CHAPTER 6
CERTAIN HIGHER ORDER QUASILINEAR DELAY
DIFFERENCE EQUATIONS

In this chapter\(^1\), we discuss about the oscillatory behavior of the certain higher order quasilinear delay difference equation. An example is given to illustrate the main results.

6.1 INTRODUCTION

We consider the following higher order quasilinear delay difference equation of the form

\[
\Delta \left( a(k) \left( \Delta^{(n-1)} x(k) \right)^{\alpha} \right) + p(k) x^{\beta} \left( \tau(k) \right) = 0, \quad k \geq k_0, \quad (6.1.1)
\]

where

\[
\sum_{s=n_0}^{\infty} \frac{1}{a^\frac{1}{\alpha}(s)} < \infty. \quad (6.1.2)
\]

By a solution of equation (6.1.1), we mean a real sequence \( \{x(k)\} \) which is defined for \( k \geq \min_{i \geq 0} \{\tau(i)\} \) and satisfies equation (6.1.1) for sufficiently large \( k \).

The purpose of this chapter is to derive some oscillation criteria for

\(^1\)The content of this chapter has been published in *International Journal of Mathematical Analysis*, 9(18), 2015, 907-915.
equation (6.1.1). Throughout this chapter, the following conditions are assumed to hold:

(i) $\alpha, \beta$ are the ratios of odd positive integers and $\beta \leq \alpha$;

(ii) $\{a(k)\}$ and $\{p(k)\}$ are positive real sequences for $n \geq n_0$;

(iii) $\{\tau(k)\}$ is a positive real-valued sequence such that $\tau(k) < k$ and $\lim_{k \to \infty} \tau(k) = \infty$.

6.2 PRELIMINARY LEMMAS

We need the following lemmas for our results.

Lemma 6.2.1. [see (Agarwal & Wong 1997)] Let $y(k)$ be defined for $k \geq k_0 \in \mathbb{N}$ and $y(k) > 0$ with $\Delta^n y(k)$ of constant sign for $k \geq k_0$ and not identically zero. Then there exists an integer $m$, $0 \leq m \leq n$ with $(n + m)$ odd for $\Delta^n y(k) \leq 0$ and $(n + m)$ even for $\Delta^n y(k) \geq 0$ such that

(i) $m \leq n - 1$ implies $(-1)^{m+i} \Delta^{i} y(k) > 0$, for all $k \geq k_0$, $m \leq i \leq n - 1$;

(ii) $m \geq 1$ implies $\Delta^{i} y(k) > 0$, for all large $k \geq k_0$, $1 \leq i \leq m - 1$.

Lemma 6.2.2. [see (Agarwal & Wong 1997)] Let $y(k)$ be defined for $k \geq k_0$ and $y(k) > 0$ with $\Delta^n y(k) \leq 0$, for $k \geq k_0$ and not identically zero. Then there exists a large integer $k_1 \geq k_0$ such that

$$y(k) \geq \frac{1}{(n-1)!} (k - k_1)^{n-1} \Delta^{n-1} y\left(2^n - m - 1, k\right), k \geq k_1,$$

where $m$ is defined as in Lemma 6.2.1. Further, if $y(k)$ is increasing, then

$$y(k) \geq \frac{1}{(n-1)!} \left(\frac{k}{2^{n-1}}\right)^{n-1} \Delta^{n-1} y(k), k \geq 2^{n-1} k_1.$$
6.3 OSCILLATION RESULTS

In this section, we study sufficient conditions for oscillation of solution of equation (6.1.1).

Theorem 6.3.1. Let $n$ be even. Assume that (6.1.2) holds and every solution of the first order delay difference equation

$$\Delta y(k) + p(k) \left[ \frac{1}{(n-1)!} \left( \frac{\tau(k)}{2^{n-1}} \right)^{n-1} \frac{1}{a^\frac{1}{\alpha}(\tau(k))} \right]^\beta y^\beta(\tau(k)) = 0, \quad (6.3.1a)$$

$k \geq k_0$ is oscillatory. Further, if

$$\limsup_{k \to \infty} \sum_{s=k_1}^{k-1} \left[ M^{\beta-\alpha} p(s) \left( \frac{1}{(n-2)!} \left( \frac{4k}{2^n} \right)^{n-2} \right)^\beta \delta^\alpha(s) - \frac{\alpha^{\alpha+1}}{(\alpha + 1)^{\alpha+1}} \frac{1}{\delta(s) a^\frac{1}{\beta}(s)} \right] = \infty, \quad (6.3.1b)$$

for every constant $M > 0$, then every solution of equation (6.1.1) is either oscillatory or tends to zero as $k \to \infty$.

