Chapter 4

Contest under Interdependent Valuations

4.1 Introduction:

Private valuation is a standard assumption in auction theory. This assumption provides a framework where the realized value of the object for any bidder is completely known to the concerned bidder only. However, this assumption can be relaxed by allowing for the possibility that bidders have only partial information regarding the value, say, in the form of a noisy signal. Indeed, it might be the case that the knowledge of the information that other bidders possess would affect the value that a particular bidder assigns to the object. The resulting information structure is called one of interdependent valuation. Each bidder is assumed to have some private information concerning the value of the object. The private information that bidder $i$ possesses is summarized as the realization of the random variable $x_i \in [0, \omega_i]$, called $i$'s signal. It is assumed that the value of the object to bidder $i$, $V_i$, can be expressed as a function of all bidders' signals and thus can be written as

$$V_i = v_i(X_1, X_2, \cdots, X_N)$$

where the function $v_i$ is bidder $i$'s valuation and is assumed to be non-decreasing and twice continuously differentiable in all its variables. In addition, it is assumed that $v_i$ is strictly increasing in $X_i$.

The presumption underlying this specification is that the value is completely determined by the signals. This means to say that, no uncertainty remains there. This does not necessarily be the case, however, and more general formulation can also be accommodated. In more general settings, we suppose that $V_1, V_2, \cdots, V_N$ denote the $N$ (unknown) values to the bidders; $X_1, X_2, \cdots, X_N$
denote the \( N \) signals available to the bidders; and \( S \) denotes a signal available only to the seller. In that case, we can define

\[
v_i(x_1, x_2, \cdots, x_n) \equiv E[V_i \mid X_1 = x_1, X_2 = x_2, \cdots, X_N = x_N]
\]

as the expected value to \( i \) conditional on all the information available to the bidders. We take this to be the effective value that bidders can use in their calculations for operational conveniences.

With either specification, we suppose that \( v_i(0, 0, \cdots, 0) = 0 \) and that \( E[V_i] < \infty \). We continue to hold the assumption that bidders are risk neutral - each bidder maximizes the expectation of \( V_i - p_i \), where \( p_i \) is the price paid.

Let us note here that this specification of the values includes the private values model as an extreme case in which \( v_i(X) = X_i \). On the other extreme lies the case of a pure common value in which all bidders assign the same value to the object. In such a situation, the valuations of all the bidders are identical. Bidders' information of course consists only of their own signals. So the \textit{ex post} value, which is common to all, is unknown to any particular bidder. A special case that is interesting both analytically and practically, implicates first specifying a distribution for the common value \( V \) and next assuming that conditional on the event \( V = v \), bidders' signals \( X_i \) is also assumed to be an unbiased estimator of \( V \); so that \( E[X_i \mid V = v] = v \). This particular specification has been used to model the information structure associated with auctions of oil-drilling leases and is sometimes termed as the \textit{"mineral rights"} model.

The decision making problem facing a bidder gets much more complicated with the presence of interdependent valuations as against independent values. Specifically, since the true value of the object is dependent on other bidders' signals as well and thus not exactly known, an \textit{a priori} estimate of this value might require to be revised as a result of events that take place during, and even after, the auction. This is caused by the fact that these events may convey valuable information about signals of other bidders. One such event is the announcement that some particular bidder has won the auction.

An interesting situation involving interdependent valuations of bidders is that of the \textit{winner's curse}. In this case, prior to the auction the only information available to a bidder, e.g. \( i \), is
just his own signal $X_i = x$. On the basis of this information alone, his estimate of the value is $E[V \mid X_i = x]$. Now let us consider the case where the object is sold using a sealed-bid first-price auction and then look at what happens when and if it is announced that bidder $i$ is, in fact, the winner. If all the bidders are symmetric and follow the same bidding function $\beta$, then this fact reveals to bidder $i$ that the highest of the $N-1$ signals is less than $x$. As a result, his estimate of the value upon learning that he is the winner is $E[V \mid X_i = x, Y_i < x]$, which is less than $E[V \mid X_i = x]$. The announcement that he is the winner leads to a reduction in his estimated value; in this sense, winning brings him “bad news”. A failure to foresee this effect and take it fully into account when formulating bidding strategies will result in what has been called the winner’s curse - the possibility that the winner pays more than the final value. We emphasize that the winner’s curse arises only if bidders do not calculate the value of winning correctly and overbid as a result - it does not arise in equilibrium.

The phenomenon is most explicitly visible in a pure common value model in which each bidder’s signal is $X_i = V + \epsilon_i$. We assume that the different $\epsilon_i$’s are independently and identically distributed and satisfy $E[\epsilon_i] = 0$. Then each bidder’s signal becomes an unbiased estimator of the common value - that is, for all $i$, $E[X_i \mid V = v]$. But now notice that even though each individual signal is an unbiased estimator of the value, the largest of $N$ such signals is not. In fact, since “max” is a convex function, $E[\max X_i \mid V = v] > \max E[X_i \mid V = v]$, showing that the expectation of the highest signal, in fact, overestimates the value. A bidder who does not take this fully into account and bids an amount $\beta(X_i)$ which is close to $X_i$ would, upon winning, pay more than the estimated worth of the object. Put another way, bidders may need to shade their bids well below their initial estimates in order to avoid the winner’s curse. Note also that the magnitude of the winner’s curse increases with the number of bidders in the auction. The news that a signal is highest of, say, 20 bidders is worse than the news that it is the highest of 10 bidders.

