CHAPTER 5

OSCILLATION THEOREMS FOR CERTAIN FOURTH ORDER NONLINEAR DIFFERENCE EQUATIONS

In this chapter, some sufficient conditions for the oscillation of all solutions of certain fourth order nonlinear difference equation are obtained. Examples are given to illustrate the results.

5.1 INTRODUCTION

We have considered the oscillatory behavior of all the solutions of the nonlinear difference equation of the form

\[ \Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right) + q_n f(y_{n-\tau_n}) = 0, \quad n = 0, 1, 2, \ldots \quad (5.1.1) \]

and the following conditions are assumed to hold:

(H1) \( \{q_n\} \) is a sequence of real numbers and \( \{p_n\} \) is a sequence of positive numbers;

(H2) \( \{\tau_n\} \) is a sequence of integers such that \( \lim_{n \to \infty} (n - \tau_n) = \infty \);

(H3) \( R_n = \sum_{k=0}^{n-1} p_k \to \infty \) as \( n \to \infty \);

\(^1\)The content of this chapter has been published in *International Journal of Mathematics Research*, 5(3), 2013, 299-312.
(H4) \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous with \( yf(y) > 0 \), for \( y \neq 0 \).

By a solution of equation (5.1.1), we mean a real sequence \( \{y_n\} \) which is defined for \( n \geq \min_{i \geq 0} (i - \tau_i) \) and satisfies equation (5.1.1) for all large \( n \).

5.2 OSCILLATION RESULTS

In this section, we are interested to present some sufficient conditions for the oscillation of all solutions of the equation (5.1.1).

**Theorem 5.2.1.** Suppose that
i) \( q_n \geq 0 \) and \( \sum_{k=0}^{\infty} q_k = \infty \);
ii) \( \lim_{n \to \infty} \inf yf(y) > 0 \).

Then every solution of equation (5.1.1) is oscillatory.

**Proof.** Suppose on the contrary that \( \{y_n\} \) is nonoscillatory solution of equation (5.1.1). We assume that \( \{y_n\} \) is eventually positive. Then there exists a positive integer \( n_0 \) such that

\[
y_{n - \tau_n} > 0, \text{ for } n \geq n_0. \tag{5.2.1}
\]

From equation (5.1.1), we have

\[
\Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right) = -q_n f(y_{n - \tau_n}) \leq 0, n \geq n_0.
\]

Then \( \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \) is an eventually non-increasing sequence.

We claim \( \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \geq 0, \text{ for } n \geq n_0. \)
We assume the contradiction that there is an $n_1 \geq n_0$ such that

$$\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) < 0, \text{ for } n \geq n_1.$$ 

That is,

$$\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) < -k_1, k_1 > 0, n_2 \geq n_1.$$ 

Summing the last inequality from $n_2$ to $(n - 1)$, we have

$$\sum_{s=n_2}^{n-1} \Delta \left( \frac{1}{p_s} \Delta^2 y_s \right) \leq \sum_{s=n_2}^{n-1} (-k_1).$$ 

This implies

$$\frac{1}{p_n} \Delta^2 y_n - \frac{1}{p_{n_2}} \Delta^2 y_{n_2} \leq (-k_1) (n - n_2).$$ 

That is,

$$\frac{1}{p_n} \Delta^2 y_n \leq \frac{1}{p_{n_2}} \Delta^2 y_{n_2} + (-k_1) (n - n_2).$$ 

Therefore,

$$\frac{1}{p_n} \Delta^2 y_n \rightarrow -\infty \text{ as } n \rightarrow \infty.$$ 

Therefore, there exists an integer $n_3 \geq n_2$ such that

$$\frac{1}{p_n} \Delta^2 y_n < -k_2, k_2 > 0, \text{ for } n \geq n_3.$$ 

Summing the last inequality from $n_3$ to $(n - 1)$, we obtain

$$\Delta y_n - \Delta y_{n_3} \leq (-k_2) \sum_{s=n_3}^{n-1} p_s,$$ 

which implies

$$\Delta y_n \leq \Delta y_{n_3} - k_2 \sum_{s=n_3}^{n-1} p_s.$$ 

That is

$$\Delta y_n \rightarrow -\infty \text{ as } n \rightarrow \infty.$$
Hence there is an integer \( n_4 \geq n_3 \) such that

\[
\Delta y_n < -k_3, \quad k_3 > 0, \quad \text{for } n \geq n_4.
\]

Summing the last inequality from \( n_4 \) to \((n - 1)\), we have

\[
y_n - y_{n_4} \leq \sum_{s=n_4}^{n-1} (-k_3).
\]

This implies

\[
y_n \leq y_{n_4} - k_3(n - n_4).
\]

Therefore

\[
y_n \to -\infty \text{ as } n \to \infty,
\]

which is a contradiction to the fact that \( y_n > 0 \), for \( n \geq n_1 \).

