Chapter 6

Standard Auction

6.1 Introduction:

One of the earliest breakthroughs in Auction Theory came in 1960, with W. Vickrey’s paper. After that, many well known economists and mathematicians worked in this field, and made auction theory an essential tool for many governmental and non-governmental decision-making. Although the literature of Auction Theory began in 60s, but the era of putting Auction Theory to work began just 12 - 13 years ago. In 1993-94, there were radio spectrum auctions in United States, which began the use of the literature in practice. By 1996, Auction Theory had become so influential that one of its original minds, Prof. Vickrey was awarded Nobel Prize in economics.

From the beginning to the recent years, Auction Theory was mainly influenced by Game Theory. The status of Game Theory in economics was a hotly debated topic. In the early stages of development, Auction Theory was derived from Game theory, by considering auctions as games. So, Auction Theory also did inherit the same controversy that Game Theory had. Although Game Theory gained increasing prominence through out the 1980s and had begun to influence the leading graduate textbooks by the early 1990s, but in case of Auction Theory there was no consensus about its relevance until in 1994, when the Federal Communications Commission conducted the first of the new spectrum auctions.

"The traditional foundations of game theory incorporate stark assumptions about the rationality of the players and the accuracy of their expectations, which are hard to reconcile with reality. Yet, based on both field data and laboratory data, the contributions of auction theory are hard to dispute. The qualitative predictions of auction theory have been striking successful in explaining patterns of
bidding for oil and gas and have fared well in other empirical studies as well. Economic laboratory tests of auction theory have uncovered many violations of the most detailed theories, but several key tendencies predicted by the theory find significant experimental support. Taken as a whole, these findings indicate that although existing theories need refinement, they capture important features of actual bidding. For real-world auction designers, the lesson is that theory can be helpful, but it needs to be supplemented by experiments to test the applicability of key propositions and by practical judgments, seasoned by experience.” - Paul Milgrom (Putting Auction Theory To Work)

In any auction there are two parties, and we call them agents of auction. The first type of agents is called sellers (who is selling or allocating one or multiple objects) and the second type of agents is called buyers. The task of the sellers is to allocate or sell, single or multiple objects to the buyers. It is very common in the theory of auction to assume that the objective of the buyers is to just maximize their own utility. Whereas, the sellers may have single or multiple objectives, e.g. efficiency, balanced budget etc. can be the objectives of the sellers (we will explain all the term in detail later on). Now any party can initiate an auction. A seller starting an auction to sell any object is very common, e.g. government wants private investment for a public sector project, by starting an auction. But it is also possible for a buyer to start an auction, this type of auction is known as procurement auction (procurement auction is one where the auctioneer is the buyer rather than the seller), e.g. any institute wants to buy some computers so they start an auction. Whoever starts the auction, we call her the mechanism designer because she sets the rules of the auction, according to her objectives. And the theory regarding designing an auction falls under an area popularly known as Mechanism Design Theory. So, one of the segments of Mechanism Design Theory, deals with designing of auctions, such that different objectives of the mechanism designer will be fulfilled. Below we discuss different objectives of mechanism designer and the inter-linkage between Auction Theory and Mechanism design theory in details.

In early stages, auction theory was primarily concerned with the question of revenue. But efficiency gained attention when there was large scale privatization of socially held assets by governments in USA and Britain. These include the sales of industrial enterprises in Eastern Europe and the former Soviet Union, radio spectra in the United States and the railway system in Britain. The primary objective of all these auctions was not revenue maximization, but rather that the socially held assets are sold and allocated efficiently.

The first mechanism that comes to mind while discussing the literature of efficient mechanism designing, is the Second Price Sealed-bid Auction (SPSA) (First Price Sealed-Bid Auction is also efficient, but SPSA attracts more attention due to its similarity to the Vickrey-Clark-Groves Mechanism. Moreover, SPSA is a truth reveling mechanism while its counter-part is not). In SPSA, the
highest bidder wins and pays the second highest bid. It is very easy to show under some simple assumptions, the optimal strategy for each bidder is to bid her own valuation.

Vickrey’s (1961) classic paper on auctions was also concerned with the question of efficiency. VCG mechanism is an incentive compatible, direct mechanism. In VCG mechanism it is always optimal for the bidder to bid truthfully (like SPSA). Moreover truthful bidding is the only bidding strategy that is always optimal, and therefore it is a dominant strategy. It is also an efficient mechanism. The mechanism achieves efficiency by imposing, on each participant, the cost of any distortion she causes. The VCG payment for any participant is set in a way that the report of that participant cannot affect the total payoff of the other parties (including the mechanism designer).

