CHAPTER 5

LIE - THEORETIC GENERATING RELATIONS
IN Volving MULTI-VARIABLE BESSEL FUNCTIONS OF
TWO INDICES

5.1. INTRODUCTION

The theory of special functions plays an important role in the formalism of mathematical physics. Bessel functions (BF) $J_n(x)$ (1.4.19), are among the most important special functions with very diverse applications to physics, engineering and mathematical analysis ranging from abstract number theory and theoretical astronomy to concrete problems of physics and engineering.

The importance of BF has been further stressed by their various generalizations. Dattoli and his co-workers introduced and discussed generalized Bessel functions (GBF) and their multi-variable, multi-index extensions within purely mathematical and applicative contexts, see for example [10,16,20,27,28,31,37,41]. GBF have proved a powerful tool to investigate the dynamical aspects of physical problems such as electron scattering by an intense linearly polarized laser wave, multi-photon processes and undulator radiation. The analytical and numerical study of GBF has revealed their interesting properties, which in some sense can be regarded as an extension of the properties of BF to a two-dimensional domain. In this connection, the relevance of GBF and their multi-variable extension in mathematical physics has been emphasized, since they provide analytical solutions to partial differential equations such as the multi-dimensional diffusion equation, the Schrödinger and Klein-Gordon equations. The algebraic structure underlying GBF can be recognized in full analogy with BF, thus providing a unifying view to the theory of both BF and GBF.

A useful complement to the theory of GBF is offered by the introduction of 2-index 5-variable Bessel functions (2IVBF), defined as ([27],[19],p.190(3.23))

$$J_{m,n}(x_1, x_2, x_3, x_4, x_5) = \sum_{s=-\infty}^{\infty} J_{m-s}(x_1, x_3) J_{n-s}(x_2, x_4) J_s(x_5),$$  \hspace{1cm} (5.1.1)
with the following generating function

\[
\sum_{m,n=\infty} J_{m,n}(x_1, x_2, x_3, x_4, x_5)u^m v^n = \exp \left[ \frac{x_1}{2} \left( u - \frac{1}{u} \right) + \frac{x_2}{2} \left( v - \frac{1}{v} \right) + \frac{x_3}{2} \left( u^2 - \frac{1}{u^2} \right) + \frac{x_4}{2} \left( v^2 - \frac{1}{v^2} \right) + \frac{x_5}{2} \left( uv - \frac{1}{uv} \right) \right].
\]  

(5.1.2)

The importance of 215VBF has been discussed in [68], in connection with problems associated to the radiation emitted by relativistic electrons moving in complex structures. We consider 2-index 5-variable 5-parameter Bessel function (215V5PBF) defined as:

\[
J_{m,n}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \sum_{m,n=-\infty}^{\infty} \xi_1^m \xi_2^n J_{m+n}(x_1, x_2; \xi_1, \xi_2) J_{n-m}(x_3, x_4, x_5; \xi_3, \xi_4, \xi_5),
\]

(5.1.3)

where \(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5\) are arbitrary complex parameters.

The generating function for 215V5PBF is given as:

\[
\sum_{m,n=-\infty}^{\infty} J_{m,n}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5)u^m v^n = \exp \left[ \frac{x_1}{2} \left( u - \frac{\xi_1}{u} \right) + \frac{x_2}{2} \left( v - \frac{\xi_2}{v} \right) + \frac{x_3}{2} \left( u^2 - \frac{\xi_3}{u^2} \right) + \frac{x_4}{2} \left( v^2 - \frac{\xi_4}{v^2} \right) + \frac{x_5}{2} \left( uv - \frac{1}{\xi_5 uv} \right) \right].
\]

(5.1.4)

This is the most convenient form to find the generating relations by using Lie theoretic approach.

The 215V5PBF \(J_{m,n}(x_i; \xi_i), (i = 1, 2, 3, 4, 5)\) defined by Eqs(5.1.3) and (5.1.4) satisfy the following differential and pure recurrence relations

\[
\frac{\partial}{\partial x_1} J_{m,n}(x_i; \xi_i) = \frac{1}{2} \left[ J_{m-1,n}(x_i; \xi_i) - \xi_1 J_{m+1,n}(x_i; \xi_i) \right],
\]

\[
\frac{\partial}{\partial x_2} J_{m,n}(x_i; \xi_i) = \frac{1}{2} \left[ J_{m,n-1}(x_i; \xi_i) - \xi_2 J_{m,n+1}(x_i; \xi_i) \right],
\]

\[
\frac{\partial}{\partial x_3} J_{m,n}(x_i; \xi_i) = \frac{1}{2} \left[ J_{m-2,n}(x_i; \xi_i) - \xi_3 J_{m+2,n}(x_i; \xi_i) \right],
\]

87
\[
\frac{\partial}{\partial x_4} J_{m,n}(x_i; \xi_i) = \frac{1}{2} \left[ J_{m,n-2}(x_i; \xi_i) - \xi_4 J_{m,n+2}(x_i; \xi_i) \right],
\]
\[
\frac{\partial}{\partial x_5} J_{m,n}(x_i; \xi_i) = \frac{1}{2} \left[ \xi_5 J_{m-1,n-1}(x_i; \xi_i) - \frac{1}{\xi_5} J_{m+1,n+1}(x_i; \xi_i) \right],
\]
\[
\frac{\partial}{\partial \xi_1} J_{m,n}(x_i; \xi_i) = -\frac{x_1}{2} J_{m+1,n}(x_i; \xi_i),
\]
\[
\frac{\partial}{\partial \xi_2} J_{m,n}(x_i; \xi_i) = -\frac{x_2}{2} J_{m,n+1}(x_i; \xi_i),
\]
\[
\frac{\partial}{\partial \xi_3} J_{m,n}(x_i; \xi_i) = -\frac{x_3}{2} J_{m+2,n}(x_i; \xi_i),
\]
\[
\frac{\partial}{\partial \xi_4} J_{m,n}(x_i; \xi_i) = -\frac{x_4}{2} J_{m,n+2}(x_i; \xi_i),
\]
\[
\frac{\partial}{\partial \xi_5} J_{m,n}(x_i; \xi_i) = \frac{x_5}{2} \left[ J_{m-1,n-1}(x_i; \xi_i) + \xi_5^2 J_{m+1,n+1}(x_i; \xi_i) \right], \quad (i = 1, 2 \cdots, 5), \quad (5.1.5)
\]

and
\[
m J_{m,n}(x_i; \xi_i) = \frac{x_1}{2} (J_{m-1,n}(x_i; \xi_i) + \xi_1 J_{m+1,n}(x_i; \xi_i))
\]
\[+ x_3 (J_{m-2,n}(x_i; \xi_i) + \xi_3 J_{m+2,n}(x_i; \xi_i))
\]
\[+ \frac{x_5}{2} \left( \xi_5 J_{m-1,n-1}(x_i; \xi_i) + \frac{1}{\xi_5} J_{m+1,n+1}(x_i; \xi_i) \right),
\]
\[
n J_{m,n}(x_i; \xi_i) = \frac{x_2}{2} (J_{m,n-1}(x_i; \xi_i) + \xi_2 J_{m,n+1}(x_i; \xi_i))
\]
\[+ x_4 (J_{m,n-2}(x_i; \xi_i) + \xi_4 J_{m,n+2}(x_i; \xi_i))
\]
\[+ \frac{x_5}{2} \left( \xi_5 J_{m-1,n-1}(x_i; \xi_i) + \frac{1}{\xi_5} J_{m+1,n+1}(x_i; \xi_i) \right), \quad (i = 1, 2 \cdots, 5). \quad (5.1.6)
\]

The differential equations satisfied by 2155PBF \( J_{m,n}(x_i; \xi_i), \; (i = 1, 2, 3, 4, 5) \) are
\[
\left( -\frac{1}{\xi_1} \frac{\partial^2}{\partial x_1^2} + \frac{4x_3^2}{\xi_1^2 x_1^2} \frac{\partial^2}{\partial x_3^2} + \frac{16\xi_3^2}{\xi_1^2} \frac{\partial^2}{\partial x_1^2} + \frac{\xi_5^2}{\xi_1^2} \frac{\partial^2}{\partial x_3^2} + \frac{\xi_5^2}{\xi_1^2} \frac{\partial^2}{\partial x_1^2} - \frac{1}{\xi_1} \frac{\partial}{\partial x_1} + \frac{4x_3}{\xi_1 x_1^2} (1 - m) \frac{\partial}{\partial x_3} \right.
\]
\[+ \frac{8\xi_3}{\xi_1 x_1^2} (2 + m) \frac{\partial}{\partial \xi_3} + \frac{\xi_5}{\xi_1 x_1^2} (1 - 2m) \frac{\partial}{\partial \xi_5} + \frac{m^2}{\xi_1 x_1^2} - 1 \right) J_{m,n}(x_i; \xi_i) = 0, \quad (i = 1, 2, 3, 4, 5) \tag{5.1.7}
\]
\[
\left( -\frac{1}{\xi_2} \frac{\partial^2}{\partial x_2^2} + \frac{4x_4^2}{\xi_2 x_2^2} \frac{\partial^2}{\partial x_4^2} + \frac{16\xi_4^2}{\xi_2^2} \frac{\partial^2}{\partial x_2^2} + \frac{\xi_5^2}{\xi_2^2} \frac{\partial^2}{\partial x_4^2} + \frac{\xi_5^2}{\xi_2^2} \frac{\partial^2}{\partial x_2^2} - \frac{1}{\xi_2} \frac{\partial}{\partial x_2} + \frac{4x_4}{\xi_2 x_2^2} (1 - n) \frac{\partial}{\partial x_4} \right.
\]