In this chapter we will briefly study the theory of contest under interdependent valuations. Valuations are interdependent if the valuation of any contestant not only depends on her own type, but also the types of all the other contestants. In this chapter we will assume a special type of interdependent valuation viz. pure common value. Here all the contestants assign same value for all the prizes. By type of a contestant, we mean the ability of that contestant. We will first describe our model by a set of assumptions. Throughout this chapter we are going to assume that performance of any contestant is exactly equal to the effort she puts in the contest. Then we will study a contest with interdependent valuations under the assumption of a linear cost function. We first derive the equilibrium bidding strategy of a contestant, then we study whether it is optimal for the contest designer to give a single “winner take all” first prize or multiple prizes, under the assumption that the contest designer can only give either a single prize or two prizes. Then we analyze a contest
with interdependent valuations under non-linear cost functions. Here, we derive the equilibrium bidding strategy of a contest. Finally, we analyze the case where the contest designer can give more than two prizes (Note that even in that case as well, the contest designer can give at most \( k \) prizes, where \( k \) is the number of contestants). We derive the equilibrium bidding strategy of a contestant and we comment that the results we have derived for the case where the contest designer can give a single prize or two prizes still holds for the case where the contest designer can give more than two prizes.

### 4.2 Contests with interdependent valuation:

Below we will formally study contests with interdependent valuations.

#### 4.2.1 The model and its assumptions:

- Consider a contest where \( p > 0 \) prizes are awarded.
- There are \( k \) contestants. The set of contestants is \( K = \{1, ..., k\} \).
- Without loss of generality we can assume that \( k \geq p \) (i.e., there are at least as many contestants as the number of prizes).
- Each contestant has some private information concerning the value of each prize. Contestant \( i \)'s private information is summarized as the realization of the random variable \( A_i \in [m, 1] \), called the \( i \)'s signal, where \( 1 > m > 0 \). The realization of \( A_i \) is nothing but the ability of the contestant \( i \).
- We assume that contestants' signals are independent and identically distributed with the distribution function \( F(.) \). We also assume that \( F(.) \) has a corresponding density \( f(.) \) and has full support.
- The value of the \( j^{th} \) prize (\( V^{(j)} \)) to contestant \( i \), can be expressed as a function of all the contestants' signals, i.e. \( V^{(j)} = \tilde{V}^{(j)}(A_1, A_2, ..., A_k) \). Immediately, one can see that we are assuming pure common value in which all the contestants assign the same values to all the prizes - the valuations of the contestants are identical. We assume that the valuation function is a common knowledge to all the contestants.
• For all \( j \in [1, p] \), \( \hat{V}^{[j]}(0, 0, ..., 0) = 0 \) and \( E \left[ \hat{V}^{[j]} \right] < \infty \).

• We assume that, for all \( j \in [1, p] \), \( \hat{V}^{[j]} \) is symmetric in the last \( k - 1 \) components. This means that from the perspective of a particular contestant, the signals of the other contestants can be interchanged without affecting the value.

• For all \( j \in [1, p - 1] \), \( \hat{V}^{[j]}(.) \geq \hat{V}^{[j+1]}(.) \) i.e. for all the contestants the valuation of the first prize is greater than or equal to the valuation of the second prize and so on.

• In the contest each player \( i \) makes an effort \( x_i \). Efforts are undertaken simultaneously.

• We assume that the cost function (a function of effort and ability), is increasing in effort and decreasing in ability. An effort \( x_i \) causes a dis-utility (or cost) denoted by \( \frac{1}{a_i} C(x_i) \). We assume that cost is a strictly increasing function of effort (i.e. \( C^{-1}(.) \) exists and is a strictly increasing function) and \( C(0) = 0 \). We will analyze contests with linear, strictly concave and strictly convex cost functions.

• The contestant with the highest performance wins the first prize \( V^{[1]} \) while the contestant with second highest performance wins the second prize \( V^{[2]} \) and so on until all the prizes are allocated. That is, the payoff of contestant \( i \) who has ability \( a_i \), and makes an effort \( x_i \) is either \( V^{[i]} - \frac{1}{a_i} C(x_i) \) if \( i \) wins prize \( j \), or \(-\frac{1}{a_i} C(x_i) \) if \( i \) does not win a prize.

• Performance \( (\phi_i) \) of player \( i \) is equal to \( i \)'s effort.

• We show that at equilibrium the contestant, whose ability is \( m \) puts in zero effort, i.e. at equilibrium the equation below must be satisfied \( x_i(m, ...) = 0 \) \( \forall i = 1, ..., k \). Note that the equilibrium effort of any contestant is a function of the abilities of all the contestants, i.e. for all \( i \in [1, k] \), \( x_i = b(a_1, a_2, ..., a_k) \). Therefore at equilibrium, we have \( \phi_i = b(a_1, a_2, ..., a_k) \).

We are interested in an increasing symmetric equilibrium only, so that at equilibrium \( b \) is a strictly increasing function of ability of the contestant \( i \), i.e. \( b_i > 0 \) throughout the domain of the function \( b \). Moreover as we have assumed above, at equilibrium an agent with the lowest ability always puts in zero effort, i.e. \( b_i(a_1, ..., a_{i-1}, m, a_{i+1}, ..., a_k) = 0 \). Therefore the performance of the contestant, whose ability is \( m \), is zero at equilibrium.

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1 The treatment of the case where \( i \)'s cost function is given by \( \delta(1/a_i)x_i \), with \( \delta \) strictly monotone increasing, is completely analogous.

2 Let \( h > 1 \) contestants tie for prize \( j \). If \( h \leq p - j + 1 \), we assume that prizes \( j, ..., j + h - 1 \) are randomly allocated among the tied players. If \( h > p - j + 1 \), then prizes \( j, ..., p \) and a total of \( h - (p - j + 1) \) zero prizes are randomly allocated among the tied players.

3 We are implicitly assuming that the performance and effort can't be negative.
• Each contestant $i$ chooses her effort in order to maximize her expected utility (given the other competitors' efforts and given the value functions of the different prizes). [Alternatively each contestant $i$ chooses her performance to maximize expected utility as $a_i$ is known to $i$].

• The contest designer determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximize the expected value of the sum of the performances. $\sum_{i=1}^k \phi_i$ (given the contestants' equilibrium performance function).