Ausubel (1997) constructed an ascending auction for the many-bidder, multiunit setting. Ausubel ascending auction possesses an ex-post efficient equilibrium in the case of private values, but in case of interdependent valuation ex-post efficiency cannot be guaranteed, unless we assume that the bidders are ex-ante symmetric, their signals are affiliated and they have flat demand schedules up to a fixed capacity.

Perry and Reny (1997, 1999) worked on the linkage principle in multiunit auctions. They provided a theory on, two-bidder, two-object Vickrey auction on interdependent values. Their analysis showed that like in private value case, Vickrey auction also gives efficiency in case of interdependent valuation.

Jehiel and Moldovanu (1998) study the ex-post efficient implementation in economic environment in which agents’ signals can be multidimensional. They showed that when the signals are of the dimension two or above, unless the marginal rates of substitutions of agents’ signals coincide, ex-post efficient implementation is typically not possible.

Dasgupta and Maskin (1999) provide a selling mechanism, which achieves ex-post efficiency, under the assumption of finite number of possible heterogeneous goods. Their selling mechanism requires the agents to submit valuation correspondences (in effect, a continuum of potential preference profiles). The designer is then required to calculate the entire set of fixed points of their products. When there are multiple fixed points, which could occur even in the equilibrium, the agents must simultaneously announce the correct one to the auctioneer.

In 2004 C. Mezzetti first introduced “Generalized Groves Mechanism”, a mechanism which under certain conditions ensures efficiency. His paper is on ‘single object auction under interdependent valuation’. Agents’ valuations are interdependent, if they depend on the signals, or types, of all
CHAPTER 6. STANDARD AUCTION

120

the agents. So, under interdependent valuation, the valuations of every agent depend not only on their own types but also depend on other agents participating in that auction. This implies that agents are not completely aware of their outcome-decision payoffs.

Many literature showed that under interdependent valuation and independent signals, no standard auction can ensure efficiency. In this respect, the paper by Mezzetti (2004) was a great breakthrough, where he showed that "Generalized Groves Mechanism" can ensure efficiency. "Generalized Groves Mechanism" is a two-stage mechanism (obviously, it is not a 'standard mechanism' as standard mechanism is always a single stage mechanism, but "Generalized Groves Mechanism" on the other hand, is a two stage mechanism), where first the final outcome (i.e. allocation of the goods) is determined, and then the agents observe their own outcome-decision payoffs, and finally final transactions are made.

6.2 Revelation Principle and Incentive Compatibility:

Consider a seller has one indivisible object to sell and there are \( N \) risk neutral buyers from the set \( \mathbb{N} = \{1, 2, \ldots, N\} \). Buyers have private valuations and the valuations are independently and identically distributed. Buyer \( i \)'s value \( X_i \) is distributed over the interval \( X_i \in [0, \omega] \) according to the distribution function \( F_i(.) \) with associated density function \( f_i(.) \). For the sake of simplicity we assume the value of the object to the seller is zero.

Let \( X = \sum_{j=1}^{N} X_i \) and \( \forall i, X_{-i} = \sum_{j \neq i}^{N} X_j \). Define \( f(X) \) to be the joint density of \( X = (X_1, \ldots, X_N) \).

Since the valuations are independently distributed, \( f(X) = \prod_{i=1}^{N} f_i(X_i) \) and \( f_{-i}(X_{-i}) = \prod_{j \neq i}^{N} f_j(X_j) \) are the joint density of \( X_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_N) \).

Any mechanism has three components, viz. a set of possible messages (\( \mathcal{B}_i \)) for each bidder \( i \), an allocation rule (\( \pi : \mathcal{B} \rightarrow \Delta \), where \( \Delta \) is the set of probability distributions over the set of buyers) and a payment rule (\( \mu : \mathcal{B} \rightarrow \mathbb{R}^N \), where \( N \) is the number of bidders). Let us denote by \( \beta_i : [0, \omega] \rightarrow \mathcal{B}_i \), the equilibrium bidding strategy of the \( i^{th} \) bidder. Revelation principle tells that there exists a direct mechanism, where \( Q(x) = \pi(\beta(x)) \), which ensures truth telling as an equilibrium. As we know that a mechanism is incentive compatible if and only if \( q_i(.) \) is non-decreasing, where \( q_i(z_i) = \int_{x_{-i}} Q_i(z_i, x_{-i}) f_{-i}(x_{-i}) dx_{-i} \).