Hence

\[
\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \geq 0, \quad \text{for } n \geq n_0.
\]

Therefore we have,

\[
y_{n-\tau_n} > 0, \Delta y_n \geq 0, \quad \frac{1}{p_n} \Delta^2 y_n \geq 0, \quad \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \geq 0 \quad \text{and} \quad \Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right) \leq 0, \quad \text{for } n \geq n_0.
\]

Let

\[
L = \lim_{n \to \infty} y_n.
\]

Then \( L > 0 \) is finite or infinite.

**Case(i): \( L > 0 \) is finite.**

Since \( f \) is a continuous function, we have

\[
\lim_{n \to \infty} f(y_{n-\tau_n}) = f(L) > 0.
\]
Then there exists a positive integer $n_5 \geq n_0$ such that

$$f(y_n - \tau_n) \geq \frac{1}{2} f(L), \text{ for } n \geq n_5. \quad (5.2.2)$$

By substituting inequality (5.2.2) in equation (5.1.1), we obtain

$$\Delta^2 \left(\frac{1}{p_n} \Delta^2 y_n\right) + \frac{1}{2} f(L) q_n \leq 0, \text{ for } n \geq n_5. \quad (5.2.3)$$

Summing the last inequality from $n_5$ to $n$, we have

$$\Delta \left(\frac{1}{p_{n+1}} \Delta^2 y_{n+1}\right) - \Delta \left(\frac{1}{p_{n_5}} \Delta^2 y_{n_5}\right) + \frac{1}{2} f(L) \sum_{s=n_5}^{n} q_s \leq 0,$$

which implies

$$\frac{1}{2} f(L) \sum_{s=n_5}^{n} q_s \leq \Delta \left(\frac{1}{p_{n_5}} \Delta^2 y_{n_5}\right), n \geq n_5,$$

which contradicts (i).

**Case (2):** $L = \infty$

From the condition (ii), we have

$$\liminf_{n \to \infty} f(y_n - \tau_n) > 0.$$

So we may choose a positive constant $c$ and a positive integer $n_6$ such that

$$f(y_n - \tau_n) \geq c, n \geq n_6. \quad (5.2.4)$$

Substituting inequality (5.2.4) in equation (5.1.1), we have

$$\Delta^2 \left(\frac{1}{p_n} \Delta^2 y_n\right) + cq_n \leq 0, \text{ for } n \geq n_6.$$


Summing the last inequality from \( n_6 \) to \( n \), we get

\[
c \sum_{s=n_6}^{n} q_s \leq \Delta \left( \frac{1}{p_{n_6}} \Delta^2 y_{n_6} \right), \quad n \geq n_6,
\]

which contradicts (i).

This completes the proof. \( \square \)

**Theorem 5.2.2.** Assume that \( q_n \geq 0 \) and \( \sum_{k=0}^{\infty} \frac{(k-N)}{R_k} q_k = \infty \).

Then every bounded solution of equation (5.1.1) is oscillatory.

**Proof.** Proceeding as in the proof of Theorem 5.2.1 with assumption that equation (5.1.1) has a nonoscillatory bounded solution \( \{y_n\} \), we get the inequality (5.2.3),

\[
\Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right) + \frac{1}{2} f(L) q_n \leq 0, \text{ for } n \geq n_5.
\]

Dividing the last inequality by \( R_n \), we obtain

\[
\frac{1}{R_n} \Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right) + \frac{1}{2R_n} f(L) q_n \leq 0. \tag{5.2.5}
\]

Consider

\[
\sum_{s=N}^{n-1} \Delta^2 \left( \frac{1}{p_s} \Delta^2 y_s \right) = \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) - \Delta \left( \frac{1}{p_N} \Delta^2 y_N \right),
\]

which implies

\[
(n-N) \Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right) = \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) - \Delta \left( \frac{1}{p_N} \Delta^2 y_N \right).
\]