88
\[ + \frac{8\xi_4}{\xi_2x_2^2} (2 + n) \frac{\partial}{\partial \xi_4} + \frac{\xi_5}{\xi_2x_2^2} (1 - 2n) \frac{\partial}{\partial \xi_5} + \frac{n^2}{\xi_2x_2^2} - 1 \] \[ J_{m,n}(x_i; \xi_i) = 0, \quad (i = 1, 2, 3, 4, 5), \] (5.1.8)

The Bessel functions of integral order have been shown to be connected with the faithful irreducible unitary representations of the real Euclidean group \( E_3 \) in the plane [145,152]. The Euclidean group \( E_3 \) is a real 3-parameter global Lie group, whose Lie algebra \( \mathcal{E}_3 \) has basis elements

\[ J_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \] (5.1.9)

with commutation relations

\[ [J_1, J_2] = 0, \quad [J_3, J_1] = J_2, \quad [J_3, J_2] = -J_1. \] (5.1.10)

The three dimensional complex local Lie group \( T_3 \) is the set of all \( 4 \times 4 \) matrices of the form (1.6.25). A basis for the Lie algebra \( T_3 = L(T_3) \) is provided by the matrices (1.6.28) with commutation relations (1.6.29).

Further, we observe that the complex matrices

\[ J^+ = -J_2 + iJ_1, \quad J^- = J_2 + iJ_1, \quad J^3 = iJ_3, \quad i = \sqrt{-1}, \] (5.1.11)

satisfy the commutation relations identical with (1.6.29). Thus we say that \( T_3 \) is the complexification of \( \mathcal{E}_3 \) and \( \mathcal{E}_3 \) is a real form of \( T_3 \) [64]. Due to this relationship between \( T_3 \) and \( \mathcal{E}_3 \), the abstract irreducible representation \( Q(\omega, m_0) \) of \( T_3 \) induces an irreducible representation of \( \mathcal{E}_3 \).

In this Chapter, we deal with the problems of framing 2I5V5PBF \( J_{m,n}(x_i; \xi_i) \), \((i = 1, 2, 3, 4, 5)\) into the context of the representation \( Q(\omega, m_0) \) of the Lie algebra \( T_3 \). In Section 5.2, we consider some special cases of 2I5V5PBF \( J_{m,n}(x_i; \xi_i) \). In Section 5.3, we derive generating relations involving 2I5V5PBF \( J_{m,n}(x_i; \xi_i) \), \((i = 1, 2, 3, 4, 5)\). In Section 5.4, we consider some applications of these generating relations. Section 5.5 is devoted to the concluding remarks.
5.2. SPECIAL CASES AND PROPERTIES OF 2I5V5PBF $J_{m,n}(x_i; \xi_i)$, 
$(i = 1, 2, 3, 4, 5)$

We note the following special cases of 2I5V5PBF $J_{m,n}(x_i; \xi_i)$, $(i = 1, 2, 3, 4, 5)$:

(1) $J_{m,n}(x_1, x_2, x_3, x_4, x_5; 1, 1, 1, 1, 1) = J_{m,n}(x_1, x_2, x_3, x_4, x_5)$, \hspace{1cm} (5.2.1)

where $J_{m,n}(x_1, x_2, x_3, x_4, x_5)$ denotes 2I5VBF defined by the generating function (5.1.2).

(2) $J_{m,n}(x_1 \to x, x_2 \to y, 0, 0, x_5 \to z; 1, 1, \xi_3, \xi_4, \xi_5 \to \xi) = J_{m,n}(x, y, z; \xi)$, \hspace{1cm} (5.2.2)

where $J_{m,n}(x, y, z; \xi)$ denotes 2-index 3-variable 1-parameter Bessel functions (2I3V1PBF) defined by the generating function ([37]; p.344(1.2)),

$$
\sum_{m,n=-\infty}^{\infty} J_{m,n}(x, y, z; \xi) u^m v^n = \exp \left[ \frac{x}{2} \left( u - \frac{1}{u} \right) + \frac{y}{2} \left( v - \frac{1}{v} \right) + \frac{z}{2} \left( \xi uv - \frac{1}{\xi uv} \right) \right].
$$

(5.2.3)

(3) $J_{m,n}(x_1 \to x, x_2 \to y, 0, 0, x_5 \to z; 1, 1, \xi_3, \xi_4, 1) = J_{m,n}(x, y, z)$, \hspace{1cm} (5.2.4)

where $J_{m,n}(x, y, z)$ denotes 2-index 3-variable Bessel functions (2I3VBF) defined by the generating function ([28]; p.3639(13)),

$$
\sum_{m,n=-\infty}^{\infty} J_{m,n}(x, y, z) u^m v^n = \exp \left[ \frac{x}{2} \left( u - \frac{1}{u} \right) + \frac{y}{2} \left( v - \frac{1}{v} \right) + \frac{z}{2} \left( uv - \frac{1}{uv} \right) \right].
$$

(5.2.5)

(4) $J_{m,n}(x_1 \to x, x_2 \to x, 0, 0, x_5 \to x; 1, 1, \xi_3, \xi_4, \xi_5 \to \xi) = J_{m,n}(x; \xi)$, \hspace{1cm} (5.2.6)

where $J_{m,n}(x; \xi)$ denotes 2-index 1-variable 1-parameter Bessel functions (2I1V1PBF) defined by the generating function ([28]; p.3648(43)),

$$
\sum_{m,n=-\infty}^{\infty} J_{m,n}(x; \xi) u^m v^n = \exp \left[ \frac{x}{2} \left( \left( u - \frac{1}{u} \right) + \left( v - \frac{1}{v} \right) + \left( \xi uv - \frac{1}{\xi uv} \right) \right) \right].
$$

(5.2.7)

(5) $J_{m,n}(x_1 \to x, x_2 \to x, 0, 0, x_5 \to x; 1, 1, \xi_3, \xi_4, 1) = J_{m,n}(x)$, \hspace{1cm} (5.2.8)

where $J_{m,n}(x)$ denotes 2-index 1-variable Bessel functions (2I1VBF) defined by the
generating function ([28]; p.3637(1)),
\[
\sum_{m,n=-\infty}^{\infty} J_{m,n}(x)u^mv^n = \exp \left[ \frac{x}{2} \left( \left( u - \frac{1}{u} \right) + \left( v - \frac{1}{v} \right) + \left( uv - \frac{1}{uv} \right) \right) \right].
\]  
(5.2.9)

(6) \quad J_{m,n}(x_1 \rightarrow x, 0, x_3 \rightarrow y, 0, 0; \xi_1 \rightarrow z, \xi_2, \xi_3 \rightarrow z', \xi_4, \xi_5) = J_m(x; z/y; z'),
\]  
(5.2.10)
or
\[
J_{m,n}(0, x_2 \rightarrow x, 0, x_4 \rightarrow y, 0; \xi_1, \xi_2 \rightarrow z, \xi_3, \xi_4 \rightarrow z', \xi_5) = J_n(x; z/y; z'),
\]  
(5.2.11)
where \( J_{m,n}(x; z/y; z') \) denotes 2-variable 2-parameter Bessel functions (2V2PBF) defined by the generating function ([37]; p.160(4.1)),
\[
\sum_{m=-\infty}^{\infty} J_m(x; z/y; z')u^m = \exp \left[ \frac{x}{2} \left( u - \frac{z}{u} \right) + \frac{y}{2} \left( u^2 - \frac{z'}{u^2} \right) \right].
\]  
(5.2.12)

(7) \quad J_{m,n}(x_1 \rightarrow x, 0, x_3 \rightarrow y, 0, 0; 1, \xi_2, 1, \xi_4, \xi_5) = J_m(x, y),
\]  
(5.2.13)
or
\[
J_{m,n}(0, x_2 \rightarrow x, 0, x_4 \rightarrow y, 0; 1, \xi_1, 1, \xi_3, 1, \xi_5) = J_n(x, y),
\]  
(5.2.14)
where \( J_m(x, y) \) denotes 2-variable Bessel functions (2VBF) defined by the generating function ([31]; p.24(1.8(a))),(5.2.15)
\[
\sum_{m=-\infty}^{\infty} J_m(x, y)u^m = \exp \left[ \frac{x}{2} \left( u - \frac{1}{u} \right) + \frac{y}{2} \left( u^2 - \frac{1}{u^2} \right) \right].
\]  