Now we know that if \( Q_i(z_i, x_{-i}) \) is non-decreasing in \( z_i \), then \( q_i(z_i) \) is also non-decreasing in \( z_i \). Assume an auction format where the highest bidder wins. Therefore in this case \( \pi(.) \) is non-decreasing. And if we are interested

\footnote{See Krishna, Auction Theory, Page 65}
in finding a symmetric increasing equilibrium bidding strategy for this auction format, we know
$\beta(.)$ is also non-decreasing. Therefore we have that $Q_l(.)$ is non-decreasing for this mechanism.
So, by revelation principle we know that there exists an incentive compatible direct mechanism
for this auction, such that truthfully reporting bidders' own valuations is a Nash equilibrium. So
for any auction where the highest bidder wins, any increasing bidding strategy satisfies incentive
compatibility condition.

6.3 Equilibrium strategies:

Here we have to specify the equilibrium strategies of all the buyers and the seller. Remember that a
strategy is a complete plan of actions. A mechanism can consist of a single stage or multiple stages.
Therefore, a buyer/seller has to calculate a bid at a particular stage at a particular valuation. And
a strategy profile of a particular buyer/seller is a collection of those bids for all valuations and all
stages.

A strategy for a bidder $i$ in $j^{th}$ stage of a game is a function $\beta_i^j : [0, \omega] \rightarrow \mathbb{R}_+$; $\forall i \in [0, 1, ..., N]$ which determines her bid for any value. We are interested in a symmetric equilibrium where in
each stage $j$, we are assuming $\beta_i^j(.) > 0$; $\forall i \in [0, 1, ..., N]$ throughout the domain of $\beta_i^j(.)$ i.e.
$\beta_i^j(.)$ is a monotonically increasing function of the valuation of the $i^{th}$ bidder at each stage $j$ of the
game.

If the game has strictly positive reserve price and/or entry fee, not all the buyers from the domain
$[0, \omega]$ will enter into the auction. Rather there exists a valuation $\underline{V}$ such that $\underline{V} > 0$ and buyers
with valuations in the domain $[\underline{V}, \omega]$ will only enter into the auction. If the mechanism has reserve
price and/or entry fee, we will also derive the value of $\underline{V}$ as a function of that reserve price and/or
entry fee.

6.4 Standard Auction Revisited:

In this section we will study standard auctions, where the rule of the auction is that the highest
bidder will always win the auction. This format is very common in theory as well as in practice.
Many familiar auction format falls into this category viz, first price sealed bid auction, second price
sealed bid auction, all pay auction, sad loser auction, Santa Clause auction etc. We will focus on the symmetric increasing and continuous bidding strategies.

We know that in case of standard auction, the idea is first to assume a monotonically increasing bid function for each bidder, so that in our analysis we can safely take the inverse of the bid function (monotonicity guarantees that the inverse of the function is also a function and not a correspondence). After that, we will derive the bid function, and we will derive a sufficient condition that will ensure the existence of an equilibrium bidding strategy which is a monotonically increasing function of the valuation. The assumptions are stated below, which retain all the assumptions of standard auction along with the possibility of positive reserve price and entry fee.

6.4.1 Assumptions:

- There are \( N \) buyers where \( N > 0 \) and a seller.
- The seller has an indivisible object.
- For simplicity we assume that the valuation of the seller is zero.
- Valuation of the \( i^{th} \) buyer is \( V_i \) for all \( i = 1, ..., N \).
- All the buyers' valuation lies in the interval \([0, \omega]\) i.e. \( \forall j = 1, ..., N \ V_j \in [0, \omega] \). The valuations are distributed with a distribution function \( F(.) \) which is increasing. \( F(.) \) admits a continuous density \( f(.) = F'(.) \) and has full support.
- All buyers are risk neutral i.e. they want to maximize their respective expected payoffs.
- Buyers are not subject to any liquidity constraint.
- All components of the model other than the valuations are assumed to be commonly known to all the bidders and the seller.
- This is a single stage game where each bidder has to submit a sealed bid.
- The highest bidder wins the auction and gets the object.
- The payment function of the \( j^{th} \) highest bidder is \( M_j(b) \) for all \( j = 1, ..., N \) where \( b \) is the bid of the \( j^{th} \) highest bidder.
- Assume that there is a reserve price \( R \geq 0 \) and an entry fee \( e \geq 0 \).
CHAPTER 6. STANDARD AUCTION

6.4.2 Equilibrium bidding strategy:

We now derive the symmetric increasing equilibrium bidding strategy of this auction and we will try to find out the sufficient condition of existence of such an equilibrium.