That is,

\[
(n-N) \Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right) \leq \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right).
\]

Then inequality (5.2.5) becomes,
\[
\frac{\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right)}{R_n (n - N)} + \frac{1}{2R_n} f(L) q_n \leq 0.
\]
That is,
\[
\frac{1}{R_n} \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) + \frac{n - N}{2R_n} f(L) q_n \leq 0. \tag{5.2.6}
\]
We know that
\[
\Delta \left( \frac{1}{R_n} \frac{1}{p_n} \Delta^2 y_n \right) = \frac{1}{R_n+1} \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) + \frac{1}{p_n} \Delta^2 y_n \Delta \frac{1}{R_n}
\]
\[
\leq \frac{1}{R_n} \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) + \frac{1}{p_n} \Delta^2 y_n \Delta \frac{1}{R_n},
\]
which implies
\[
\frac{1}{R_n} \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \geq \Delta \left( \frac{1}{R_n} \frac{1}{p_n} \Delta^2 y_n \right) - \frac{1}{p_n} \Delta^2 y_n \Delta \frac{1}{R_n}. \tag{5.2.7}
\]
Substituting (5.2.7) in (5.2.6), we have
\[
\frac{n - N}{2R_n} f(L) q_n \leq -\frac{1}{R_n} \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right)
\]
\[
\leq -\Delta \left( \frac{1}{R_n} \frac{1}{p_n} \Delta^2 y_n \right) + \frac{1}{p_n} \Delta^2 y_n \Delta \frac{1}{R_n}
\]
\[
\leq -\Delta \left( \frac{1}{R_n} \frac{1}{p_n} \Delta^2 y_n \right) + \Delta^2 y_n \left( -\frac{1}{R^2_{n+1}} \right)
\]
\[
\leq -\Delta \left( \frac{1}{R_n} \frac{1}{p_n} \Delta^2 y_n \right) + \Delta^2 y_n.
\]
That is,
\[
\Delta \left( \frac{1}{R_n} \frac{1}{p_n} \Delta^2 y_n \right) - \Delta^2 y_n + \frac{n - N}{2R_n} f(L) q_n \leq 0.
\]
Summing the last inequality from \(n_5\) to \(n\), we get
\[
\sum_{s=n_5}^{n} \Delta \left( \frac{1}{R_s} \frac{1}{p_s} \Delta^2 y_s \right) - \sum_{s=n_5}^{n} \Delta^2 y_s + \frac{1}{2} f(L) \sum_{s=n_5}^{n} \frac{s - N}{R_s} q_s \leq 0.
\]
This implies
\[ \frac{1}{R_{n+1}} \frac{1}{p_{n+1}} \Delta^2 y_{n+1} - \frac{1}{R_{n_5}} \frac{1}{p_{n_5}} \Delta^2 y_{n_5} - \Delta y_{n+1} + \Delta y_{n_5} + \frac{1}{2} f(L) \sum_{s=n_5}^{n} \frac{s-N}{R_s} q_s \leq 0. \]

That is,
\[ \frac{1}{2} f(L) \sum_{s=n_5}^{n} \frac{s-N}{R_s} q_s \leq \frac{1}{R_{n_5}} \frac{1}{p_{n_5}} \Delta^2 y_{n_5} + \Delta y_{n+1} - \Delta y_{n_5} \leq \frac{1}{R_{n_5}} \frac{1}{p_{n_5}} \Delta^2 y_{n_5} + y_{n+2} - y_{n+1} - \Delta y_{n_5}. \]

Since \( \{y_n\} \) is bounded, we may choose a positive constant \( c \) such that
\[ \sum_{s=n_5}^{n} \frac{s-N}{R_s} q_s \leq c, \text{ for } n \geq n_5, \]
which is a contradiction to the assumption of the theorem.

\[ \square \]

**Theorem 5.2.3.** Assume that \( n - \tau_n \) is non-decreasing, where \( \tau_n \in \{0, 1, 2, \ldots\} \) and
\[ \sum_{n=0}^{\infty} q_n = \infty. \] (5.2.8)

Then the difference \( \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \) of every solution \( \{y_n\} \) of equation (5.1.1) oscillates.