(8) \quad J_{m,n}(x_1 \rightarrow x, 0, x_3 \rightarrow y, 0, 0; -1, \xi_2, -1, \xi_4, \xi_5) = I_m(x, y),
\]  
(5.2.16)
or
\[
J_{m,n}(0, x_2 \rightarrow x, 0, x_4 \rightarrow y, 0; -1, \xi_1, -1, \xi_3, -1, \xi_5) = I_n(x, y),
\]  
(5.2.17)
where \( I_m(x, y) \) denotes 2-variable modified Bessel functions (2VMBF) defined by the generating function ([20]; p.331(2.11(a))),
\[
\sum_{m=-\infty}^{\infty} I_m(x, y)u^m = \exp \left[ \frac{x}{2} \left( u + \frac{1}{u} \right) + \frac{y}{2} \left( u^2 + \frac{1}{u^2} \right) \right].
\]  
(5.2.18)
\begin{align}
\tag{9}
J_{m,n}(x_1 \rightarrow x, 0, x_3 \rightarrow y, 0, 0; 0, \xi_2, 0, \xi_4, \xi_5) &= \frac{1}{m!} H_m \left( \frac{x}{2}, \frac{y}{2} \right), \quad (5.2.19)
\end{align}

or

\begin{align}
J_{m,n}(0, x_2 \rightarrow x, 0, x_4 \rightarrow y, 0; \xi_1, 0, \xi_3, 0, \xi_5) &= \frac{1}{n!} H_n \left( \frac{x}{2}, \frac{y}{2} \right),
\end{align}

where \(H_m(x, y)\) denotes 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) defined by the generating function ([37]; p.151(3.1)),

\begin{align}
\sum_{m=0}^{\infty} H_m(x, y) \frac{y^m}{m!} &= \exp(xy + yu^2).
\end{align}

\begin{align}
\tag{10}
J_{m,n}(x_1 \rightarrow x, 0, x_3 \rightarrow y, 0, 0; 0, \xi_2, \xi_3 \rightarrow \xi, \xi_4, \xi_5) &= \sum_{s=0}^{\lfloor m/2 \rfloor} \frac{(x/2)^{m-2s}}{(m-2s)!} J_s(y; \xi),
\end{align}

or

\begin{align}
J_{m,n}(0, x_2 \rightarrow x, 0, x_4 \rightarrow y, 0; \xi_1, 0, \xi_3, \xi_4 \rightarrow \xi, \xi_5) &= \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(x/2)^{n-2r}}{(n-2r)!} J_r(y; \xi),
\end{align}

where \(J_m(y; \xi)\) denotes 1-variable 1-parameter Bessel functions (1V1PBF) ([37]; p.162).

\begin{align}
\tag{11}
J_{m,n}(x_1 \rightarrow x, 0, x_3 \rightarrow y, 0, 0; i, \xi_2, i, \xi_4, \xi_5)
&= \exp\left(-\frac{im\pi}{4}\right) J_m \left( x \exp\left(\frac{\pi i}{4}\right), y \exp\left(\frac{\pi i}{4}\right), \frac{\pi i}{4} \right),
\end{align}

or

\begin{align}
J_{m,n}(0, x_2 \rightarrow x, 0, x_4 \rightarrow y, 0; \xi_1, i, \xi_3, i, \xi_5)
&= \exp\left(-\frac{in\pi}{4}\right) J_n \left( x \exp\left(\frac{\pi i}{4}\right), y \exp\left(\frac{\pi i}{4}\right), \frac{\pi i}{4} \right),
\end{align}

which can be viewed as a kind of generalized Kelvin function ([37]; p.164).

Further, the special case

\begin{align}
J_{m,n}(x_1 \rightarrow x, 0, x_3 \rightarrow y, 0, 0; i, \xi_2, 1, \xi_4, \xi_5) &= \exp\left(-\frac{im\pi}{4}\right) J_m \left( x \exp\left(\frac{\pi i}{4}\right), \frac{\pi i}{4}, y \right),
\end{align}

or

\begin{align}
J_{m,n}(0, x_2 \rightarrow x, 0, x_4 \rightarrow y, 0; \xi_1, i, \xi_3, 1, \xi_5) &= \exp\left(-\frac{in\pi}{4}\right) J_n \left( x \exp\left(\frac{\pi i}{4}\right), \frac{\pi i}{4}, y \right),
\end{align}

92
provides another interesting form ([37]; p.164).

\[ J_{m,n}(x_1 \rightarrow x, 0, 0, 0; 1, \xi_2, \xi_3, \xi_4, \xi_5) = J_m(x), \]  \hspace{1cm} (5.2.28)

or

\[ J_{m,n}(0, x_2 \rightarrow x, 0, 0; \xi_1, 1, \xi_3, \xi_4, \xi_5) = J_n(x), \]  \hspace{1cm} (5.2.29)

where \( J_m(x) \) denotes ordinary Bessel function defined by the generating function (1.4.19).

\[ J_{m,n}(x_1 \rightarrow x, 0, 0, 0; -1, \xi_2, \xi_3, \xi_4, \xi_5) = I_m(x), \]  \hspace{1cm} (5.2.30)

or

\[ J_{m,n}(0, x_2 \rightarrow x, 0, 0; \xi_1, -1, \xi_3, \xi_4, \xi_5) = I_n(x), \]  \hspace{1cm} (5.2.31)

where \( I_m(x) \) denotes modified Bessel functions (MBF) defined by the generating function ([37]; p.23(5.1),

\[ \sum_{m=-\infty}^{\infty} I_m(x) u^m = \exp \left[ \frac{x}{2} \left( u + \frac{1}{u} \right) \right]. \]  \hspace{1cm} (5.2.32)

\[ J_{m,n}(x_1 \rightarrow 4x, 0, -2, 0; 0, \xi_2, \xi_3, \xi_4, \xi_5) = H_m(x), \]  \hspace{1cm} (5.2.33)

or

\[ J_{m,n}(0, x_2 \rightarrow 4x, 0, -2, 0; \xi_1, 0, \xi_3, 0, \xi_5) = H_n(x), \]  \hspace{1cm} (5.2.34)

where \( H_m(x) \) denotes Hermite polynomials defined by the generating function (1.4.29).

It is worth to mention the following expansions of \( J_{m,n}(x_i; \xi_i), (i = 1, 2, 3, 4, 5) \) in terms of 2-index 5-variable Hermite Kampé de Fériet polynomials ([15VHKdFP])

\[ H_{m,n}(x_1, x_2, x_3, x_4, x_5) \sum_{m,n=0}^{\infty} \frac{u^m v^n}{m! n!} = \exp(x_1 u + x_2 v + x_3 u^2 + x_4 v^2 + x_5 u v), \]

\[ J_{m,n}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \sum_{r,s=0}^{\infty} \frac{H_{m+r,n+s}}{(m+r)! (n+s)!} \left( \frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}, \frac{x_4}{2}, \frac{x_5}{2} \right) \]

\[ \times \frac{H_{r,s}}{r! s!} \left( \frac{-x_1 \xi_1}{2}, \frac{-x_2 \xi_2}{2}, \frac{-x_3 \xi_3}{2}, \frac{-x_4 \xi_4}{2}, \frac{-x_5 \xi_5}{2} \right). \]  \hspace{1cm} (5.2.35)
The addition and multiplication theorems have noticeable relevance and are particularly useful for numerical evaluation of 215V5PBF $J_{m,n}(x_i;\xi_i), (i = 1, 2, 3, 4, 5)$. The Neumann addition theorem for $J_{m,n}(x_i;\xi_i), (i = 1, 2, 3, 4, 5)$ is given by

$$J_{m,n}(x_1\pm y_1, x_2\pm y_2, x_3\pm y_3, x_4\pm y_4, x_5\pm y_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \sum_{p,q=-\infty}^{\infty} J_{m+p,n+q}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) J_{p,q}(y_1, y_2, y_3, y_4, y_5; \xi_1, x_2, \xi_3, \xi_4, \xi_5)$$  \hspace{2cm} (5.2.36)

and multiplication theorem for $J_{m,n}(x_i;\xi_i), (i = 1, 2, 3, 4, 5)$ is given in the following way:

$$J_{m,n}(\lambda x_1, \mu x_2, \alpha x_3, \beta x_4, \mu x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \lambda^m \mu^n \sum_{p,q=0}^{\infty} \frac{\lambda^p \mu^q}{p! q!} \times J_{m+p,n+q}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \times H_{p,q}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7)$$ \hspace{2cm} (5.2.37)

5.3. GENERATING RELATIONS INVOLVING 215V5PBF $J_{m,n}(x_i;\xi_i), (i = 1, 2, 3, 4, 5)$

We derive generating relation involving 215V5PBF $J_{m,n}(x_i;\xi_i), (i = 1, 2, 3, 4, 5)$ by using the representation $Q(w, m_0)$ of the Lie algebra $T_5$. We note that $w, m_0 \notin C$ such that $w \neq 0$ and $0 \leq Re m_0 < 1$ and the spectrum $S$ of this representation is the set $\{m_0 + k : k \in \mathbb{Z}\}$. In order to find a realization of this representation, we look for the functions

$$f_{m,n}(x_i, u; \xi_i) = Z_{m,n}(x_i;\xi_i)u^m v^n, \quad (i = 1, 2, 3, 4, 5), \hspace{2cm} (5.3.1)$$

such that

$$K^3 f_{m,n} = mf_{m,n}, \quad K^+ f_{m,n} = \omega f_{m+1,n}, \quad K^- f_{m,n} = \omega f_{m-1,n}$$

$$C_{0,0} f_{m,n} = (K^+ K^-) f_{m,n} = \omega^2 f_{m,n}, \quad (\omega \neq 0; m \in \mathbb{Z}) \hspace{2cm} (5.3.2)$$

94
and also
\[ K^3 f_{m,n} = n f_{m,n}, \quad K^{+'} f_{m,n} = \omega f_{m,n+1}, \quad K^{-'} f_{m,n} = \omega f_{m,n-1}, \]
\[ C_{0,0} f_{m,n} = (K^{+'} K^{-'}) f_{m,n} = \omega^2 f_{m,n}, \quad (\omega \neq 0; n \in s) \] (5.3.3)

The sets of operators \( \{K^3, K^+, K^-\} \) and \( \{K^{3'}, K^{+'}, K^{-'}\} \) satisfy the commutation relations identical to (4.3.2).