Proof. Assume the contradiction that equation (6.1.1) has a nonoscillatory solution $\{x(k)\}$ and also assume that $\{x(k)\}$ is eventually positive. Furthermore, suppose that $\lim_{k \to \infty} x(k) \neq 0$. Then it follows from equation (6.1.1) that there are two possible cases:

(i) $x(k) > 0$, $\Delta^{(n-1)}x(k) > 0$, $\Delta^{(n)}x(k) < 0$ and
$$\Delta \left[ a(k) \left( \Delta^{(n-1)}x(k) \right)^\alpha \right] < 0;$$

(ii) $x(k) > 0$, $\Delta^{(n-2)}x(k) > 0$, $\Delta^{(n-1)}x(k) < 0$ and
$$\Delta \left[ a(k) \left( \Delta^{(n-1)}x(k) \right)^\alpha \right] < 0, \text{ for } k \geq k_1.$$

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case(i):
In this case, we have \( x(k) > 0 \) and \( \Delta^{(n)}x(k) \) is of constant sign and not identically zero for \( k \geq k_1 \).

According to Lemma 6.2.1, there exists a \( k_2 \geq k_1 \) such that

\[
(-1)^{m+i} \Delta^i x(k) > 0 \quad (i = 1, 2, 3, \ldots, n-1), \quad \text{for } k \geq k_2.
\]

In particular, since \( \Delta x(k) > 0 \), for \( k \geq k_2 \), we have \( x(k) \) is increasing.

Since \( x(k) \) is increasing, by the instructions in the second part of Lemma 6.2.2, there exists a \( k_3 \geq k_2 \) such that

\[
x(k) \geq \frac{1}{(n-1)!} \left( \frac{k}{2^{n-1}} \right)^{n-1} \Delta^{n-1} x(k), \quad \text{for } k \geq k_3.
\]

That is,

\[
x(k) \geq \frac{1}{(n-1)!} \left( \frac{k}{2^{n-1}} \right)^{n-1} \left( \frac{1}{a^{\frac{1}{a}}(k)} \right) \left( a^{\frac{1}{a}}(k) \Delta^{n-1} x(k) \right), \quad \text{for } k \geq k_3.
\]

There exists a \( k_4 \geq k_3 \) such that

\[
x(\tau(k)) \geq \frac{1}{(n-1)!} \left( \frac{\tau(k)}{2^{n-1}} \right)^{n-1} \left( \frac{1}{a^{\frac{1}{a}}(\tau(k))} \right) \left( a^{\frac{1}{a}}(\tau(k)) \Delta^{n-1} x(\tau(k)) \right),
\]

(6.3.2)

for \( k \geq k_4 \).

Let

\[
y(k) = a(k) \left( \Delta^{(n-1)} x(k) \right)^\alpha > 0.
\]

(6.3.3)

Consider the difference inequality of the equation (6.1.1)

\[
\Delta \left( a(k) \left( \Delta^{(n-1)} x(k) \right)^\alpha \right) + p(k) x^\beta (\tau(k)) \leq 0.
\]

(6.3.4)
Using (6.3.3) in the inequality (6.3.4), we have

\[ \Delta y(k) + p(k)x^\beta(\tau(k)) \leq 0. \]

This implies,

\[ \Delta y(k) \leq -p(k)x^\beta(\tau(k)). \]

Using (6.3.2) in the above inequality, we have

\[ \Delta y(k) \leq -p(k)x^\beta(\tau(k)) \left[ \frac{1}{(n-1)!} \left( \frac{\tau(k)}{2^n} \right)^{n-1} \frac{1}{a_1^\alpha(\tau(k))} \right]^\beta y_+^\beta(\tau(k)). \]

That is,

\[ \Delta y(k) + p(k) \left[ \frac{1}{(n-1)!} \left( \frac{\tau(k)}{2^n} \right)^{n-1} \frac{1}{a_1^\alpha(\tau(k))} \right]^\beta y_+^\beta(\tau(k)) \leq 0, \quad (6.3.5) \]

for \( k \geq k_4 \).

By (6.3.3), we can see that the inequality (6.3.5) has an eventually positive solution \( y(k) \). Then by the result in (Grzegorczyk \\& Werbowski 2001; Ladas et al. 1989), the difference equation (6.3.1a) also has an eventually positive solution, which is a contradiction to the fact that equation (6.3.1a) is oscillatory.