Payoff of buyer(bidder) \( i \) is given below, when the bid is \( b_i \)

\[
\Pi_i = \begin{cases} 
  v_i - M_i(b_i) - e & \text{if } b_i > \max_{j \neq i} b_j \\
  -M_i(b_i) - e & \text{if } b_i < \max_{j \neq i} b_j
\end{cases}
\]  

(6.4.1)

If there is more than one highest bidder then each bidder will get the object with equal probability.

Suppose that bidders \( j \neq i \) follow the symmetric, increasing and differentiable equilibrium strategy \( \beta^j \equiv \beta \). Suppose bidder \( i \) bids \( b_i \). We wish to determine the optimal \( \beta \).

First, notice that it can never be optimal to choose a bid \( b > \beta(\omega) \) since in that case, bidder \( i \) would win for sure and could do better by reducing his bid slightly so that he still wins for sure but pays less. So we only need to consider bids such that \( b \leq \beta(\omega) \). Secondly no bidder will enter into the auction unless his expected payoff from this auction is at-least zero. Let us now introduce one notation that we are going to use frequently in this chapter.

Let us define \( G_i^j(a) = \binom{N-1}{j-1} f(a)^{(N-j)}(1 - F(a))^{(j-1)} \) is the probability that the \( i^{th} \) bidder is the \( j^{th} \) highest bidder if he bids \( a \). Note that

\[
g_i^j(a) = G_i^{ij}(a) = \binom{N-1}{j-1} \left[ (N-j)F(a)^{(N-j-1)}(1 - F(a))^{(j-1)}f(a) - (j-1)F(a)^{(N-j)}(1 - F(a))^{(j-2)}f(a) \right]
\]

Bidder \( i \) wins the auction whenever he submits the highest bid - that is, whenever \( \max_{j \neq i} \beta(V_i) < b_i \). Since at equilibrium \( \beta \) is an increasing function, \( \max_{j \neq i} \beta(V_i) = \beta(\max_{j \neq i} V_i) = \beta(Y_1) \), where \( Y_1 \equiv Y_1^{(N-1)} \), the highest of \( N - 1 \) values. Bidder \( i \) wins whenever \( \beta(Y_1) < b_i \) or equivalently, whenever \( Y_1 < \beta^{-1}(b_i) \). His expected payoff is therefore
As we are only interested in those equilibria where \( b_i \) is an increasing function of \( V_i \), therefore if we can show that \( EP_i(b_i(V_i)) = 0 \) has a unique solution in \( V_i \) then no bidder whose valuation is less than \( V_i \) will participate in this auction. Let us call that valuation \( V_c \), that is a bidder with valuation \( V \) will participate in this auction only if \( V \geq V_c \). We will show that there is indeed a unique solution of the equation \( EP_i(b_i(V_i)) = 0 \). Therefore, we have \( \beta(V_c) = R \), as \( \beta \) is increasing and continuous.

For the time being let us concentrate on the equilibrium bidding strategy of this auction. We are going to derive the first order necessary condition of expected payoff maximization. Maximizing the equation 6.4.1 with respect to \( b_i \) yields the first order condition:

\[
V_i \frac{\partial}{\partial b_i} g_i^1(\beta^{-1}(b_i)) - \sum_{j=1}^{N} \left[ M_j^i(b_i) g_j^1(\beta^{-1}(b_i)) + M_j^{ii}(b_i) G_j^1(\beta^{-1}(b_i)) \right] = 0
\]

\[
\frac{1}{\beta'(\beta^{-1}(b_i))} \left[ g_i^1(\beta^{-1}(b_i)) V_i - \sum_{j=1}^{N} M_j^i(b_i) g_j^1(\beta^{-1}(b_i)) \right] = \sum_{j=1}^{N} \left[ M_j^{ii}(b_i) G_j^1(\beta^{-1}(b_i)) \right]
\]

\[
\left[ g_i^1(\beta^{-1}(b_i)) V_i - \sum_{j=1}^{N} M_j^i(b_i) g_j^1(\beta^{-1}(b_i)) \right] = \beta'(\beta^{-1}(b_i)) \sum_{j=1}^{N} \left[ M_j^{ii}(b_i) G_j^1(\beta^{-1}(b_i)) \right]
\]

\[
g_i^1(\beta^{-1}(b_i)) V_i = \sum_{j=1}^{N} M_j^i(b_i) g_j^1(\beta^{-1}(b_i)) + \beta'(\beta^{-1}(b_i)) \sum_{j=1}^{N} \left[ M_j^{ii}(b_i) G_j^1(\beta^{-1}(b_i)) \right]
\]