There are numerous possible solutions of Eq (4.3.2). We assume that the sets of linear differential operators \( \{K^3, K^+, K^-\} \) and \( \{K^{3'}, K^{+'}, K^{-'}\} \) take the forms
\[
K^3 = \frac{u}{\partial u},
\]
\[
K^+ = u \left[ \frac{1}{\xi_1 \partial x_1} + \frac{2x_3}{\xi_1 x_1} \frac{\partial}{\partial x_3} - \frac{4\xi_3}{\xi_1 x_1} \frac{\partial}{\partial \xi_3} + \frac{\xi_5}{\xi_1 x_1} \frac{\partial}{\partial \xi_5} - \frac{u}{\xi_1 x_1} \frac{\partial}{\partial u} \right],
\]
\[
K^- = u^{-1} \left[ \frac{\partial}{\partial x_1} + \frac{2x_3}{x_1} \frac{\partial}{\partial x_3} - \frac{4\xi_3}{x_1} \frac{\partial}{\partial \xi_3} + \frac{\xi_5}{x_1} \frac{\partial}{\partial \xi_5} - \frac{u}{x_1} \frac{\partial}{\partial u} \right]
\]
and
\[
K^{3'} = \frac{v}{\partial v},
\]
\[
K^{+'} = v \left[ \frac{1}{\xi_2 \partial x_2} + \frac{2x_4}{\xi_2 x_2} \frac{\partial}{\partial x_4} - \frac{4\xi_4}{\xi_2 x_2} \frac{\partial}{\partial \xi_4} + \frac{\xi_5}{\xi_2 x_2} \frac{\partial}{\partial \xi_5} - \frac{v}{\xi_2 x_2} \frac{\partial}{\partial v} \right],
\]
\[
K^{-'} = v^{-1} \left[ \frac{\partial}{\partial x_2} + \frac{2x_4}{x_2} \frac{\partial}{\partial x_4} - \frac{4\xi_4}{x_2} \frac{\partial}{\partial \xi_4} + \frac{\xi_5}{x_2} \frac{\partial}{\partial \xi_5} - \frac{v}{x_2} \frac{\partial}{\partial v} \right],
\]
respectively. The operators in Eqs. (5.3.4) and (5.3.5) satisfy the commutation relations (4.3.2).

In terms of the functions \( Z_{m,n}(x_i; \xi_i), (i = 1, 2, 3, 4, 5) \) and using operators (5.3.4) and (5.3.5) relations (5.3.2) and (5.3.3) reduce to

(i) \[ \left[ \frac{1}{\xi_1 \partial x_1} + \frac{2x_3}{\xi_1 x_1} \frac{\partial}{\partial x_3} - \frac{4\xi_3}{\xi_1 x_1} \frac{\partial}{\partial \xi_3} + \frac{\xi_5}{\xi_1 x_1} \frac{\partial}{\partial \xi_5} - \frac{m}{\xi_1 x_1} \frac{\partial}{\partial m} \right] Z_{m,n}(x_i; \xi_i) = \omega Z_{m+1,n}(x_i; \xi_i), \]

(ii) \[ \left[ -\frac{\partial}{\partial x_1} + \frac{2x_3}{x_1} \frac{\partial}{\partial x_3} - \frac{4\xi_3}{x_1} \frac{\partial}{\partial \xi_3} + \frac{\xi_5}{x_1} \frac{\partial}{\partial \xi_5} - \frac{m}{x_1} \frac{\partial}{\partial m} \right] Z_{m,n}(x_i; \xi_i) = \omega Z_{m-1,n}(x_i; \xi_i), \]
\[
\begin{align*}
(iii) & \quad \left[ -\frac{1}{\xi_1} \frac{\partial^2}{\partial x_1^2} + \frac{4x_3^2}{\xi_1x_1^3} \frac{\partial^2}{\partial x_3^2} + \frac{16\xi_3^2}{\xi_1^2x_1^2} \frac{\partial^2}{\partial \xi_3^2} + \frac{\xi_5^2}{\xi_1x_1^2} \frac{\partial^2}{\partial \xi_5^2} - \frac{1}{\xi_1} \frac{\partial}{\partial x_1} + \frac{x_3}{\xi_1} (1 - m) \frac{\partial}{\partial x_3} 
\right. \\
& \quad + \frac{8\xi_5}{\xi_1x_1^2} (2 + m) \frac{\partial}{\partial \xi_5} + \frac{\xi_5}{\xi_1x_1^2} (1 - 2m) \frac{\partial}{\partial \xi_5} + \frac{m^2}{\xi_1^2x_1^2} \right] Z_{m,n}(x_i; \xi_i) \\
& \quad = \omega^2 Z_{m,n}(x_i; \xi_i), \quad (i = 1, 2, 3, 4, 5) \\
\end{align*}
\]

and
\[
\begin{align*}
(i) & \quad \left[ \frac{1}{\xi_2} \frac{\partial}{\partial x_2} + \frac{2x_4}{x_2} \frac{\partial}{\partial x_4} - \frac{4x_4}{x_2} \frac{\partial}{\partial x_4} + \frac{\xi_5}{\xi_2x_2} \frac{\partial}{\partial \xi_5} - \frac{n}{\xi_2x_2} \right] Z_{m,n}(x_i; \xi_i) = \omega Z_{m,n+1}(x_i; \xi_i), \\
(ii) & \quad \left[ -\frac{\partial}{\partial x_2} + \frac{2x_4}{x_2} \frac{\partial}{\partial x_4} - \frac{4x_4}{x_2} \frac{\partial}{\partial x_4} + \frac{\xi_5}{\xi_2x_2} \frac{\partial}{\partial \xi_5} - \frac{n}{x_2} \right] Z_{m,n}(x_i; \xi_i) = \omega Z_{m,n-1}(x_i; \xi_i), \\
(iii) & \quad \left[ -\frac{1}{\xi_2} \frac{\partial^2}{\partial x_2^2} + \frac{4x_4^2}{\xi_2x_2^3} \frac{\partial^2}{\partial x_4^2} + \frac{16\xi_4^2}{\xi_2^2x_2^2} \frac{\partial^2}{\partial \xi_4^2} + \frac{\xi_5^2}{\xi_2x_2^2} \frac{\partial^2}{\partial \xi_5^2} - \frac{1}{\xi_2} \frac{\partial}{\partial x_2} + \frac{4x_4}{\xi_2x_2} (1 - n) \frac{\partial}{\partial x_4} 
\right. \\
& \quad + \frac{8\xi_4}{\xi_2x_2^2} (2 + n) \frac{\partial}{\partial \xi_4} + \frac{\xi_5}{\xi_2x_2^2} (1 - 2n) \frac{\partial}{\partial \xi_5} + \frac{n^2}{\xi_2x_2^2} \right] Z_{m,n}(x_i; \xi_i) \\
& \quad = \omega^2 Z_{m,n}(x_i; \xi_i), \quad (i = 1, 2, 3, 4, 5),
\end{align*}
\]
respectively. The complex constant \( \omega \) in these equations and in Eqs. (5.3.2),(5.3.3) is clearly nonessential. Hence we can take \( \omega = -1 \).

For this choice of \( \omega \), we see that (iii) of Eqs. (5.3.6) and (5.3.7) coincide with the differential equations (5.1.7) and (5.1.8) respectively of 215V5PBF \( J_{m,n}(x_i; \xi_i), \quad (i = 1, 2, 3, 4, 5) \). In fact, for all \( m, n \in \mathbb{S} \) the choice for \( Z_{m,n}(x_i; \xi_i) = J_{m,n}(x_i; \xi_i), \quad (i = 1, 2, 3, 4, 5) \) satisfy Eqs. (5.3.6) and (5.3.7). It follows from the above discussion that the functions \( f_{m,n}(x_i, u, v; \xi_i) = J_{m,n}(x_i; \xi_i)u^m v^n, \quad (i = 1, 2, 3, 4, 5) \) \( m, n \in \mathbb{S} \) form a basis for a realization of the representation \( Q(-1, m_0) \) of \( \mathcal{T}_3 \). By using Theorem 1.7.1, this representation of \( \mathcal{T}_3 \) can be extended to a local multiplier representation \( T \) of \( \mathcal{T}_3 \), defined on \( \mathcal{F} \), the space of all functions analytic in a neighbourhood of the point \((x_1^0, x_2^0, x_3^0, x_4^0, x_5^0, u^0, v^0, \xi_1^0, \xi_2^0, \xi_3^0, \xi_4^0, \xi_5^0) = (1, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1) \).