• Let us define the function

$$v^{[i]}(a_i, y_j) = \begin{cases} E \left[ V^{[i]} \mid A = a_i, Y_1 = y_1 \right] & \text{if } j = 1 \\ E \left[ V^{[j]} \mid A = a_i, Y_j = y_j, Y_{j-1} = y_{j-1}, a_i < y_{j-1} \right] & \text{otherwise} \end{cases}$$

to be the expectation of the value of the contestant $i$ for the prize $j$, when the signal he or she receives is $a_i$ and the $j^{th}$ highest signal among the other contestants, $Y_j$ is $y_j$. Because of symmetry, these functions are the same for all the contestants. For all $j \in [1, p]$, let us assume $v^{[i]}$ are non-decreasing functions in $a_i$ and $y$. Let us also assume that for all $j \in [1, p]$, $v^{[j]}$ are strictly increasing in $a_i$. Moreover, since for all $j \in [1, p]$, $\bar{V}^{[j]}(m, m, ..., m) = 0$, $v^{[j]}(m, m) = 0$ and assume that for all $j \in [1, p-1]$, $v^{[j]}(.) \geq v^{[j+1]}(.)$. Finally, we assume, for all $j \in [1, p]$, $v^{[j]}(z, .) > 0 \forall z > m$.

Let us assume $p = 2$ and $k \geq 3$. We now derive the equilibrium bidding strategy of a contestant. Let us define, $\bar{v}^{[j]}(a_i, y_1) = E \left[ V^{[2]} \mid A = a_i, Y_1 = y_1 \right]$. We assume for all $a_i > m$, $\bar{v}^{[i]}(a_i, .) \geq v^{[i]}(a_i, .)$ i.e. the expected value of the second prize, given the next highest ability, when there are two prizes and the contestant gets second prize due to the fact that she finishes the contest with the second position, is less than or equal to the expected value of the second prize, given the next highest ability, when the contestant gets the second prize along with the first prize due to the fact that there is a single prize that equals the sum of the first and the second prizes and she wins the contest. Let us suppose that all the contestants except the contestant $i$ follow the increasing and differentiable strategy $b(.)$. Clearly it is not optimal for any contestant to put in an effort more than $b(1)$. Let us define $G^{[i]}(.) = \binom{k-1}{i-1} \frac{1}{(k-1)!} (F(a))^{i-1} (1-F(a))^{k-i}$ as the probability of winning the $j$th prize and let $g^{[j]}(.)$ be the first derivative of $G^{[j]}(.)$ with respect to $a$.

First let us assume a linear cost function (i.e. the cost function is given by $\frac{x_i}{a_i}$). The expected payoff of the contestant $i$ when her signal is $a_i$ and she puts in an effort equal to $b^{[2]}(z)$ is
The first order condition for the expected profit maximization is

\[ v^{[1]}(a_i, z)g^{[1]}(z) + v^{[2]}(a_i, z)g^{[2]}(z) - \frac{b^{[2]}/(z)}{a_i} = 0 \]

At a symmetric equilibrium, it is optimal to bid according to \( z = a_i \); so setting \( z = a_i \) in the first-order condition, we obtain the differential equation:

\[ v^{[1]}(a_i, a_i)g^{[1]}(a_i) + v^{[2]}(a_i, a_i)a_ig^{[2]}(a_i) = \frac{b^{[2]}/(a_i)}{a_i} \quad (4.2.1) \]

Since, by assumption, \( v(m, m) = 0 \), it is the case that \( b(m) = 0 \). Thus associated with the above equation (4.2.1) we have the boundary condition \( b(m) = 0 \). The solution to the differential equation (4.2.1) together with the boundary condition \( b(m) = 0 \), as stated in the proposition below, constitutes a symmetric equilibrium.

**Proposition 4.2.1.** Symmetric equilibrium strategies in the contest are given by

\[ b^{[2]}(a_i) = \int_m^{a_i} v^{[1]}(z, z)zg^{[1]}(z)dz + \int_m^{a_i} v^{[2]}(z, z)zg^{[2]}(z)dz \]

**proof:**

Note that, because \( v^{[1]}(a_i, a_i)a_ig^{[1]}(a_i) + v^{[2]}(a_i, a_i)a_ig^{[2]}(a_i) = b^{[2]}/(a_i) \) and because all \( a_i > m \) imply for all \( j \in [1, p] \), \( v^{[j]}(a_i, .) > 0 \), we have

\[
\begin{align*}
& v^{[1]}(a_i, a_i)a_ig^{[1]}(a_i) + v^{[2]}(a_i, a_i)a_ig^{[2]}(a_i) \\
& \geq \left[ g^{[1]}(a_i) + g^{[2]}(a_i) \right] a_ig^{[2]}(a_i) \\
& = (k - 1)(k - 2)F(a_i)^{k-3}(1 - F(a_i))F'(a_i)a_ig^{[2]}(a_i, a_i) \\
& > 0
\end{align*}
\]
Now consider a contestant who bids $b^{[2]}(z)$ when her signal is $a_i$. The expected profit from such a bid can be written as

$$\Pi(z, a_i) = \int_m^z v^{[1]}(a_i, y_1)g^{[1]}(y_1)dy_1 + \int_m^z v^{[2]}(a_i, y_2)g^{[2]}(y_2)dy_2 - \frac{b^{[2]}(z)}{a_i}$$

since $b^{[2]}$ is increasing.

Differentiating with respect to $z$ yields

$$\frac{\partial \Pi}{\partial z} = v^{[1]}(a_i, z)g^{[1]}(z) + v^{[2]}(a_i, z)g^{[2]}(z) - \frac{b^{[2]}(z)}{a_i}$$

If $z < a_i$ then since $v^{[1]}(a_i, z) > v^{[1]}(z, z)$ and $v^{[2]}(a_i, z) > v^{[2]}(z, z)$, we obtain that

$$\frac{\partial \Pi}{\partial z} > v^{[1]}(z, z)g^{[1]}(z) + v^{[2]}(z, z)g^{[2]}(z) - \frac{b^{[2]}(z)}{a_i}$$

$$> v^{[1]}(z, z)g^{[1]}(z) + v^{[2]}(z, z)g^{[2]}(z) - \frac{b^{[2]}(z)}{z}$$

$$= 0$$

using equation (4.2.1). Similarly, if $z > a_i$, then $\frac{\partial \Pi}{\partial z} < 0$. Thus, $\Pi(z, x)$ is maximized by choosing $z = x$.