**Proof.** Assume the contradiction that equation (5.1.1) has a solution \( \{y_n\} \) such that its difference \( \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \) is nonoscillatory. Assume first that the sequence \( \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \) is eventually negative. Then we may choose a positive integer \( n_0 \) such that \( \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) < 0. \)
Set

\[ v_n = \frac{\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right)}{f(y_n - \tau_n)}, \text{ for } n \geq n_1 \geq n_0. \]  
(5.2.9)

Then

\[ \Delta v_n = \frac{f(y_n - \tau_n) \Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right) - \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \Delta f(y_n - \tau_n)}{f(y_n - \tau_n) f(y_{n+1} - \tau_{n+1})}. \]  
(5.2.10)

Now,

\[ \Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right) = \Delta \left( \frac{1}{p_{n+1}} \Delta^2 y_{n+1} \right) - \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \]
\[ = \frac{1}{p_{n+2}} \Delta^2 y_{n+2} - \frac{1}{p_{n+1}} \Delta^2 y_{n+1} - \frac{1}{p_{n+1}} \Delta^2 y_{n+1} + \frac{1}{p_n} \Delta^2 y_n \]
\[ = \frac{1}{p_{n+2}} \Delta^2 y_{n+2} - \frac{2}{p_{n+1}} \Delta^2 y_{n+1} + \frac{1}{p_n} \Delta^2 y_n \]
and

\[ \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \Delta f(y_n - \tau_n) = \left( \frac{1}{p_{n+1}} \Delta^2 y_{n+1} - \frac{1}{p_n} \Delta^2 y_n \right) \left( f(y_{n+1} - \tau_{n+1}) - f(y_n - \tau_n) \right) \]
\[ = \frac{1}{p_{n+1}} \Delta^2 y_{n+1} f(y_{n+1} - \tau_{n+1}) - \frac{1}{p_n} \Delta^2 y_n f(y_{n+1} - \tau_{n+1}) \]
\[ - \frac{1}{p_{n+1}} \Delta^2 y_{n+1} f(y_{n-\tau_n}) + \frac{1}{p_n} \Delta^2 y_n \Delta f(y_n - \tau_n). \]
Therefore (5.2.10) becomes

\[ 
\Delta v_n = \frac{1}{p_n+2} \Delta^2 y_{n+2} \frac{f (y_{n+1} - \tau_{n+1})}{f (y_{n+1} - \tau_n)} - \frac{1}{p_{n+1}} \Delta^2 y_{n+1} \frac{f (y_{n+1} - \tau_{n+1})}{f (y_{n} - \tau_n)} + \frac{1}{p_n} \Delta^2 y_n \frac{f (y_{n+1} - \tau_{n+1})}{f (y_{n} - \tau_n)} 
\]

\[ = \Delta \left( \frac{1}{p_{n+1}} \Delta^2 y_{n+1} \right) \frac{f (y_{n+1} - \tau_{n+1})}{f (y_{n} - \tau_n)} - \Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \frac{f (y_{n+1} - \tau_{n+1})}{f (y_{n} - \tau_n)} 
\]

\[ = \frac{\Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right)}{f (y_{n} - \tau_n)} + \Delta \left( \frac{1}{p_{n+1}} \Delta^2 y_{n+1} \right) \left[ \frac{f (y_n - \tau_n) - f (y_{n+1} - \tau_{n+1})}{f (y_{n} - \tau_n) f (y_{n+1} - \tau_{n+1})} \right] 
\]

\[ \leq \frac{\Delta^2 \left( \frac{1}{p_n} \Delta^2 y_n \right)}{f (y_{n} - \tau_n)} = -q_n. 
\]

That is,

\[ \Delta v_n \leq -q_n. \quad (5.2.11) \]

Summing up (5.2.11) from \( n_1 \) to \( n \), we have

\[ v_{n+1} - v_{n_1} \leq - \sum_{s=n_1}^{n} q_s. \]

By assumption (5.2.8), we have

\[ \lim_{n \to +\infty} v_n = -\infty, \quad (5.2.12) \]

Then (5.2.9) implies,

\[ f (y_n - \tau_n) > 0 \text{ and hence } y_n - \tau_n > 0. \quad (5.2.13) \]

By equation (5.2.12), there exists a positive integer \( n_2 \geq n_1 \) such that

\[ v_n \leq -M + 1, M > 0, \text{ for } n \geq n_2. \]
That is,\[
\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \leq v_n \leq -M + 1.
\]
This implies\[
\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) + (M + 1) f (y_{n-\tau}) \leq 0, \text{ for } n \geq n_2. \tag{5.2.14}
\]
Let\[
\lim_{n \to \infty} y_n = L.
\]
Then \( L \geq 0 \).