Using operators (5.3.4) the action of the one parameter groups \( \exp(\tau \mathcal{J}^3) \), \( \exp(\beta \mathcal{J}^+) \), \( \exp(\alpha \mathcal{J}^-) \) are obtained by integrating the following differential equations
\[
\frac{dx_i}{d\tau} = 0, \quad \frac{d\xi_i}{d\tau} = 0, \quad (i = 1, 2, 3, 4, 5) \quad \frac{d\mu}{d\tau} = \mu(\tau), \quad \frac{du}{d\tau} = 0, \quad \frac{dv}{d\tau} = 0,
\]

96
\[
\begin{align*}
\frac{dx_1(b)}{db} &= \frac{u(b)}{\xi_1(b)}, \quad \frac{dx_2(b)}{db} = 0, \quad \frac{dx_3(b)}{db} = 2x_3(b)u(b), \quad \frac{dx_4(b)}{db} = 0, \quad \frac{dx_5(b)}{db} = 0, \\
\frac{d\xi_1(b)}{db} &= 0, \quad \frac{d\xi_2(b)}{db} = 0, \quad \frac{d\xi_3(b)}{db} = -4\xi_3(b)u(b), \quad \frac{d\xi_4(b)}{db} = 0, \quad \frac{d\xi_5(b)}{db} = \frac{\xi_5(b)}{\xi_1(b)x_1(b)}, \\
\frac{du(b)}{db} &= \frac{-u(b)}{\xi_1(b)x_1(b)}, \quad \frac{dv(b)}{db} = 0, \quad \frac{dv(b)}{db} = 0, \\
\frac{dx_1(c)}{dc} &= -\frac{1}{u(c)}, \quad \frac{dx_2(c)}{dc} = 0, \quad \frac{dx_3(c)}{dc} = 2x_3(c), \quad \frac{dx_4(c)}{dc} = 0, \quad \frac{dx_5(c)}{dc} = 0, \\
\frac{d\xi_1(c)}{dc} &= 0, \quad \frac{d\xi_2(c)}{dc} = 0, \quad \frac{d\xi_3(c)}{dc} = -4\xi_3(c), \quad \frac{d\xi_4(c)}{dc} = 0, \quad \frac{d\xi_5(c)}{dc} = \frac{\xi_5(c)}{u(c)x_1(c)}, \\
\frac{du(c)}{dc} &= -\frac{1}{x_1(c)}, \quad \frac{dv(c)}{dc} = 0, \quad \frac{dv(c)}{dc} = 0, \\
\end{align*}
\]

subject to the conditions \(x_i(0) = x_i, \; \xi_i(0) = \xi_i, \; u(0) = u, \; \nu(0) = \nu, \; \nu(0) = 1, \; (i = 1, 2, 3, 4, 5),\) where \(\nu\) is multiplier of the representation.

Hence, the values of the multiplier representations of \(\exp(\tau \mathcal{J}^3), \exp(b\mathcal{J}^+), \exp(c\mathcal{J}^-)\) are given by

\[
\begin{align*}
[T(\exp \tau \mathcal{J}^3)f](x_1, x_2, x_3, x_4, x_5, u, v; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) &= f(x_1(x_1 + \frac{2bu}{\xi_1x_1})^{1/2}, x_2, \\
x_3(x_1 + \frac{2bu}{\xi_1x_1})^{-1}, x_4, x_5, u(x_1 + \frac{2bu}{\xi_1x_1})^{-1/2}, v; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \\
[T(\exp b \mathcal{J}^+)f](x_1, x_2, x_3, x_4, x_5, u; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) &= f(x_1(1 - \frac{2c}{ux_1})^{1/2}, x_2, \\
x_3(1 - \frac{2c}{ux_1})^{-1}, x_4, x_5, u(1 - \frac{2c}{ux_1})^{1/2}, v; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, (1 - \frac{2c}{ux_1})^{-1/2}, \xi_4, \xi_5), \\
\end{align*}
\]

for \(f \in \mathcal{F}.\) If \(g \in T_3\) is given by by Eq. (1.625), then

\[
g = (\exp b \mathcal{J}^+)(\exp c \mathcal{J}^-)(\exp a \mathcal{J}^-).
\]

97
Therefore, for \( f \in \mathcal{F} \) and \( g \) in a sufficiently small neighbourhood of the identity we have

\[
[T(g)f](x_1, x_2, x_3, x_4, x_5, u, v; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = [T(\exp b\mathcal{F}^+)T(\exp c\mathcal{F}^-)T(\exp a\mathcal{F}^3)f](x_1, x_2, x_3, x_4, x_5, u, v; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5)
\]

and hence we obtain

\[
[T(g)f](x_1, x_2, x_3, x_4, x_5, u, v; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = f \left( x_1 \left( 1 + \frac{2bu}{\xi_1 x_1} \right) \frac{1}{2} \left( 1 - \frac{2c}{ux_1} \right) \right),
\]

\[
x_2, x_3 \left( 1 + \frac{2bu}{\xi_1 x_1} \right) \left( 1 - \frac{2c}{ux_1} \right)^{-1}, x_4, x_5, e^u \left( 1 + \frac{2bu}{\xi_1 x_1} \right) \left( 1 - \frac{2c}{ux_1} \right)^{1/2},
\]

\[
v; \xi_1, \xi_2, \xi_3 \left( 1 + \frac{2bu}{\xi_1 x_1} \right)^{-2} \left( 1 - \frac{2c}{ux_1} \right)^2, \xi_4, \xi_5 \left( 1 + \frac{2bu}{\xi_1 x_1} \right)^{1/2} \left( 1 - \frac{2c}{ux_1} \right)^{-1/2},
\]

\[
\left| \frac{2bu}{\xi_1 x_1} \right| < 1, \left| \frac{2c}{ux_1} \right| < 1.
\]

The matrix elements of \( T(g) \) with respect to the analytic basis \( (f_{m,n})_{m,n \in \mathbb{S}} \) are the functions \( A_{mk}(g) \) uniquely determined by \( Q(-1, m_0) \) of \( \mathcal{T}_3 \) and are defined by

\[
[T(g)f_{m_0+k,n}](x_1, u, v; \xi_i) = \sum_{l=-\infty}^{\infty} A_{lk}(g)f_{m_0+l,n}(x_1, u, v; \xi_i),
\]

\[
k = 0, \pm 1, \pm 2, \pm 3 \cdots, \quad (i = 1, 2, 3, 4, 5).
\]

Therefore, we prove the following result:

**Theorem 5.3.1** The following generating equation holds:

\[
\left( \frac{1 - (2c/ux_1)}{1 + (2bu/\xi_1 x_1)} \right)^{m/2} J_{m,n} \left( x_1 \left( 1 + \frac{2bu}{\xi_1 x_1} \right) \frac{1}{2} \left( 1 - \frac{2c}{ux_1} \right) \right), x_2,
\]

\[
x_3 \left( 1 + \frac{2bu}{\xi_1 x_1} \right) \left( 1 - \frac{2c}{ux_1} \right)^{-1}, x_4, x_5; \xi_1, \xi_2, \xi_3 \left( 1 + \frac{2bu}{\xi_1 x_1} \right)^{-2} \left( 1 - \frac{2c}{ux_1} \right)^2, \xi_4,
\]

\[
\xi_5 \left( 1 + \frac{2bu}{\xi_1 x_1} \right)^{1/2} \left( 1 - \frac{2c}{ux_1} \right)^{-1/2} = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} e^{-p+|p|/2} b^{p+|p|/2} _0F_1[-; |p|+1; bc]
\]

98
\[ \times J_{m,p,n}(x_1, x_2, x_3, x_4, x_5, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5)u^p, \quad \frac{2bu}{x_1} < 1, \quad \frac{2c}{ux_1} < 1. \quad (5.3.12) \]

**Proof.** Using (5.3.10), we obtain

\[
e^{mr} \left( \frac{1 - (2c/ux_1)}{1 + (2bu/\xi_1 x_1)} \right)^{m/2} J_{m,n} \left( x_1 \left( 1 + \frac{2bu}{\xi_1 x_1} \right)^{1/2} \left( 1 - \frac{2c}{ux_1} \right)^{1/2} \right), x_2,
\]

\[
x_3 \left( 1 + \frac{2bu}{\xi_1 x_1} \right) \left( 1 - \frac{2c}{ux_1} \right)^{-1}, x_4, x_5; \xi_1, \xi_2, \xi_3 \left( 1 + \frac{2bu}{\xi_1 x_1} \right)^{-2} \left( 1 - \frac{2c}{ux_1} \right)^{2},
\]

\[
\xi_4, \xi_5 \left( 1 + \frac{2bu}{\xi_1 x_1} \right)^{1/2} \left( 1 - \frac{2c}{ux_1} \right)^{-1/2} \right)
\]

\[= \sum_{l=-\infty}^{\infty} A_{l,m-m_0}(g) J_{m_0+l,n}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5)u^{m_0+l-m} \quad (5.3.13)\]

and the matrix element \(A_{lk}(g)\) are given by (4.3.14). Substituting the value of \(A_{lk}(g)\) given by (4.3.14) into (5.3.13), we obtain the result (5.3.12).