We know that at a symmetric equilibrium, \( b_i = \beta(V_i) \) and thus we have

\[
g_i^1(V_i) V_i = \sum_{j=1}^{N} M_j^i(\beta(V_i)) g_j^1(V_i) + \beta'(V_i) \sum_{j=1}^{N} \left[ M_j^{ii}(\beta(V_i)) G_j^1(V_i) \right]
\]

\[
g_i^1(V_i) V_i = \frac{d}{dV_i} \sum_{j=1}^{N} M_j^i(\beta(V_i)) G_j^1(V_i)
\]
and as we stated earlier that $\beta(V_c) = R$, we have

$$
\int_{V_c}^{V_i} g_1^i(t) dt = \int_{V_c}^{V_i} \left[ \sum_{j=1}^{N} M_j^i(\beta(t))G_j^i(t) \right] dt
$$

$$
\int_{V_c}^{V_i} g_1^i(t) dt = \sum_{j=1}^{N} M_j^i(\beta(V_i))G_j^i(V_i) - \sum_{j=1}^{N} M_j^i(\beta(V_c))G_j^i(V_c)
$$

$$
\sum_{j=1}^{N} M_j^i(R)G_j^i(V_c) + \int_{V_c}^{V_i} g_1^i(t) dt = \sum_{j=1}^{N} M_j^i(\beta(V_i))G_j^i(V_i)
$$

$$
V_i G_i^1(V_i) - \int_{V_c}^{V_i} G_1^i(t) dt - \left[ V_i G_i^1(V_c) - \sum_{j=1}^{N} M_j^i(R)G_j^i(V_c) \right] = \sum_{j=1}^{N} M_j^i(\beta(V_i))G_j^i(V_i)
$$

$$
V_i G_i^1(V_i) - \int_{V_c}^{V_i} G_1^i(t) dt - EP_i(V_c) - e = \sum_{j=1}^{N} M_j^i(\beta(V_i))G_j^i(V_i)
$$

$$
V_i G_i^1(V_i) - \int_{V_c}^{V_i} G_1^i(t) dt - e = \sum_{j=1}^{N} M_j^i(\beta(V_i))G_j^i(V_i)
$$

Note that the left side of the above equation is independent of $\beta$. Therefore, we have one equation and one unknown (i.e. $\beta$), so it can be solved. Note that from the above equation it is unclear that whether the derived equilibrium bidding strategy is an increasing function of the valuation or not. Below we derive a sufficient condition for an increasing equilibrium bidding strategy.

Let us differentiate the above equation with respect to $V_i$

$$
g_1^i(V_i)V_i = \sum_{j=1}^{N} M_j^i(\beta(V_i))g_j^i(V_i) + \beta'(V_i) \sum_{j=1}^{N} \left[ M_j^i(\beta(V_i))G_j^i(V_i) \right]
$$

$$
g_1^i(V_i)V_i - \sum_{j=1}^{N} M_j^i(\beta(V_i))g_j^i(V_i) \frac{\sum_{j=1}^{N} \left[ M_j^i(\beta(V_i))G_j^i(V_i) \right]} = \beta'(V_i)
$$

Note that
\[
\sum_{j=1}^{N} M_i^j (\beta(V_i)) g_i^j (V_i) \\
= \sum_{j=1}^{N} M_i^j (\beta(V_i)) \left( \begin{array}{c} N - 1 \\ j - 1 \end{array} \right) [(N - j) F(V_i)^{(N-j-1)}(1 - F(V_i))^{(j-1)} f(V_i)] \\
- \sum_{j=1}^{N} M_i^j (\beta(V_i)) \left( \begin{array}{c} N - 1 \\ j - 1 \end{array} \right) [(j - 1) F(V_i)^{(N-j-1)}(1 - F(V_i))^{(j-2)} f(V_i)] \\
= \sum_{j=1}^{N} M_i^j (\beta(V_i)) \left( \begin{array}{c} N - 1 \\ j - 1 \end{array} \right) [(N - 1) F(V_i)^{(N-j-1)}(1 - F(V_i))^{(j-1)} f(V_i)] \\
- \sum_{j=1}^{N} M_i^j (\beta(V_i)) \left( \begin{array}{c} N - 1 \\ j - 1 \end{array} \right) [(j - 1) F(V_i)^{(N-j-1)}(1 - F(V_i))^{(j-2)} f(V_i)] \\
= \sum_{j=1}^{N} (N - 1) \frac{f(V_i)}{F(V_i)} G_i^j (V_i) M_i^j (\beta(V_i)) - \sum_{j=1}^{N} (j - 1) \frac{f(V_i)}{F(V_i)(1 - F(V_i))} G_i^j (V_i) M_i^j (\beta(V_i))
\]