Later in this chapter we will show that our argument here, can be easily generalized for the case $p > 2$.

Now consider a contest in which the highest performer gets both the prizes i.e. there is a single first prize ($v$) only, such that $v(.) = v^{[1]}(.) + v^{[2]}(.)$. Then the payoff of contestant $i$, when the equilibrium bidding strategy is $b^{[1]}$, is given by

$$\Pi(z, a_i) = \int_m^z v(a_i, y_1)g^{[1]}(y_1)dy_1 - \frac{b^{[1]}(z)}{a_i}$$

$$= \int_m^z [v^{[1]}(a_i, y_1) + v^{[2]}(a_i, y_1)] g^{[1]}(y_1)dy_1 - \frac{b^{[1]}(z)}{a_i}$$
It is routine to check that in this case the symmetric equilibrium bidding strategy of a contestant with ability $a_i$ is given by

$$b_i^1(a_i) = \int_m^{a_i} \left[ v_1^1(z, z) + \delta_i^2(z, z) \right] z g_1^1(z) \, dz$$

The expected sum of performances of all the contestants in the case of two prizes is

$$\Phi_2 = k \int_m^1 b_i^2(a_i) f(a_i) \, da_i$$
$$= k \int_m^1 \left[ \int_m^{a_i} v_1^1(z, z) z g_1^1(z) \, dz + \int_m^{a_i} v_2^2(z, z) z g_2^2(z) \, dz \right] f(a_i) \, da_i$$
$$= k \int_m^1 \left[ \int_m^{a_i} v_1^1(z, z) z g_1^1(z) \, dz \right] f(a_i) \, da_i + k \int_m^1 \left[ \int_m^{a_i} v_2^2(z, z) z g_2^2(z) \, dz \right] f(a_i) \, da_i$$

Similarly, the expected sum of performances of all the contestants in the case of a single first prize is

$$\Phi_1 = k \int_m^1 b_i^1(a_i) f(a_i) \, da_i$$
$$= k \int_m^1 \left[ \int_m^{a_i} \left[ v_1^1(z, z) + v_2^2(z, z) \right] z g_1^1(z) \, dz \right] f(a_i) \, da_i$$
$$= k \int_m^1 \left[ \int_m^{a_i} v_1^1(z, z) z g_1^1(z) \, dz \right] f(a_i) \, da_i + k \int_m^1 \left[ \int_m^{a_i} v_2^2(z, z) z g_1^1(z) \, dz \right] f(a_i) \, da_i$$

Let us define $A(a_i) = \int_m^{a_i} v_2^2(z, z) z g_1^1(z) \, dz$ and $B(a_i) = \int_m^{a_i} v_2^2(z, z) z g_2^2(z) \, dz$. Therefore, we have

$$\Phi_1 \geq k \int_m^1 \left[ \int_m^{a_i} v_1^1(z, z) z g_1^1(z) \, dz \right] f(a_i) \, da_i + k \int_m^1 A(a_i) f(a_i) \, da_i$$
$$\Phi_2 = k \int_m^1 \left[ \int_m^{a_i} v_1^1(z, z) z g_1^1(z) \, dz \right] f(a_i) \, da_i$$

Below we will prove seven lemmas which will play an important role in proving our main results. These seven lemmas are very similar with the seven lemmas we have proved in chapter 2, and we will proceed exactly in the same way as in chapter 2.
Lemma. Under the assumptions stated above, the seven conditions below hold.

1. \( A(m) = B(m) = 0 \).
2. \( \forall c \in (m, 1) \ A(c) > 0, \ A'(c) > 0 \).
3. Let \( c^* \) be such that \( F(c^*) = \frac{k-2}{k-1} \) then \( B'(c^*) = 0 \); \( \forall c \in [m, c^*) \ B'(c) > 0 \) and \( \forall c \in (c^*, 1] \ B'(c) < 0 \).
4. \( |B'(c)| > |A'(c)| \) for \( c \) in a neighborhood of \( m \).
5. \( A(1) > B(1) \).
6. For any \( k \geq 3 \), there exists a unique point \( c^* \neq m \) such that \( A(c^*) = B(c^*) \).
7. \( \int_m^1 (B(c) - A(c))F'(c)dc < 0 \).

Proof:
Lemma 1., 2. and 3. are trivial from the definitions of \( A(c) \), \( B(c) \) and their respective derivatives.

Note that

\[
A(c) = (k - 1) \int_m^c v^2(a, a) F(a)^{k-2} F'(a) da \\
B(c) = (k - 1) \int_m^c v^2(a, a) F(a)^{k-3} F'(a) [(k - 2) - (k - 1)F(a)] da \\
A'(c) = (k - 1)v^2(c, c)cF(c)^{k-2} F'(c) \\
B'(c) = (k - 1)v^2(c, c)cF(c)^{k-3} F'(c) [(k - 2) - (k - 1)F(c)]
\]

4. If \( c \in (m, c^*) \) then

\[
|B'(c)| - |A'(c)| = B'(c) - A'(c) = (k - 1)v^2(c, c)cF(c)^{k-3} F'(c) [(k - 2) - kF(c)] > 0
\]

if \( c \) is close enough to \( m \) such that \( F(c) < \frac{k-2}{k} \).