Similarly, for the operators (5.3.5), we have the following result:

**Theorem 5.3.2.** The following generating equation holds:

\[
\left( \frac{1 - (2c'/ux_2)}{1 + (2b'v/\xi_2 x_2)} \right)^{n/2} J_{m,n} \left( x_1, x_2 \left( 1 + \frac{2b'v}{\xi_2 x_2} \right)^{1/2} \left( 1 - \frac{2c'}{vx_2} \right)^{1/2} \right), x_3,
\]

\[
x_4 \left( 1 + \frac{2b'v}{\xi_2 x_2} \right) \left( 1 - \frac{2c'}{vx_2} \right)^{1/2}, x_5; \xi_1, \xi_2, \xi_3, \xi_4 \left( 1 + \frac{2b'v}{\xi_2 x_2} \right)^{-2} \left( 1 - \frac{2c'}{vx_2} \right)^{2},
\]

\[
\xi_5 \left( 1 + \frac{2b'v}{\xi_2 x_2} \right)^{1/2} \left( 1 - \frac{2c'}{vx_2} \right)^{-1/2} \right) = \sum_{q=-\infty}^{\infty} \frac{(-1)^{|q|}}{|q|!} c'\left(-q+|q|\right)/2 b'(q+|q|)/2 \Gamma[-; |q|+1; b'c']
\]

\[\times J_{m,n+q}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5)u^q, \quad \frac{2b'v}{\xi_2 x_2} < 1, \quad \frac{2c'}{vx_2} < 1. \quad (5.3.14)\]

The following corollaries are immediate consequences of Theorems 5.3.1 and 5.3.2.

**Corollary 5.3.1.** The following generating equation holds:

\[
\left( \frac{1 + (r/ux_1)}{1 + (rux/\xi_1 x_1)} \right)^{m/2} J_{m,n} \left( x_1 \left( 1 + \frac{rux}{\xi_1 x_1} \right)^{1/2} \left( 1 + \frac{r}{ux_1} \right)^{1/2} \right), x_2,
\]

99
\[
x_3 \left( 1 + \frac{ruv}{\xi_1 x_1} \right) \left( 1 + \frac{r}{uw x_1} \right)^{-1}, x_4, x_5; \xi_1, \xi_2, \xi_3 \left( 1 + \frac{ruv}{\xi_1 x_1} \right)^{-2} \left( 1 + \frac{r}{uw x_1} \right)^2, \xi_4, \xi_5 \left( 1 + \frac{ruv}{\xi_1 x_1} \right)^{1/2} \left( 1 + \frac{r}{uw x_1} \right)^{-1/2}, \right.
\]

\[
= \sum_{p=-\infty}^{\infty} (-\nu)^p J_p(r) J_{m+p,n}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) u^p, \quad \left| \frac{r}{uw x_1} \right| < 1, \quad \left| \frac{ruv}{\xi_1 x_1} \right| < 1.
\]

(5.3.15)

Proof. If \( bc \neq 0 \), we can introduce the co-ordinates \( r \) and \( v \) such that \( b = rv/2 \) and \( c = -(r/2\nu) \), with these new co-ordinates the matrix elements (4.3.14) can be expressed as

\[
A_{lk}(g) = \exp((m_0 + k)r)(-v)^{-k} J_{l-k}(r), \quad k = 0, \pm 1, \pm 2, \ldots
\]

(5.3.16)

and the generating relation (5.3.12) yields (5.3.15).

Corollary 5.3.2. The following generating equation holds:

\[
\left( 1 + \frac{(r'/uv' x_2)}{1 + (r'/uv' x_2 \xi_2)} \right)^{n/2} J_{m,n}(x_1, x_2; 1 + \frac{r'uv'}{\xi_2 x_2})^{1/2} \left( 1 + \frac{r'}{uv' x_2} \right)^{1/2}, x_3,
\]

\[
x_4 \left( 1 + \frac{r'uv'}{\xi_2 x_2} \right) \left( 1 + \frac{r'}{uv' x_2} \right)^{-1}, x_5, \xi_1, \xi_2, \xi_3, \xi_4 \left( 1 + \frac{r'uv'}{\xi_2 x_2} \right)^{-2} \left( 1 + \frac{r'}{uv' x_2} \right)^2, \xi_5 \left( 1 + \frac{r'uv'}{\xi_2 x_2} \right)^{1/2} \left( 1 + \frac{r'}{uv' x_2} \right)^{-1/2},
\]

\[
= \sum_{q=-\infty}^{\infty} (-\nu)^q J_q(r') J_{m,n+q}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) u^q,
\]

\[
\left| \frac{r'}{uv' x_2} \right| < 1, \quad \left| \frac{r'uv'}{\xi_2 x_2} \right| < 1.
\]

(5.3.17)

5.4. APPLICATIONS

We discuss some applications of the generating relations obtained in the preceding section.
I. Taking \( c = 0 \) and \( u = 1 \) in generating relation (5.3.12), we get

\[
\left( 1 + \frac{2b}{\xi_1 x_1} \right)^{-m/2} J_{m,n} \left( x_1 \left( 1 + \frac{2b}{\xi_1 x_1} \right)^{1/2}, x_2, x_3 \left( 1 + \frac{2b}{\xi_1 x_1} \right), x_4, x_5; \xi_1, \xi_2, \xi_3 \left( 1 + \frac{2b}{\xi_1 x_1} \right)^{1/2} \right) \\
= \sum_{p=0}^{\infty} \frac{(-b)^p}{p!} J_{m+p,n}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \quad \left| \frac{2b}{\xi_1 x_1} \right| < 1. \tag{5.4.1}
\]

Again taking \( b = 0 \) and \( u = 1 \) in generating relation (5.3.12), we get

\[
\left( 1 - \frac{2c}{x_1} \right)^{m/2} J_{m,n} \left( x_1 \left( 1 - \frac{2c}{x_1} \right)^{1/2}, x_2, x_3 \left( 1 - \frac{2c}{x_1} \right)^{-1}, x_4, x_5; \xi_1, \xi_2, \xi_3 \left( 1 - \frac{2c}{x_1} \right)^{1/2} \right) \\
= \sum_{p=0}^{\infty} \frac{(c)^p}{p!} J_{m-p,n}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \quad \left| \frac{2c}{x_1} \right| < 1. \tag{5.4.2}
\]

Further taking \( x_2 = x_3 = x_4 = x_5 = 0 \) and \( \xi_1 = 1 \) in generating relations (5.4.1) and (5.4.2), we obtain the formulas of Lommel ([113], p.62(3.30,3.31)) respectively. Similarly we can obtain results corresponding to generating relation (5.3.14).

II. Taking \( \xi_i (i = 1,2,3,4,5) = u = v = 1 \) in generating relation (5.3.15), we obtain the following result:

\[
J_{m,n} \left( x_1 \left( 1 + \frac{r}{x_1} \right), x_2, x_3, x_4, x_5 \right) = \sum_{p=-\infty}^{\infty} J_p(r) J_{m+p,n}(x_1, x_2, x_3, x_4, x_5), \quad \left| \frac{r}{x_1} \right| < 1, \tag{5.4.3}
\]

where \( J_{m,n}(x_1, x_2, x_3, x_4, x_5) \) denotes 215VBF given by Eq. (5.1.2). Similar results can be obtained from generating relations (5.3.12), (5.3.14) and (5.3.17).

III. Taking \( x_3 = x_4 = 0, \xi_1 = \xi_2 = u = v = 1 \) and replacing \( x_1 \) by \( x \), \( x_2 \) by \( y \), \( x_5 \) by \( z \) and \( \xi_5 \) by \( \xi \) in generating relation (5.3.15), we get

\[
J_{m,n} \left( x \left( 1 + \frac{r}{x} \right), y, z; \xi \right) = \sum_{p=-\infty}^{\infty} J_p(r) J_{m+p,n}(x, y, z; \xi), \quad \left| \frac{r}{x} \right| < 1, \tag{5.4.4}
\]

101
where \( J_{m,n}(x, y, z; \xi) \) denotes 2I3V1PBF given by Eq. (5.2.3).

Further, taking \( \xi = 1 \) in generating relation (5.4.4), we obtain ([101]; p.308(4.3))

\[
J_{m,n}(x \left(1 + \frac{r}{x}\right), y, z) = \sum_{p=-\infty}^{\infty} J_p(r) J_{m+p,n}(x, y, z), \quad \left|\frac{r}{x}\right| < 1
\]

(5.4.5)

where \( J_{m,n}(x, y, z) \) denotes 2I3VBF given by Eq. (5.2.5). Similar results can be obtained from generating relations (5.3.12), (5.3.14) and (5.3.17).