Therefore we have,

\[
\beta' (V_i) = \frac{g_i^j (V_i) V_i - \sum_{j=1}^{N} M_i^j (\beta(V_i)) g_i^j (V_i)}{\sum_{j=1}^{N} \left[ M_i^j (\beta(V_i)) G_i^j (V_i) \right]} \\
\beta' (V_i) = \frac{g_i^j (V_i) V_i - \sum_{j=1}^{N} (N - 1) \frac{f(V_i)}{F(V_i)} G_i^j (V_i) M_i^j (\beta(V_i)) + \sum_{j=1}^{N} (j - 1) \frac{f(V_i)}{F(V_i)(1 - F(V_i))} G_i^j (V_i) M_i^j (\beta(V_i))}{\sum_{j=1}^{N} \left[ M_i^j (\beta(V_i)) G_i^j (V_i) \right]} \\
\beta' (V_i) = \frac{(N - 1) \frac{f(V_i)}{F(V_i)} \left[ G_i^j (V_i) V_i - \sum_{j=1}^{N} G_i^j (V_i) M_i^j (\beta(V_i)) \right] + \frac{f(V_i)}{F(V_i)(1 - F(V_i))} \sum_{j=1}^{N} (j - 1) G_i^j (V_i) M_i^j (\beta(V_i))}{\sum_{j=1}^{N} \left[ M_i^j (\beta(V_i)) G_i^j (V_i) \right]}
\]

(6.4.3)

Note that,

\[
(N - 1) \frac{f(V_i)}{F(V_i)} \left[ G_i^j (V_i) V_i - \sum_{j=1}^{N} G_i^j (V_i) M_i^j (\beta(V_i)) \right] > 0 \quad \forall V_i > V_c
\]

Remark 6.4.1. Therefore the sufficient condition for the existence of an increasing bidding strategy is that if the denominator is positive then the second summation of the numerator should not be a very large negative number and if the denominator is negative then that summation should be
a sufficiently large negative number. Note that all the commonly known standard auctions (viz. \(N^{th}\) price sealed bid auction, all pay auction, Santa Clause auction, sad loser auction etc satisfies this condition).

The derivation of equilibrium \(\beta\) is only heuristic because the above condition is merely a necessary condition – we have not formally established that if the other \(N - 1\) bidders follow \(\beta\), then it is indeed optimal for a bidder with value \(V_i\) to bid \(\beta(V_i)\). Below we show that this is indeed correct as long as the sufficient condition stated above holds.

Suppose that all but bidder \(i\) follow the strategy \(\beta^i \equiv \beta\) given by the equation (6.4.2). We will argue that in that case it is optimal for bidder \(i\) to follow \(\beta\) also. First, suppose that the above sufficient condition is satisfied, so that we have an increasing \(\beta\) function. Thus in equilibrium the bidder with the highest valuation submits the highest bid and wins the auction. Therefore the above sufficiency condition also ensure efficiency. It is not optimal for the bidder to bid a \(b > \beta(\omega)\). The expected payoff of bidder \(i\) with valuation \(V_i\) if he bids an amount \(b \leq \beta(\omega)\) is calculated as follows. Denote by \(z = \beta^{-1}(b)\) the value for which \(b\) is the equilibrium bid – that is, \(\beta(z) = b\). Then we can write bidder \(i\)'s expected payoff from bidding \(\beta(z)\) when his value is \(V_i\) as follows:

\[
\Pi(b, V_i) = V_i G_i^1(z) - \sum_{j=1}^{N} M_j^i(\beta(z)) G_j^i(z) - e
\]

\[
= V_i G_i^1(z) - \int_{V_c}^{z} g_i(t) dt - \sum_{j=1}^{N} M_j^i(R) G_j^i(V_c) - e
\]

\[
= (V_i - z) G_i^1(z) + \int_{V_c}^{z} G_i^1(t) dt + V_i G_i^1(V_c) - \sum_{j=1}^{N} M_j^i(R) G_j^i(V_c) - e
\]

\[
= (V_i - z) G_i^1(z) + \int_{V_c}^{z} G_i^1(t) dt + \text{EP}_i(V_c)
\]

\[
= (V_i - z) G_i^1(z) + \int_{V_c}^{z} G_i^1(t) dt
\]

We thus obtain that

\[
\Pi(\beta(V_i), V_i) - \Pi(\beta(z), V_i) = \int_{V_c}^{V_i} G_i^1(t) dt - (V_i - z) G_i^1(z) - \int_{V_c}^{z} G_i^1(t) dt
\]

\[
= \int_{z}^{V_i} G_i^1(t) dt - (V_i - z) G_i^1(z) > 0
\]
regardless of whether \( z \geq V_i \) or \( z \leq V_i \).