**case(ii):**

Define the function \( v(k) \) by

\[ v(k) = a(k) \left( \Delta^{(n-1)}x(k) \right)^\alpha \left( \Delta^{(n-2)}x(k) \right)^\alpha, \quad \text{for } k \geq k_1. \quad (6.3.6) \]

Then

\[ v(k) < 0, \quad \text{for } k \geq k_1. \]
Since \( a(k) \left( \Delta^{(n-1)}x(k) \right)^{\alpha} \) is decreasing, we have

\[
a(s) \left( \Delta^{(n-1)}x(s) \right)^{\alpha} \leq a(k) \left( \Delta^{(n-1)}x(k) \right)^{\alpha}, \quad \text{for } s \geq k \geq k_1.
\]

That is,

\[
a^{\frac{1}{\alpha}}(s) \Delta^{(n-1)}x(s) \leq a^{\frac{1}{\alpha}}(k) \Delta^{(n-1)}x(k), \quad \text{for } s \geq k \geq k_1.
\]

Dividing the above inequality by \( a^{\frac{1}{\alpha}}(s) \), we obtain

\[
\Delta^{(n-1)}x(s) \leq a^{\frac{1}{\alpha}}(k) \Delta^{(n-1)}x(k) \frac{1}{a^{\frac{1}{\alpha}}(s)}.
\]

Summing the above inequality from \( k \) to \( u-1 \), we get

\[
\Delta^{(n-2)}x(u) \leq \Delta^{(n-2)}x(k) + a^{\frac{1}{\alpha}}(k) \Delta^{(n-1)}x(k) \sum_{s=k}^{u-1} \frac{1}{a^{\frac{1}{\alpha}}(s)}.
\]

Letting \( u \to \infty \) in the above inequality, we have

\[
0 \leq \Delta^{(n-2)}x(k) + a^{\frac{1}{\alpha}}(k) \Delta^{(n-1)}x(k) \delta(k),
\]

where

\[
\delta(k) = \sum_{s=k}^{\infty} \frac{1}{a^{\frac{1}{\alpha}}(s)},
\]

which implies,

\[
- \frac{a^{\frac{1}{\alpha}}(k) \Delta^{(n-1)}x(k)}{\Delta^{(n-2)}x(k)} \delta(k) \leq 1, \quad \text{for } k \geq k_1.
\]

Using (6.3.6) in the above inequality, we have

\[
v(k) \delta^{\alpha}(k) \leq 1. \tag{6.3.7}
\]
From (6.3.6), we can get
\[ \Delta v(k) = \frac{\Delta \left( a(k) \Delta^{(n-1)} x(k) \right)^\alpha}{(\Delta^{(n-2)} x(k))^\alpha} - \alpha \frac{a(k) \Delta^{(n-1)} x(k)}{(\Delta^{(n-2)} x(k))^\alpha+1} \cdot \]

Using (6.1.1) and (6.3.6) in the above equation, we obtain
\[ \Delta v(k) = -\frac{p(k) x^\beta \tau(k)}{(\Delta^{(n-2)} x(k))^\alpha} - \alpha \frac{v^{\alpha+1}(k)}{a^\alpha(k)} . \]

By Lemma 6.2.2, we get
\[ x(k) \geq \frac{1}{(n-2)!} \left( \frac{k}{2^{n-2}} \right)^{n-2} \Delta^{n-2} x(k) = \frac{1}{(n-2)!} \left( \frac{4k}{2^n} \right)^{n-2} \Delta^{n-2} x(k), \]
for \( k \geq k_1 \). Then, we can find a constant \( M > 0 \) such that
\[ \Delta v(k) \leq -M^{\beta-\alpha} p(k) \left[ \frac{1}{(n-2)!} \left( \frac{4k}{2^n} \right)^{n-2} \right]^\beta - \alpha \frac{v^{\alpha+1}(k)}{a^\alpha(k)} . \]

Multiplying the above inequality by \( \delta^{\alpha}(k) \) and summing from \( k_1 \) to \( k-1 \), we get
\[ \delta^{\alpha}(k) v(k) - \delta^{\alpha}(k_1) v(k_1) + \sum_{s=k_1}^{k-1} M^{\beta-\alpha} p(s) \left[ \frac{1}{(n-2)!} \left( \frac{4k}{2^n} \right)^{n-2} \right]^\beta \delta^{\alpha}(s) \]
\[ + \sum_{s=k_1}^{k-1} \alpha \frac{v^{\alpha+1}(s)}{a^\alpha(s)} \delta^{\alpha}(s) + \alpha \sum_{s=k_1}^{k-1} \delta^{\alpha-1}(s) v(s) \leq 0. \]