IV. Taking \( x_2 = x_5 = x_1; \quad x_3 = x_4 = 0; \quad \xi_1 = \xi_2 = 1 \) and replacing \( x_1 \) by \( x \) and \( \xi_5 \) by \( \xi \) in generating relation (5.3.15), we get ([101]; p.309(4.5))

\[
\left(1 + \frac{r}{uw/x}ight)^{m/2} \left(1 + \frac{ruw}{x}\right)^{1/2} J_{m,n}(x \left(1 + \frac{r}{uw/x}\right)^{1/2} \left(1 + \frac{ruw}{x}\right)^{1/2}, x, x; \xi \left(1 + \frac{r}{uw/x}\right)^{-1/2} \left(1 + \frac{ruw}{x}\right)^{1/2})
\]

\[
= \sum_{p=-\infty}^{\infty} (-\nu)^p J_p(r) J_{m+p,n}(x; \xi) u^p, \quad \left|\frac{r}{uw/x}\right| < 1, \quad \left|\frac{ruw}{x}\right| < 1,
\]

(5.4.6)

where \( J_{m,n}(x; \xi) \) denotes 2II1V1PBF given by Eq. (5.2.7).

Further, taking \( \xi = u = \nu = 1 \) in generating relation (5.4.6), we obtain ([101]; p.309(4.5))

\[
J_{m,n}(x \left(1 + \frac{r}{x}\right)) = \sum_{p=-\infty}^{\infty} J_p(r) J_{m+p,n}(x), \quad \left|\frac{r}{x}\right| < 1.
\]

(5.4.7)

where \( J_{m,n}(x) \) denotes 2II1VBF given by Eq. (5.2.9). Similarly we can obtain results corresponding to generating relations (5.3.12), (5.3.14) and (5.3.17).

V. Taking \( x_2 = x_4 = x_5 = 0 \) and replacing \( x_1 \) by \( x \), \( x_3 \) by \( y \), \( \xi_1 \) by \( z \) and \( \xi_2 \) by \( z' \) in generating relation (5.3.12) and using Eq(5.2.10), we get ([98]; p.507(3.12))

\[
\left(1 - \frac{2c}{ux}ight)^{m/2} J_m \left(x \left(1 + \frac{2bu}{xz}\right)^{1/2} \left(1 - \frac{2c}{ux}\right)^{1/2}, z\right)\left(1 + \frac{2bu}{xz}\right) \left(1 - \frac{2c}{ux}\right)^{-1};
\]

\[
z' \left(1 + \frac{2bu}{xz}\right)^{-2} \left(1 - \frac{2c}{ux}\right)^{2};
\]

\[
= \sum_{p=-\infty}^{\infty} \frac{(-1)^p}{p!} \left(c^{-p+|p|/2} b^{p+|p|/2}\right) _0F_1[-; \, |p| + 1; \, bc] J_{m+p}(x; z|y; z'),
\]

(5.4.8)
where \( J_m(x; z/y; z') \) denotes 2V2PBF given by Eq. (5.2.12). Similar results can be obtained from generating relations (5.3.14), (5.3.15) and (5.3.17).

Further taking \( b = -c \) and \( z = z' = u = 1 \) in generating relation (5.4.8) and using Eq(5.2.13), we obtain ([98]; p.508(4.4))

\[
J_m \left( x \left( 1 - \frac{2c}{x} \right), y \right) = \sum_{p=\infty}^{\infty} \frac{(-1)^{3p+1}x^{|p|}}{|p|!} \frac{|p|}{c} \cdot F_1[-; |p|+1; \cdot c^2] J_{m+p}(x, y), \quad \left| \frac{2c}{x} \right| < 1.
\]

(5.4.9)

where \( J_m(x, y) \) denotes 2VBF given by Eq. (5.2.15). Similar results can be obtained from generating relations (5.3.14), (5.3.15) and (5.3.17).

VI. Taking \( x_2 = x_4 = x_5 = 0, \xi_1 = \xi_3 = -1; b = c \) and replacing \( x_1 \) by \( x \), \( x_3 \) by \( y \) in generating relation (5.3.17) and using Eq(5.2.16), we get ([98]; p.509(4.6))

\[
I_m \left( x \left( 1 - \frac{2c}{x} \right), y \right) = \sum_{p=\infty}^{\infty} \frac{(-c)^{|p|}}{|p|!} \cdot F_1[-; |p|+1; \cdot c^2] I_{m+p}(x, y), \quad \left| \frac{2c}{x} \right| < 1
\]

(5.4.10)

where \( I_m(x, y) \) denotes 2VMBF given by Eq. (5.2.18). Similar results can be obtained from generating relations (5.3.14), (5.3.15) and (5.3.17).

Further, taking \( y = 0 \) in generating relation (5.4.10) and using Eq(5.2.30), we get

\[
I_m \left( x \left( 1 - \frac{2c}{x} \right) \right) = \sum_{p=\infty}^{\infty} \frac{(-c)^{|p|}}{|p|!} \cdot F_1[-; |p|+1; \cdot c^2] I_{m+p}(x),
\]

(5.4.11)

where \( I_m(x) \) denotes MBF given by Eq. (5.2.32).

VII. Taking \( x_2 = x_4 = x_5 = \xi_1 = \xi_3 = b = 0; u = 1 \) and replacing \( x_1 \) by \( x \), \( x_3 \) by \( y \) in the generating relation (5.3.12) and using Eq(5.2.19), we obtain ([98]; p.508(4.3))

\[
\left( 1 - \frac{2c}{x} \right)^{m/2} H_m \left( x \frac{1 - \frac{2c}{x}}{2}, \frac{y}{2} \left( 1 - \frac{2c}{x} \right)^{-1} \right)
\]

\[
= \sum_{p=0}^{\infty} \frac{m}{p} (-c)^p H_{m-p} \left( \frac{x}{2}, \frac{y}{2} \right), \quad \left| \frac{2c}{x} \right| < 1
\]

(5.4.12)

where \( H_m(x, y) \) denotes 2VHKdFP given by Eq. (5.2.21). Similar results can be obtained from generating relations (5.3.14), (5.3.15) and (5.3.17).
Further, taking \( y = 0 \) in generating relation (5.4.12) and using Eq.(5.2.33), we get

\[
\left(1 - \frac{2c}{x}\right)^{m/2} H_m \left( x \frac{\left(1 - \frac{2c}{x}\right)^{1/2}}{2} \right) = \sum_{p=0}^{\infty} \binom{m}{p} (-c)^p H_{m-p} \left( x \frac{2c}{x} \right), \quad \left| \frac{2c}{x} \right| < 1,
\]

where \( H_m(x) \) denotes Hermite polynomials given by Eq. (1.4.29).

**VIII.** Taking \( x_2 = x_3 = x_4 = x_5 = 0; \, \xi_1 = 1 \) and replacing \( x_1 \) by \( x \) in generating relation (5.3.12), we obtain ([113]; p.62(3.29))

\[
\left(1 - \frac{2c}{ux}\right)^{m/2} J_m \left( x \left(1 - \frac{2c}{ux}\right)^{1/2} \left(1 + \frac{2bu}{x}\right)^{1/2} \right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} c^{|p|/2} \times b^{(p+|p|)/2} F_1(-; |p| + 1; bc) J_{m+p} \left( x \frac{2bu}{x} \right), \quad \left| \frac{2bu}{x} \right| < 1, \quad \left| \frac{2c}{ux} \right| < 1,
\]

where \( J_m(x) \) denotes ordinary Bessel function given by Eq. (1.4.19). Again taking \( x_2 = x_3 = x_4 = x_5 = 0; \, \xi_1 = u = 1 \) and replacing \( x_1 \) by \( x \) in generating relation (5.3.15), we get a generalization of Graf's addition theorem ([113], p.63(3.32))

\[
\left(1 + \frac{r}{ux}\right)^{m/2} J_m \left( x \left(1 + \frac{r}{ux}\right)^{1/2} \left(1 + \frac{r\nu}{x}\right)^{1/2} \right) = \sum_{p=-\infty}^{\infty} (-\nu)^p J_p(r) J_{m+p} \left( x \frac{r\nu}{x} \right), \quad \left| \frac{r\nu}{x} \right| < 1, \quad \left| \frac{r}{ux} \right| < 1.
\]

Similar results can be obtained from generating relations (5.3.14) and (5.3.17) respectively.

**5.5. CONCLUDING REMARKS**

We note that the expressions (5.3.11) are valid only for group elements \( g \) in a sufficiently small neighbourhood of the identity element of the Lie group \( T_3 \). However, we can also use the operators (5.3.4) to derive generating relations for 215V5PBF and related functions associated with group elements bounded away from the identity.