The preceding argument shows that bidding an amount \( \beta(z') > \beta(z) \) rather than \( \beta(z) \) results in a loss equal to the shaded area to the right in Figure 6.1; similarly, bidding an amount \( \beta(z'') > \beta(z) \) results in a loss equal to the area to the left.

We have thus argued that if all other bidders are following the strategy \( \beta \), a bidder with a valuation \( V_i \) cannot benefit by bidding anything other than \( \beta(V_i) \); and this implies that \( \beta \) is a symmetric equilibrium strategy.

Finally, we will show that equilibrium payoff function is strictly increasing function of valuations so that there is a valuation, \( V_c \geq 0 \) such that \( EP_i(V_c, b_i(V_c)) = 0 \Rightarrow EP_i(V_c) = 0 \) and the valuation is unique because \( \forall V < V_c \ EP_i(b_i(V)) < 0 \) and \( \forall V > V_c \ EP_i(b_i(V)) > 0 \). Therefore, at equilibrium \( \beta(V_c) = R \).

Note that the equilibrium bidding function is given by the equation below

\[
\sum_{j=1}^{N} M_i^j(R)G_i^j(V_c) + \int_{V_c}^{V_i} g_i^j(t)t \, dt = \sum_{j=1}^{N} M_i^j(\beta(V_i))G_i^j(V_i) \tag{6.4.4}
\]
CHAPTER 6. STANDARD AUCTION

Putting this equation in the payoff function we get

\[ EP_i(V_i) = V_i G_1^i(V_i) - \sum_{j=1}^{N} M_j^i(\beta(V_i))G_j^i(V_i) - e \]

\[ = V_i G_1^i(V_i) - \left[ \sum_{j=1}^{N} M_j^i(R)G_j^i(V_c) + \int_{V_c}^{V_i} g_j^i(t) dt \right] - e \]

\[ = V_i G_1^i(V_i) - [V_i G_1^i(V_i) - V_c G_1^i(V_c)] - \left[ \sum_{j=1}^{N} M_j^i(R)G_j^i(V_c) - \int_{V_c}^{V_i} G_j^i(t) dt \right] - e \]

\[ = \int_{V_c}^{V_i} G_1^i(t) dt + V_c G_1^i(V_c) - \sum_{j=1}^{N} M_j^i(R)G_j^i(V_c) - e \]

\[ = \int_{V_c}^{V_i} G_1^i(t) dt + EP_i(V_c) \]

\[ = \int_{V_c}^{V_i} G_1^i(t) dt \]

![Figure 6.2: The equilibrium payoff function](image)

Note that \( EP_i'(V_i) = G_1^i(V_i) > 0 \quad \forall V_i > 0 \), therefore we have proved that the equilibrium payoff function is strictly increasing function of valuations (see Figure 6.2 above).

Thus we have prove the following proposition.
**Proposition 6.4.2.** An increasing bidding strategy constitutes a symmetric Bayesian Nash equilibrium if and only if the bidding strategy, derived from the first order condition of expected payoff maximization, is increasing in valuations.

That is, if in any such auction we can show that the sufficient condition we have derived above, is satisfied then there is no need to check the second order condition. We will finish this section by showing two examples (viz. First Price Sealed Bid Auction and Santa Clause Auction) as a special case of our general setup.

**Example 6.4.3.** First Price Sealed Bid Auction:

In a first price sealed bid auction, each bidder submits a sealed bid of $b_i$, and given these bids, the payoffs are

$$\Pi(V_i, b_i) = \begin{cases} V_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

If there are more than one bidders with the highest bid the object goes to each bidder with equal probability. Bidder $i$ wins the auction whenever he/she submits the highest bid. We also assume $R = e = 0$ i.e. there is no reserve price and entry fee, therefore we have $V_c = 0$.