5. Let \( \bar{c} \) be such that \( F(\bar{c}) = \frac{k-2}{k} \) and because by assumption \( v^2(z, z) > 0 \ \forall z > m \), we have
\[ A(1) - B(1) \]
\[
= (k - 1) \int_{m}^{1} v^{[2]}(t, t) t F(t)^{k-3} F'(t) [kF(t) - (k - 2)] dt 
\]
\[
= (k - 1) \int_{m}^{1} v^{[2]}(t, t) t F(t)^{k-3} F'(t) [kF(t) - (k - 2)] dt 
+ (k - 1) \int_{\tilde{c}}^{1} v^{[2]}(t, t) t F(t)^{k-3} F'(t) [kF(t) - (k - 2)] dt 
\]
\[
> (k - 1) v^{[2]}(\bar{c}, \bar{c}) \tilde{c} \left[ \int_{m}^{1} F(t)^{k-3} F'(t) [kF(t) - (k - 2)] dt + \int_{\tilde{c}}^{1} F(t)^{k-3} F'(t) [kF(t) - (k - 2)] dt \right] 
\]
\[
= (k - 1) v^{[2]}(\bar{c}, \bar{c}) \tilde{c} \int_{m}^{1} F(t)^{k-3} F'(t) [kF(t) - (k - 2)] dt 
\]
\[
= (k - 1) v^{[2]}(\bar{c}, \bar{c}) \tilde{c} \int_{0}^{1} \frac{k}{k-1} [z^{k-1} - z^{k-2}] \] 
\[
= v^{[2]}(\bar{c}, \bar{c}) \tilde{c} > 0 
\]

The fourth equality is obtained by assuming \( z = F(t) \). Then it follows that

\begin{itemize}
  \item \( a = m \) implies \( z = F(m) = 0 \)
  \item \( a = 1 \) implies \( z = F(1) = 1 \)
  \item \( dz = F'(t) dt \)
\end{itemize}

6.

The first result of the lemma shows that both \( A(c) \) and \( B(c) \) curves start from the origin. The second result of the lemma implies that the curve \( A(c) \) always lies in the positive quadrant and it is strictly increasing for all \( c > m \). The third result illustrates that \( B(c) \) is a concave function. That is to say that initially \( B(c) \) increases with the ability, and reaches a maximum at ability equal to \( c^* \) and after that it decreases. The fourth result shows that the curve \( B(c) \) at the neighborhood of its starting point lies above \( A(c) \). The fifth result shows that \( B(c) \) again intersects \( A(c) \) at ability \( c^{**} \) where \( c^{**} < 1 \). And after that \( B(c) \) is always less than \( A(c) \). Combining all these results we have the next result, which claims that except at \( c = m \) the curves \( A(c) \) and \( B(c) \) will intersect at a point \( c^{**} > m \) as shown in the figure 4.1.
This follows by considering all the properties above, see the figure 4.1.

7. We know that $B(c) - A(c) > 0 \forall c \in [m, c^{**})$ and that $B(c) - A(c) < 0 \forall c \in (c^{**}, 1]$.

\[
\int_{m}^{c^{**}} (B(c) - A(c))F'(c)dc + \int_{c^{**}}^{1} (B(c) - A(c))F'(c)dc
\]
\[
= (k - 1) \int_{m}^{c^{**}} \left( \int_{m}^{c} v^{[2]}(a,a)aF(a)k^{-3}[(k - 2) - kF(a)]F'(a)da \right) F'(c)dc
\]
\[
+ (k - 1) \int_{c^{**}}^{1} \left( \int_{m}^{c} v^{[2]}(a,a)aF(a)k^{-3}[(k - 2) - kF(a)]F'(a)da \right) F'(c)dc
\]
\[
< (k - 1)v^{[2]}(c^{**}, c^{**})c^{**} \int_{m}^{c^{**}} \left( \int_{m}^{c} F(a)k^{-3}[(k - 2) - kF(a)]F'(a)da \right) F'(c)dc
\]
\[
+ (k - 1)v^{[2]}(c^{**}, c^{**})c^{**} \int_{c^{**}}^{1} \left( \int_{m}^{c} F(a)k^{-3}[(k - 2) - kF(a)]F'(a)da \right) F'(c)dc
\]
\[
= (k - 1)v^{[2]}(c^{**}, c^{**})c^{**} \int_{m}^{c^{**}} \left[ \int_{m}^{c} F(a)k^{-3}[(k - 2) - kF(a)]F'(a)da \right] F'(c)dc
\]
\[
= (k - 1)v^{[2]}(c^{**}, c^{**})c^{**} \int_{m}^{c^{**}} \left[ \int_{0}^{\frac{F(c)}{k}} z^{k-3}[(k - 2) - kz]dz \right] F'(c)dc
\]
\[= (k - 1)\psi^2(c^*, c^*)(c^*) \int_m^1 \left[z^{k-2} - \frac{k}{k-1} z^{k-1}\right] F'(c) dc\]

\[= (k - 1)\psi^2(c^*, c^*)(c^*) \int_m^1 \left[F(c)^{k-2} - \frac{k}{k-1} F(c)^{k-1}\right] F'(c) dc\]

\[= (k - 1)\psi^2(c^*, c^*)(c^*) \int_0^1 \left[v^{k-2} - \frac{k}{k-1} v^{k-1}\right] dv\]

\[= (k - 1)\psi^2(c^*, c^*)(c^*) \left[\frac{v^{k-1}}{k-1} - \frac{v^k}{k-1}\right]_0^1 = 0\]

The fourth equality is obtained by assuming \(z = F(a)\). Then it follows that

- \(a = m\) implies \(z = F(m) = 0\)
- \(a = c\) implies \(z = F(c)\)
- \(dz = F'(a) da\)

The seventh equality is obtained by assuming \(v = F(c)\). Then it follows that

- \(c = m\) implies \(v = F(m) = 0\)
- \(c = 1\) implies \(v = F(1) = 1\)
- \(dv = F'(c) dc\)

Also note that by assumption \(\psi^2(., .)\) is non-decreasing in its second argument and strictly increasing in its first argument.

We now prove our next main result.

**Proposition 4.2.2.** Under the setup discussed above it is always optimal to give a single first prize.

Proof:
\[(\Phi_1 - \Phi_2)/k\]
\[\geq \int_m^1 \left[ \int_m^{a_i} v^{[1]}(z, z) zg^{[1]}(z) \, dz \right] f(a_i) \, da_i + \int_m^1 A(a_i) f(a_i) \, da_i \]
\[\quad - \int_m^1 \left[ \int_m^{a_i} v^{[1]}(z, z) zg^{[1]}(z) \, dz \right] f(a_i) \, da_i + \int_m^1 B(a_i) f(a_i) \, da_i \]
\[= \int_m^1 A(a_i) f(a_i) \, da_i - \int_m^1 B(a_i) f(a_i) \, da_i \]
\[= \int_m^1 [A(a_i) - B(a_i)] f(a_i) \, da_i \]

And we know from Lemma 7 that \(\int_m^1 [A(a_i) - B(a_i)] f(a_i) \, da_i > 0\).

