left( \int_a^b f^1(t, x, \dot{x}, y, \dot{y}) dt, \ldots, \int_a^b f^p(t, x, \dot{x}, y, \dot{y}) dt \right) \]

subject to

\[
x(a) = 0 = x(b),
\]

\[
y(a) = 0 = y(b),
\]

\[
\left( \lambda^T f_y (t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_y (t, y, y') - D \left( \lambda^T f_y (t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_y (t, y, y') \right) \right) \leq 0, t \in I,
\]

\[
\int_a^b \left( \lambda^T f_y (t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_y (t, y, y') - D \left( \lambda^T f_y (t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_y (t, y, y') \right) \right) dt \geq 0,
\]

\[
\int_a^b \mu(t)^T q(t, y, y') dt \geq 0.
\]
\[ p(t, x, \dot{x}) \geq 0, \quad t \in I, \]
\[ x(t) \geq 0, \quad \mu(t) \geq 0, \quad t \in I, \]

\[ \lambda > 0. \]

**VD:** Maximize \( \left( \int_{a}^{b} f^{1}(t, u, \dot{u}, v, \dot{v}) dt, \ldots, \int_{a}^{b} f^{p}(t, u, \dot{u}, v, \dot{v}) dt \right) \)

subject to

\[ u(a) = 0 = u(b), \]
\[ v(a) = 0 = v(b), \]
\[ \left( \lambda^{T} f_{u}(t, u, \dot{u}, v, \dot{v}) - \gamma(t)^{T} p(t, u, \dot{u}) - D \left( \lambda^{T} f_{u}(t, u, \dot{u}, v, \dot{v}) - \gamma(t)^{T} p(t, u, \dot{u}) \right) \right) \geq 0, \quad t \in I, \]
\[ \int_{a}^{b} u(t) \left[ \lambda^{T} f_{u}(t, u, \dot{u}, v, \dot{v}) - \gamma(t)^{T} p(t, u, \dot{u}) - D \left( \lambda^{T} f_{u}(t, u, \dot{u}, v, \dot{v}) - \gamma(t)^{T} p(t, u, \dot{u}) \right) \right] dt \leq 0, \]

\[ \int_{a}^{b} \gamma(t)^{T} q(t, v, \dot{v}) dt \leq 0, \]

\[ q(t, v, \dot{v}) \leq 0, \quad t \in I, \]

\[ v(t) \geq 0, \quad \gamma(t) \geq 0, \quad t \in I, \]

\[ \lambda > 0. \]

The above problems are symmetric and more general than those already studied in the literature.

In **section 7.3**, we have established various duality theorems under suitable generalized convexity assumptions. Self duality is established in the **section 7.4**. In **section 7.5**, it is shown that it is possible to formulate a pair symmetric dual multiobjective variational problem with natural boundary values rather than fixed end points. In **section 7.6**, it is pointed out that if the problems are independent of \( t \) then we have a pair of nonlinear symmetric dual multiobjective programming problems studied by Chandra and Prasad (1992).