Note that here $M^1_i(b_i) = b_i$ and $M^j_i(b_i) = 0 \ \forall j \geq 2$. Therefore, we have,

$$\sum_{j=1}^{N} M^j_i(R)G^j_i(V_c) + \int_{V_c}^{V_i} g^j_i(t)tdt = \sum_{j=1}^{N} M^j_i(\beta(V_i))G^j_i(V_i)$$

$$\int_{0}^{V_i} g^1_i(t)tdt = b_i G^1_i(V_i)$$

$$b_i = \beta(V_i) = \frac{1}{G^1_i(V_i)} \int_{0}^{V_i} g^1_i(t)tdt$$

Note that $\beta(V_i)$ is an increasing function of $V_i$,

$$\beta'(V_i) = \frac{g^1_i(V_i)V_i}{G^1_i(V_i)^2} - \frac{g^1_i(V_i)}{[G^1_i(V_i)]^2} \int_{0}^{V_i} g^1_i(t)tdt$$

$$\beta'(V_i) = \frac{g^1_i(V_i)}{[G^1_i(V_i)]^2} \left[ V_i G^1_i(V_i) - \int_{0}^{V_i} g^1_i(t)tdt \right]$$
CHAPTER 6. STANDARD AUCTION

Let us concentrate on the function

$$\Omega(V_i) = \left[ V_i G_i^1(V_i) - \int_0^{V_i} g_i^1(t)dt \right]$$

Our objective is to minimize this function with respect to $V_i$. The first order condition gives

$$\Omega'(V_i) = G_i^1(V_i) + V_i g_i^1(V_i) - V_i g_i^1(V_i) = 0$$

OR, $G_i^1(V_i) = 0$

OR, $V_i = 0$

Second order condition is satisfied because

$$\Omega''(V_i) = g_i^1(V_i)|_{V_i=0} > 0$$

Therefore, the above function is minimized at $V_i = 0$ and the minimum value of the function is 0. Therefore for any $V_i > 0$ we have $\beta'(V_i) > 0$. As we stated earlier if we can find an increasing symmetric bidding strategy from the equation 6.4.3 then we have no need to check the second order condition. Therefore, the symmetric increasing equilibrium bidding strategy in first price sealed bid auction is given by $\beta(V_i) = \frac{1}{G_i(V_i)} \int_0^{V_i} g_i^1(t)dt$.

Example 6.4.4. Santa Clause Auction²:

In a Santa Clause auction, each bidder submits a sealed bid of $b_i$, and given these bids, the payoffs are

$$\Pi(V_i, b_i) = \begin{cases} 
V_i - b_i + \int_0^{b_i} G_i^1(t)dt & \text{if } b_i > \max_{j \neq i} b_j \\
\int_0^{b_i} G_i^1(t)dt & \text{if } b_i < \max_{j \neq i} b_j 
\end{cases}$$

If there are more than one bidders with the highest bid the object goes to each bidder with equal probability. Bidder $i$ wins the auction whenever he/she submits the highest bid. We also assume $R = e = 0$ i.e. there is no reserve price and entry fee; therefore we have $V_e = 0$.

²Riley and Samuelson, 1981 American Economic Review
CHAPTER 6. STANDARD AUCTION

Note that here $M_l^j(b_i) = b_i - \int_0^{b_i} G_i^j(t)\,dt$ and $M_l^j(b_i) = -\int_0^{b_i} G_i^j(t)\,dt$ $\forall j \geq 2$. Therefore we have

$$\sum_{j=1}^N M_l^j(R)G_i^j(V_c) + \int_{V_c}^{V_i} g_l^j(t)\,dt = \sum_{j=1}^N M_l^j(\beta(V_i))G_i^j(V_i)$$

$$\int_0^{V_i} g_l^j(t)\,dt = b_iG_i^j(V_i) - \int_0^{b_i} G_i^j(t)\,dt$$

$$V_iG_i^j(V_i) - \int_0^{V_i} G_i^j(t)\,dt = b_iG_i^j(V_i) - \int_0^{b_i} G_i^j(t)\,dt$$

$$(V_i - b_i)G_i^j(V_i) - \int_{b_i}^{V_i} G_i^j(t)\,dt = 0$$

$b_i = V_i$

Note that $\beta(V_i)$ is an increasing function of $V_i$, in particular $\beta'(V_i) = 1$. Again we don't have to check the second order condition. Therefore, the symmetric increasing bidding strategy in Santa Clause auction is given by $\beta(V_i) = V_i$.

6.5 Conclusion

In this short chapter we have studied standard auctions in a very general format. A auction will fall under the category of standard auction if the rule of the auction dictates that highest bidder wins the auction. Here we found out the necessary and sufficient condition for existence of a symmetric and increasing Nash equilibrium. Our result shows that if the bidding strategy generated from the first order condition of the expected payoff maximization problem of a bidder, is strictly increasing throughout the domain of the valuations, then that bidding strategy constitutes a symmetric and increasing Nash equilibrium of that standard auction. Finally, we illustrate our result by showing two examples.