Now consider a case where we assume for all \(a_i > m\), \(v^{[2]}(a_i, \cdot) \leq v^{[2]}(a_i, \cdot)\) i.e. the expected value of the second prize, given the next highest ability, is strictly lower if the contestant gets the second prize along with the first prize in the case where there is a single prize that equals the sum of the first and the second prizes and she wins the contest, compared to the case where there are two prizes and the contestant gets second prize due to the fact that she finishes the contest with the second position.

The equilibrium bidding strategy of a contestant and \(\Phi_2\) do not change, but we have

\[\Phi_1 = k \int_m^1 b^{[1]}(a_i) f(a_i) \, da_i\]
\[\leq k \int_m^1 \left[ \int_m^{a_i} v^{[1]}(z, z) zg^{[1]}(z) \, dz \right] f(a_i) \, da_i + k \int_m^1 \left[ \int_m^{a_i} v^{[2]}(z, z) zg^{[1]}(z) \, dz \right] f(a_i) \, da_i\]

And therefore,

\[(\Phi_1 - \Phi_2)/k\]
\[\leq \int_m^1 \left[ \int_m^{a_i} v^{[1]}(z, z) zg^{[1]}(z) \, dz \right] f(a_i) \, da_i + \int_m^1 A(a_i) f(a_i) \, da_i \]
\[\quad - \int_m^1 \left[ \int_m^{a_i} v^{[1]}(z, z) zg^{[1]}(z) \, dz \right] f(a_i) \, da_i + \int_m^1 B(a_i) f(a_i) \, da_i\]
\[= \int_m^1 [A(a_i) - B(a_i)] f(a_i) \, da_i > 0\]
Chapter 4. Contest under Interdependent Valuations

So in this case, either $\Phi_1 > \Phi_2$ or $\Phi_1 < \Phi_2$ or they are equal. If $\Phi_1 > \Phi_2$ holds, then it is better to give a single first prize; on the other hand, if $\Phi_1 < \Phi_2$ then it is better to give multiple prizes. Therefore, we have shown that even if both the cost and the performance functions are linear then also in case of interdependent valuations, multiple prizes may be better.

Example: Let us assume that independent and identically distributed continuous random variables are $A_1, ..., A_N$. The distribution function is $F(.)$ and the density function is $f(.)$. Let $Y_1 = \max_1 \{ A_1, ..., A_{j-1}, A_{j+1}, ..., A_N \}$ and $Y_2 = \max_2 \{ A_1, ..., A_{j-1}, A_{j+1}, ..., A_N \}$ be the highest and second highest order statistics. Suppose $V^{[2]} = \sum_{i=1}^{N} \theta_i (A_i - m)$, where $\theta_i$s are parameters (which is a common knowledge to all the contestants). Our objective is to find out the expressions for $\bar{v}^{[2]} = E \{ V^{[2]} | A_i = a_i, Y_1 = y_1 \}$ and $v^{[2]} = E \{ V^{[2]} | A_i = a_i, Y_2 = y_2, Y_1 = y_1, a_i < y_1 \}$.

First we will calculate $E \{ V^{[2]} | A_i = a_i, Y_1 = y_1 \}$.

$$E \{ V^{[2]} | A_i = a_i, Y_1 = y_1 \}$$

$$= E \left\{ \sum_{j=1}^{N} \theta_j (A_j - m) | A_i = a_i, Y_1 = y_1 \right\}$$

$$= \sum_{j=1}^{N} \theta_j E \{ A_j | A_i = a_i, Y_1 = y_1 \} - m \sum_{i=1}^{N} \theta_i$$

$$= \theta_i a_i + \sum_{j=1}^{N} \theta_j E \{ A_j | A_i = a_i, Y_1 = y_1 \} - m \sum_{i=1}^{N} \theta_i$$

$$= \sum_{j=1}^{N} \theta_j E \{ A_j | Y_1 = y_1 \} - m \sum_{i=1}^{N} \theta_i$$

$$= \theta_i a_i + \sum_{j=1}^{N} \theta_j \int_{m}^{a_i} \frac{z f_{A_j, Y_1}(z, y_1)}{f_{Y_1}(y_1)} dz - m \sum_{i=1}^{N} \theta_i$$

$$= \theta_i a_i + \frac{1}{f_{Y_1}(y_1)} \sum_{j=1}^{N} \theta_j \int_{m}^{a_i} z f_{A_j, Y_1}(z, y_1) dz - m \sum_{i=1}^{N} \theta_i$$
The eighth equality follows because the probability that $A_j < Y_1$ is zero due to the fact that contestant $i$ wins the contest. The ninth equality is again due to the fact that the random variables $(A_i)$ are IID therefore $A_j$ and $Y_1$ are independent.