Set \( A = a^{-\frac{1}{\alpha}}(s) \delta^{\alpha-1}(s), B = \frac{\delta^{\alpha}(s)}{a^{\alpha}(s)}, u = -v(s) \).
Then we have the inequality,
\[ Au - Bu \leq \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha + 1}} A^{\alpha + 1} B^\alpha, \quad B > 0. \] (6.3.9)

Using the inequalities (6.3.7) and (6.3.9) in the inequality (6.3.8), we have
\[
\sum_{s=k_1}^{k-1} \left( M^{\beta - \alpha} p(s) \left[ \frac{1}{(n-2)!} \left( \frac{4k^2}{2^n} \right)^{n-2} \right]^\beta \delta^\alpha(s) \right) - \sum_{s=k_1}^{k-1} \left( \frac{\alpha^{\alpha + 1}}{(\alpha + 1)^{\alpha + 1}} \frac{1}{\delta(s) a^\frac{1}{\alpha}(s)} \right) \leq \delta^\alpha(k_1) v(k_1) + 1,
\]
which contradicts (6.3.1b). This completes the proof of the theorem. \(\square\)

**Corollary 6.3.2.** Let \( n \) be even. Assume that (6.1.2) holds. Moreover assume that \( \alpha = \beta \) and \( 2 < e < 3 \). If
\[
\liminf_{k \to \infty} \sum_{s=\tau(k)}^{k-1} \left[ \frac{(\tau(s))^{n-1}}{a(\tau(s))} \right]^\alpha p(s) > \frac{(n-1)!^\alpha}{e}
\]
holds for \( k \geq k_0 \) and
\[
\limsup_{k \to \infty} \sum_{s=k_1}^{k-1} \left[ p(s) \left[ \frac{1}{(n-2)!} \left( \frac{4k^2}{2^n} \right)^{n-2} \right]^\beta \delta^\alpha(s) - \frac{\alpha^{\alpha + 1}}{(\alpha + 1)^{\alpha + 1}} \frac{1}{\delta(s) a^\frac{1}{\alpha}(s)} \right] = \infty,
\]
then every solution of equation (6.1.1) is oscillatory or tends to zero.

**Corollary 6.3.3.** Let \( n \) be even. Assume that (6.1.2) holds. Moreover assume that \( \alpha > \beta \), \( \tau \) is strictly increasing function and
\[
\limsup_{k \to \infty} \sum_{s=\tau(k)}^{k} \left[ \left( \frac{\tau(s)}{n-1} \right)^{\beta} \right] p(s) > 0
\]
holds for \( k \geq k_0 \). If (6.3.1b) holds for all constants \( M > 0 \), then every solution
of equation (6.1.1) is oscillatory or tends to zero.

The following example justifies the results obtained in section 6.3.

6.4 EXAMPLE

Example 6.4.1. Consider the second order difference equation

\[ \Delta \left( k^6 [\Delta x(k)]^3 \right) + 8 \left[ (k+1)^6 + k^6 \right] x^3(k-1) = 0. \]  
(E.6.1)

In the above equation, \( a(k) = k^6 \), \( p(k) = 8 \left[ (k+1)^6 + k^6 \right] \) which are positive and \( \tau(k) = k - 1 \).

Also,

\[ \sum_{s=n_0}^{\infty} \frac{1}{a_1(s)} = \sum_{s=n_0}^{\infty} \frac{1}{s^2} < \infty. \]

For sufficiently large \( k \), every solution of the first order difference equation

\[ \Delta y(k) + 8 \left[ (k+1)^6 + k^6 \right] \left[ \frac{1}{(n-1)!} \left( \frac{k-1}{2^{n-1}} \right) \right] \frac{1}{(k-1)^2} y(k-1) = 0 \]

is oscillatory, because

\[ \lim_{k \to \infty} \inf \sum_{s=\tau(k)}^{k-1} \left[ \frac{(s-1)^{n-1}}{s-1} \right] 8 \left[ (k+1)^6 + k^6 \right] > \frac{1}{e}, \]

for \( k \geq k_0 + 1 \). Thus all the conditions of Theorem 6.3.1 are satisfied and hence every solution of equation (E.6.1) is oscillatory. Its one of the solutions is \( x(k) = (-1)^k \). □