**Case (i):** \( L > 0 \)

If \( L > 0 \), by the continuity of \( f \), we have\[
\lim_{n \to \infty} f (y_{n-\tau}) = f (L) > 0.
\]
So we can choose \( n_3 \) such that\[
f (y_{n-\tau}) > \frac{1}{2} f (L), \text{ for } n \geq n_3. \tag{5.2.15}
\]
Substituting (5.2.15) in (5.2.14), we get\[
\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) + \frac{1}{2} f (L) (M + 1) \leq 0,
\]
which implies\[
\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \leq -\frac{1}{2} f (L) (M + 1).
\]
That is,\[
\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) \leq -k_1, \text{ where } k_1 = \frac{1}{2} f (L) (M + 1) > 0.
\]
Summing the last inequality from \( n_3 \) to \( n - 1 \), we have\[
\frac{1}{p_n} \Delta^2 y_n \leq \frac{1}{p_{n_3}} \Delta^2 y_{n_3} - k_1 (n - n_3).
\]
This implies
\[ \frac{1}{p_n} \Delta^2 y_n \to -\infty \text{ as } n \to \infty. \]
Proceeding in this way, finally we get
\[ y_n \to -\infty \text{ as } n \to \infty, \]
which contradicts (5.2.13).

**Case (ii):** \( L = 0 \).

If \( L = 0 \), then \( \lim_{n \to \infty} y_n = 0 \).
That is, \( y_n \to 0 \) as \( n \to \infty \),
which contradicts (5.2.13).
The proof of the Theorem is now complete.

### 5.3 **EXAMPLES**

**Example 5.3.1.** Consider the difference equation
\[ \Delta^2 (n \Delta^2 y_n) + 16 (n + 1) y_{n+1}^3 = 0. \] (E.5.1)

Here \( p_n = \frac{1}{n} \), \( q_n = 16(n + 1) \), \( f(y) = y^3 \) and \( \sum_{\infty} q_n = \infty \).
All the conditions of Theorem 5.2.1 are satisfied. Hence all the solutions of the equation (E.5.1) are oscillatory.
One of the solutions is \( y_n = (-1)^n \). \( \Box \)

**Example 5.3.2.** Consider the difference equation
\[ \Delta^2 ((n + 3) \Delta^2 y_n) + \frac{16(16n + 72)}{3} y_{n+1} = 0. \] (E.5.2)
Here \( p_n = \frac{1}{n+3}, \quad q_n = \frac{16(16n+72)}{3}, \quad f(y) = y \) and

\[
\sum_{k=0}^{\infty} \frac{(k-N)}{R_k} q_k = \sum_{k=0}^{\infty} \frac{16(k-N)(k+3)(16k+72)}{3} = \infty.
\]

All the conditions of Theorem 5.2.2 are satisfied. Hence all the solutions of the equation (E.5.2) are oscillatory.

One of the solutions is \( y_n = (-3)^n \). □

**Example 5.3.3.** Consider the difference equation

\[
\Delta^2 \left( \frac{1}{n} \Delta^2 y_n \right) + \frac{16 \left( 4n^3 + 12n^2 + 8n + 1 \right)}{n^4 + 4n^3 + 4n^2 + n} y_{n-3}^3 = 0.
\]  \quad (E.5.3)

Here \( p_n = n, \quad q_n = \frac{16(4n^3+12n^2+8n+1)}{n^4+4n^3+4n^2+n}, \quad f(y) = y^3 \) and \( \sum_{n=0}^{\infty} q_n = \infty \).

All the conditions of Theorem 5.2.3 are satisfied. Hence the difference

\[
\Delta \left( \frac{1}{p_n} \Delta^2 y_n \right) = 2 (-1)^{n+4} \left[ \frac{4n^3 + 12n^2 + 8n + 1}{n^4 + 4n^3 + 4n^2 + n} \right]
\]

of the solution \( y_n = \frac{(-1)^n}{2} \) of the equation (E.5.3) oscillates. □