Similarly now consider $E[V|A_i = a_i, Y_2 = y_2, Y_1 = y_1, a_i < y_1]$.

\[
E[V|A_i = a_i, Y_2 = y_2, Y_1 = y_1, a_i < y_1] = E\left[\sum_{j=1}^{N} \theta_j (A_j - m) \mid A_i = a_i, Y_2 = y_2, Y_1 = y_1, a_i < y_1\right]
= \sum_{j=1}^{N} \theta_j E[A_j \mid A_i = a_i, Y_2 = y_2, Y_1 = y_1, a_i < y_1] - m \sum_{i=1}^{N} \theta_i
\]
\[\begin{align*}
\theta_i a_i &= \sum_{j=1}^{N} \theta_j E[A_j \mid A_i = a_i, Y_2 = y_2, Y_1 = y_1, a_i < y_1] - m \sum_{i=1}^{N} \theta_i \\
\theta_i a_i &= \sum_{j=1, j \neq i}^{N} \theta_j E[A_j \mid Y_2 = y_2, Y_1 = y_1, a_i < y_1] - m \sum_{i=1}^{N} \theta_i \\
\theta_i a_i &= \sum_{j=1}^{N} \theta_j \int_{m}^{a_i} \frac{f_{A_j,Y_2}(z, y_2 \mid Y_1 = y_1, a_i < y_1)}{f_{Y_2}(y_2 \mid Y_1 = y_1, a_i < y_1)} \, dz - m \sum_{i=1}^{N} \theta_i \\
\theta_i a_i &= \frac{1}{f_{Y_2}(y_2 \mid Y_1 = y_1, a_i < y_1)} \sum_{j=1}^{N} \theta_j \int_{m}^{a_i} z f_{A_j,Y_2}(z, y_2 \mid Y_1 = y_1, a_i < y_1) \, dz - m \sum_{i=1}^{N} \theta_i \\
\theta_i a_i &= \frac{1}{f_{Y_2}(y_2 \mid Y_1 = y_1, a_i < y_1)} \sum_{j=1, j \neq i}^{N} \theta_j \int_{Y_2}^{a_i} z f_{A_j,Y_2}(z, y_2 \mid Y_1 = y_1, a_i < y_1) \, dz - m \sum_{i=1}^{N} \theta_i \\
\theta_i a_i &= \frac{1}{f_{Y_2}(y_2 \mid Y_1 = y_1, a_i < y_1)} \sum_{j=1}^{N} \theta_j \int_{m}^{a_i} z f(z) \, dz - m \sum_{i=1}^{N} \theta_i \\
\theta_i a_i &= \frac{f_{Y_2}(y_2 \mid Y_1 = y_1, a_i < y_1)}{f_{Y_2}(y_2 \mid Y_1 = y_1, a_i < y_1)} \sum_{j=1}^{N} \theta_j \int_{m}^{a_i} z f(z) \, dz - m \sum_{i=1}^{N} \theta_i \\
\theta_i a_i &= \frac{f_{Y_2}(y_2 \mid Y_1 = y_1, a_i < y_1)}{f_{Y_2}(y_2 \mid Y_1 = y_1, a_i < y_1)} \sum_{j=1, j \neq i}^{N} \theta_j \int_{m}^{a_i} z f(z) \, dz - m \sum_{i=1}^{N} \theta_i 
\end{align*}\]
The eighth equality follows because the probability that $A_j < Y_2$ is zero due to the fact that contestant $i$ wins the contest. The ninth equality is again due to the fact that the random variables $(A_i)$ are IID therefore $A_j$ and $Y_2$ are independent. Now let us define

$$
\zeta_1 = \int_m^1 \int_m^{a_i} \theta_i a_i + \sum_{j=1}^{N} \theta_j \int_m^{z} x f(x) dx - m \sum_{i=1}^{N} \theta_i z g^{[i]}(z) dz f(a_i) da_i
$$

$$
= \int_m^1 \theta_i a_i \int_m^{a_i} z g^{[i]}(z) dz f(a_i) da_i + \int_m^1 \int_m^{a_i} \sum_{j=1}^{N} \theta_j \int_m^{z} x f(x) dx z g^{[i]}(z) dz f(a_i) da_i
$$

$$
- m \sum_{i=1}^{N} \theta_i \int_m^1 \int_m^{a_i} z g^{[i]}(z) dz f(a_i) da_i
$$

and
\[ \zeta_2 = \int_m^a \int_m^{a_i} \left[ \theta_i a_i + \sum_{j=1}^{N} \theta_j \int_m^{z} x f(x) \, dx - m \sum_{i=1}^{N} \theta_i \right] z g^{[2]}(z) \, dz f(a_i) \, da_i \]

\[ = \int_m^a \theta_i a_i \int_m^{a_i} z g^{[2]}(z) \, dz f(a_i) \, da_i + \int_m^a \int_m^{a_i} \sum_{j=1}^{N} \theta_j \int_m^{z} x f(x) \, dx \, dz g^{[2]}(z) \, dz f(a_i) \, da_i \]

\[ - m \sum_{i=1}^{N} \theta_i \int_m^{a_i} z g^{[2]}(z) \, dz f(a_i) \, da_i \]

If \( \zeta_1 \geq \zeta_2 \) then a single first prize is better, otherwise it is better to give multiple prizes.

Now we are going to concentrate on non-linear cost functions. In case of a non-linear cost function, \( C \) can take any shape, but we will only focus on concave and convex cost functions.

**Proposition 4.2.3.** The symmetric equilibrium bidding strategy is given by

\[ b^{[2]}(a_i) = C^{-1} \left( \int_m^{a_i} v^{[1]}(z, z) z g^{[1]}(z) \, dz + \int_m^{a_i} v^{[2]}(z, z) z g^{[2]}(z) \, dz \right) \]

**proof:**

First note that \( C^{-1} \) exists because by assumption \( C \) is a strictly increasing function. Also we know that \( C^{-1} \) is a strictly increasing function. The expected payoff of the contestant \( i \) when her signal is \( a_i \) and she puts in effort \( b^{[2]}(z) \) is

\[ \Pi(z, a_i) = \int_m^z v^{[1]}(a_i, y_1) g^{[1]}(y_1) \, dy_1 + \int_m^z v^{[2]}(a_i, y_2) g^{[2]}(y_2) \, dy_2 - \frac{C(b^{[2]}(z))}{a_i} \]

The first order condition is
At a symmetric equilibrium, it is optimal to bid according to \( z \) first-order condition, we obtain the differential equation:

\[
v^{[1]}(a_i, z)g^{[1]}(z) + v^{[2]}(a_i, z)g^{[2]}(z) - C'(b^{[2]}(z)) \frac{b^{[2]'}}{a_i} = 0
\]

\[
\Leftrightarrow v^{[1]}(a_i, z)g^{[1]}(z) + v^{[2]}(a_i, z)g^{[2]}(z) = C'(b^{[2]}(z)) \frac{b^{[2]'}}{a_i}
\]

\[
\Leftrightarrow v^{[1]}(a_i, z)a_i g^{[1]}(z) + v^{[2]}(a_i, z)a_i g^{[2]}(z) = C'(b^{[2]}(z))b^{[2]'}(z)
\]