If \( f(x_i, u, \nu; \xi_1), (i = 1, 2, 3, 4, 5) \) is a solution of the equation \( C_{0,0}f = \omega^2 f \), i.e.,

\[
\left( -\frac{1}{\xi_1} \frac{\partial^2}{\partial x_1^2} - \frac{1}{\xi_1 x_1} \frac{\partial}{\partial x_1} + \frac{4x_3^2}{\xi_1 x_1} \frac{\partial^2}{\partial x_3^2} + \frac{4x_3}{\xi_1 x_1} (1 - m) \frac{\partial}{\partial x_3} + \frac{16\xi_3^2}{\xi_1 x_1} \frac{\partial^2}{\partial \xi_3^2} + \frac{8\xi_3}{\xi_1 x_1} (2 + m) \frac{\partial}{\partial \xi_3} \right) f = \omega^2 f,
\]

104
\[
\frac{\xi_5^2}{\xi_1 x_1^2} \frac{\partial^2}{\partial \xi_5^2} + \frac{\xi_5}{\xi_1 x_1^2} \left(1 - 2m \right) \frac{\partial}{\partial \xi_5} + \frac{m^2}{\xi_1 x_1^2}\right) f(x_i, u, v; \xi_i) = \omega^2 f(x_i, u, v; \xi_i), (i = 1, 2, 3, 4, 5),
\]

(5.5.1)

then the function \( T(g)f \) given by (5.3.10) satisfies the equation

\[
C_{0,0}(T(g)f) = \omega^2 (T(g)f).
\]

This follows from the fact that \( C_{0,0} \) commutes with the operators \( K^+, K^- \) and \( K^3 \).

Now if \( f \) is a solution of the equation

\[
(\alpha_1 K^+ + \alpha_2 K^- + \alpha_3 K^3)f(x_i, u, v; \xi_i) = \lambda f(x_i, u, v; \xi_i), \quad (i = 1, 2, 3, 4, 5),
\]

(5.5.2)

for constants \( \alpha_1, \alpha_2, \alpha_3 \) and \( \lambda \), then \( T(g)f \) is a solution of the equation

\[
[T(g)(\alpha_1 K^+ + \alpha_2 K^- + \alpha_3 K^3)T(g^{-1})][T(g)f] = \lambda[T(g)f].
\]

(5.5.3)

The inner automorphism \( \mu_g \) of Lie group \( T_3 \) defined by

\[
\mu_g(h) = ghg^{-1}, \quad h \in T_3,
\]

(5.5.4)

induces an automorphism \( \mu_g^* \) of Lie algebra \( T_3 \) where

\[
\mu_g^*(\alpha) = g\alpha g^{-1}, \quad \alpha \in T_3.
\]

If \( \alpha = \alpha_1 J^+ + \alpha_2 J^- + \alpha_3 J^3 \) where \( J^+, J^- \) and \( J^3 \) are given by Eq. (1.6.28) and \( g \) is given by Eq. (1.6.25), then we have

\[
\mu_g^*(\alpha) = (\alpha_1 e^\tau - b\alpha_3)J^+ + (\alpha_2 e^{-\tau} + c\alpha_3)J^- + \alpha_3 J^3,
\]

(5.5.5)

as a consequence of which, we can write

\[
T(g)(\alpha_1 K^+ + \alpha_2 K^- + \alpha_3 K^3)T(g^{-1}) = (\alpha_1 e^\tau - b\alpha_3 K^+ + (\alpha_2 e^{-\tau} + c\alpha_3)K^- + \alpha_3 K^3.
\]

(5.5.6)

To give an example of the application of these remarks, we consider the function

\[
f(x_i, u, v; \xi_i) = J_{m,n}(x_i; \xi_i)u^m v^n, \quad (i = 1, 2, 3, 4, 5), \quad m \in \mathbb{C}.
\]

Since \( C_{0,0}f = f \) and \( K^3f = m f \), so the function

\[
[T(g)f](x_1, x_2, x_3, x_4, x_5, u, v; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = e^{\text{ext}} \left( \frac{u^2 - (2c u/x_1)}{1 + (2b u/\xi_1 x_1)} \right)^{m/2} v^n
\]

105
\[ \times J_{m,n} \left( \left( x_1 + \frac{2bu}{\xi_1} \right)^{1/2} \left( x_1 - \frac{2c}{u} \right)^{1/2}, x_2, x_3 \left( 1 + \frac{2bu}{\xi_1 x_1} \right) \left( 1 - \frac{2c}{x_1 u} \right)^{-1}, x_4, x_5; \right) \]
\[ \xi_1, \xi_2, \xi_3 \left( 1 + \frac{2bu}{\xi_1 x_1} \right)^{-2} \left( 1 - \frac{2c}{x_1 u} \right)^2, \xi_4, \xi_5 \left( 1 + \frac{2bu}{\xi_1 x_1} \right)^{1/2} \left( 1 - \frac{2c}{x_1 u} \right)^{-1/2}, \right) \]
\[ \] satisfies the equations
\[ C_{0,0}[T(g)f] = [T(g)f], \] (5.5.8)
\[ (-bK^+ + cK^- + K^3)[T(g)f] = m[T(g)f]. \] (5.5.9)

For \( \tau = b = 0 \) and \( c = -1 \), we can express the function (5.5.7) in the form
\[ h(x_1, x_2, x_3, x_4, x_5, u, v; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \left( u^2 + \frac{2u}{x_1} \right)^{m/2} v^n J_{m,n} \left( \left( \frac{x_2^2 + 2x_1^2}{x_1 u} \right)^{1/2}, x_2, \right. \]
\[ x_3 \left( 1 + \frac{2}{ux_1} \right)^{-1}, x_4, x_5; \xi_1, \xi_2, \xi_3 \left( 1 + \frac{2}{x_1 u} \right)^{2}, \xi_4, \xi_5 \left( 1 + \frac{2}{x_1 u} \right)^{-1/2}. \] (5.5.10)

Now using the Laurent expansion
\[ h(x_i, u, v; \xi_i) = \sum_{k=-\infty}^{\infty} h_{k,n}(x_i; \xi_i) u^k v^n, \quad (i = 1, 2, 3, 4, 5) \quad |x_1 u| < 2, \]
in Eq. (5.5.7), we note that \( h_{k,n}(x_i; \xi_i) \) (\( i = 1, 2, 3, 4, 5 \)), is a solution of differential equation (5.1.7) for each integer \( k \). Since the function \( h(x_i, u, v; \xi_i), (i = 1, 2, 3, 4, 5) \) is bounded for \( x_i = 0, (i = 1, 2, 3, 4, 5) \), therefore we have
\[ h_{k,n}(x_i; \xi_i) = c_k J_{k,n}(x_i; \xi_i), \quad (i = 1, 2, 3, 4, 5), \quad c_k \in \mathbb{C}. \]

Thus
\[ h_{k,n}(x_i, u, v; \xi_i) = \sum_{k=-\infty}^{\infty} c_k J_{k,n}(x_i; \xi_i) u^k v^n, \quad (i = 1, 2, 3, 4, 5). \] (5.5.11)

Now from Eq. (5.5.9), we have
\[ (-K^- + K^3)h(x_i, u, v; \xi_i) = m h(x_i, u, v; \xi_i), \quad (i = 1, 2, 3, 4, 5) \]
and therefore it follows that

\[ c_{k+1} = (m - k)c_k. \]

Further, taking \( x_i = 0 \) (\( i = 1, 2, 3, 4, 5 \)) in Eq(5.5.10), and using Eq(5.5.11), we get

\[ c_0 = \frac{1}{\Gamma(m+1)}, \]

and hence

\[ c_k = \frac{1}{\Gamma(m - k + 1)}. \]

Thus we obtain the following result

\[
\left( u^2 + \frac{2u}{x_1} \right)^{m/2} J_{m,n} \left( \left( x_1^2 + \frac{2x_1}{u} \right)^{1/2}, x_2, x_3 \left( 1 + \frac{2}{x_1u} \right)^{-1}, x_4, x_5; \xi_1, \xi_2, \xi_3 \left( 1 + \frac{2}{x_1u} \right)^{2} \right),
\]

\[
\xi_4, \xi_5 \left( 1 + \frac{2}{x_1u} \right)^{-1/2} \right) = \sum_{k=-\infty}^{\infty} \frac{J_{k,n}(x_1, x_2, x_3, x_4, x_5; \xi_1, \xi_2, \xi_3, \xi_4, \xi_5)u^k}{\Gamma(m - k + 1)}, \quad |x_1u| < 2,
\]

which is not a special case of generating relation (5.3.12). The result (5.5.12) is obtained by using operators (5.3.4). We can obtain another result by using operators (5.3.5).

The theory of BF is rich and wide, and certainly provides an inexhaustible field of research. A large number of functions are recognized as belonging to the BF family. Many variable BF were introduced at the beginning of the last century, see for example [2,70], forgotten for many years and reconsidered within the context of various physical applications at the end of the last century, see for example [3,37].

We have considered GBF within the group representation formalism. The 215V5PBF \( J_{m,n}(x_i; \xi_i) \) (\( i = 1, 2, 3, 4, 5 \)) appeared as basis functions for a realization of the representation \( Q(-1, m_0) \) of the Lie algebra \( \mathcal{F}_5 \). The analysis presented in this Chapter confirms the possibility of extending this approach to other useful forms of GBF.