Since, by assumption, \( v(m, m) = 0 \), it is the case that \( b(m) = 0 \). Thus associated with the above equation (4.2.2) we have the boundary condition \( b(m) = 0 \). The solution to the differential equation (4.2.2) together with the boundary condition \( b(m) = 0 \) constitutes a symmetric equilibrium, which is given below.

\[
b^{[2]}(a_i) = C^{-1} \left[ \int_m^{a_i} v^{[1]}(z, z) z g^{[1]}(z) dz + \int_m^{a_i} v^{[2]}(z, z) z g^{[2]}(z) dz \right]
\]

Note that the equilibrium bidding function is a strictly increasing function of ability, because \( C^{-1}(\cdot) \) is strictly increasing and we have already proved in the section on linear cost function that \( \int_m^{a_i} v^{[1]}(z, z) z g^{[1]}(z) dz + \int_m^{a_i} v^{[2]}(z, z) z g^{[2]}(z) dz \) is a strictly increasing function of ability in the domain \((m, 1]\). Because cost is a strictly increasing function of effort, one can check the second order condition in the way shown in the section on linear cost function.

Now let us calculate the expected sum of performances in case of non-linear cost functions.

The expected sum of performances of all the contestants in the case of two prizes is given by

\[
\Phi_2 = k \int_m^{a_i} b^{[2]}(a_i) f(a_i) da_i
\]

\[
= k \int_m^{a_i} C^{-1} \left[ \int_m^{a_i} v^{[1]}(z, z) z g^{[1]}(z) dz + \int_m^{a_i} v^{[2]}(z, z) z g^{[2]}(z) dz \right] f(a_i) da_i
\]
In this section we will briefly show that the results we derived above can be extended to the case where \( k > p > 2 \), very easily. We will proceed exactly the same way as we did above, i.e. we will first derive the equilibrium bidding strategy of a contestant in case of \( k > p > 2 \) and then we will show that proposition 2 still holds.

**Proposition 4.2.4.** Suppose \( k > p > 2 \), then the symmetric equilibrium strategies in the contest are given by

\[
b^{[j]}(a_i) = C^{-1} \left[ \sum_{j=1}^{p} \int_{m}^{a_i} v^{[j]}(z, z) zg^{[j]}(z) dz \right]
\]

**proof:**

Note that in this case the payoff function of the contestant \( i \) when her signal is \( a_i \) and she puts in effort \( b^{[j]}(z) \), is given by

\[
\Pi(z, a_i) = \sum_{j=1}^{p} \int_{m}^{z} v^{[j]}(a_i, y_j)g^{[j]}(y_j)dy_j - \frac{C[b^{[j]}(z)]}{a_i},
\]

The rest of the proof is similar to \( p = 2 \) case presented above. ♦

The expected performance of each contestant is given by

\[
\Phi_p = \int_{m}^{1} b^{[j]}(a_i)f(a_i)da_i
\]

\[
= \int_{m}^{1} C^{-1} \left[ \sum_{j=1}^{p} \int_{m}^{a_i} v^{[j]}(z, z) zg^{[j]}(z) dz \right] f(a_i)da_i
\]

Let us assume again that \( C^{-1} \) is a linear function, such that \( \Phi_p = \int_{m}^{1} \left[ \sum_{j=1}^{p} \int_{m}^{a_i} v^{[j]}(z, z) zg^{[j]}(z) dz \right] f(a_i)da_i. \) Again if we also assume that \( v^{[j]}(a_i, .) \geq v^{[j]}(a_i, .) \forall j \in [2, p] \) where \( v^{[j]}(a_i, y_1) = E \left[ V^{[j]} \mid A_i = a_i, Y_j = y_1 \right] \), then it is routine to check that all the seven lemmas above hold for each \( j \in [2, p] \) and therefore it is optimal to give a single prize. On the other hand, if it is the case that \( v^{[j]}(a_i, .) < v^{[j]}(a_i, .) \forall j \in [2, p] \) then even if we assume a linear cost function, it may be the case that giving multiple prizes is
optimal for the contest designer. In case of private valuation $v^{[j]}(a_{i,.}) = v^{[j]}(a_{i,.}) \forall j \in [2,p]$ so that
it is optimal to give a single prize (see Moldovanu and Sela, 2001), but interdependent valuation makes things much more complicated even in the case of linear cost functions.

4.3 Conclusion:

Contest under interdependent valuations is much more complicated than that under private values. We analyze contest under interdependent valuations with multiple prizes; here we assume that the valuations of all the contestants for all the prizes are functions of the abilities of all the contestants.

We first derive equilibrium bidding strategy of a contestant where the contest designer can give at most two prizes. Consider two cases. In both the cases the cost function is linear in effort. Now, in the first case, there are two prizes $P_1$ and $P_2$ and the expected valuation of $P_2$ to the second prize winner of the contest is $V_2$. In the second case, there is a single prize $P$ which is a bundle of goods consists of $P_1$ and $P_2$, and the expected valuation of $P_2$ to the first prize winner of the contest is $\bar{V}_2$. We have shown that if $\bar{V}_2 > V_2$ then it is optimal to give single prize, otherwise it may optimal to give multiple prizes. In case of private value $\bar{V}_2 = V_2$, therefore, it is optimal to give single prizes. So private value case is a special case of interdependent valuations.

Next we derived the equilibrium bidding strategy of a contestant with a general cost function. Finally, we derived the equilibrium bidding strategy of a contestant where the contest designer can offer more than two prizes. We have argued that it may be the case that even with a linear cost function, giving multiple prizes is optimal. This stands in contrast to the private value case.

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4 Both the prizes are indivisible in nature.