CHAPTER 4

LIE-THEORETIC GENERATING RELATIONS INVOLVING MULTI-VARIABLE TRICOMI FUNCTIONS

4.1. INTRODUCTION

The function \[ C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r! (n + r)!} \] (4.1.1)
is a Bessel like function known as Tricomi function and is characterized by the following link with the ordinary Bessel function \( J_n(x) \) (1.4.19):
\[ C_n(x) = x^{-n/2} J_n(2\sqrt{x}). \] (4.1.2)
or
\[ J_n(x) = \left( \frac{x}{2} \right)^n C_n\left( \frac{x^2}{4} \right). \] (4.1.3)

The Tricomi function \( C_n(x) \) satisfies the generating function
\[ \exp \left( t - \frac{x}{b} \right) \sum_{n=-\infty}^{\infty} C_n(x) \frac{t^n}{n!}, \] (4.1.4)
which yield the recurrences
\[ \frac{d}{dx} C_n(x) = -C_{n+1}(x), \]
\[ x C_{n+1}(x) - n C_n(x) + C_{n-1}(x) = 0. \] (4.1.5)

On combining the above recurrence relations, we get the following differential equation satisfied by \( C_n(x) \):
\[ \left( x \frac{d^2}{dx^2} + (n + 1) \frac{d}{dx} + 1 \right) C_n(x) = 0. \] (4.1.6)

Equation (4.1.6) ensures that \( C_n(x) \) are eigen functions of the operator
\[ \mathcal{O}_n(x) = -\frac{d}{dx} x \frac{d}{dx} - n \frac{d}{dx}. \] (4.1.7)
The above operator, whose importance has been recognized within the frame work of theory of monomiality of Laguerre polynomials [12], can be viewed as a generalization of the ordinary derivative so that $C_n(x)$ can be considered a generalization of the ordinary exponential function, it does not possess the semigroup property and indeed $C_m(x + y) \neq C_m(x)C_m(y)$. This fact far from being a limitation, allows the possibility of introducing other families of Bessel like functions.

The study of the properties of multi-variable generalized special functions has provided new means of analysis for the derivation of the solution of large classes of partial differential equations often encountered in physical problems. The relevance of the special functions in physics is well established. Most of the special functions of mathematical physics as well as their generalizations have been suggested by physical problems.

In order to further stress the usefulness of the generalized special functions, Dattoli et al. [25] have introduced the three variable two parameter extension of Tricomi functions defined as:

$$C_n(x, y, z; \tau_1, \tau_2) = \sum_{l=-\infty}^{\infty} \tau_1^l \tau_2^l C_{n-2l}(x, y; \tau_1) C_l(z). \quad (4.1.8)$$

The generating function for 3-variable 2-parameter Tricomi functions (3V2PTF) $C_n(x, y, z; \tau_1, \tau_2)$ is given as:

$$\exp \left( t - \frac{x}{t} + t^2 \tau_1 - \frac{y}{t^2 \tau_1} + \frac{t^3 \tau_2}{t^3 \tau_2} \right) = \sum_{n=-\infty}^{\infty} t^n C_n(x, y, z; \tau_1, \tau_2). \quad (4.1.9)$$

The 3V2PTF $C_n(x, y, z; \tau_1, \tau_2)$ is related to the 3-variable 2-parameter Bessel function 3V2PBF $J_n(x, y, z; \tau_1, \tau_2)$ by [25]

$$C_n(x, y, z; \tau_1, \tau_2) = x^{-n/2} J_n \left( 2\sqrt{x}, 2\sqrt{y}, 2\sqrt{z}; \frac{x}{\sqrt{y}}, \tau_1, \tau_2 \sqrt{\frac{x^3}{z}} \right), \quad (4.1.10)$$

or

$$J_n(x, y, z; \tau_1, \tau_2) = \left( \frac{x}{2} \right)^n C_n \left( \frac{x^2}{4}, \frac{y^2}{4}, \frac{z^2}{4}; \frac{2y}{x^2} \tau_1, \frac{4z}{x^3} \tau_2 \right), \quad (4.1.11)$$
where $J_n(x, y, z; \tau_1, \tau_2)$ is given by [25]

$$J_n(x, y, z; \tau_1, \tau_2) = \sum_{l=-\infty}^{\infty} \tau_2^l J_{n-l}(x, y; \tau_1) J_l(z).$$

(4.1.12)

with the generating function

$$\sum_{n=-\infty}^{\infty} t^n J_n(x, y, z; \tau_1, \tau_2) = \exp \left( \frac{x}{2} \left( t - \frac{1}{t} \right) + \frac{y}{2} \left( t^2 \tau_1 - \frac{1}{t^2 \tau_1} \right) + \frac{z}{2} \left( t^3 \tau_2 - \frac{1}{t^3 \tau_2} \right) \right).$$

(4.1.13)

In this Chapter, we derive generating relations involving 3V2PTF $C_n(x, y, z; \tau_1, \tau_2)$ by using Lie-algebraic methods. In Section 4.2, we give a review of the basic properties of 3V2PTF $C_n(x, y, z; \tau_1, \tau_2)$ and their special cases. In Section 4.3, we derive generating relations involving 3V2PTF by using the representation $Q(w, m_0)$ of the Lie algebra $\mathfrak{sl}_3$. In Section 4.4, we obtain certain new and known generating relations involving various forms of Tricomi and Bessel functions. Finally, in Section 4.5, some concluding remarks are given.

4.2. PROPERTIES AND SPECIAL CASES OF 3V2PTF $C_n(x, y, z; \tau_1, \tau_2)$

The 3V2PTF $C_n(x, y, z; \tau_1, \tau_2)$ defined by Eqs. (4.1.8), (4.1.9) satisfy the following differential and pure recurrence relations:

$$\frac{\partial}{\partial x} C_n(x, y, z; \tau_1, \tau_2) = - C_{n+1}(x, y, z; \tau_1, \tau_2),$$

$$\frac{\partial}{\partial y} C_n(x, y, z; \tau_1, \tau_2) = - \frac{1}{\tau_1} C_{n+2}(x, y, z; \tau_1, \tau_2),$$

$$\frac{\partial}{\partial z} C_n(x, y, z; \tau_1, \tau_2) = - \frac{1}{\tau_2} C_{n+3}(x, y, z; \tau_1, \tau_2),$$

$$\frac{\partial}{\partial \tau_1} C_n(x, y, z; \tau_1, \tau_2) = C_{n-2}(x, y, z; \tau_1, \tau_2) + \frac{y}{\tau_1^2} C_{n+2}(x, y, z; \tau_1, \tau_2),$$

$$\frac{\partial}{\partial \tau_2} C_n(x, y, z; \tau_1, \tau_2) = C_{n-3}(x, y, z; \tau_1, \tau_2) + \frac{z}{\tau_2^2} C_{n+3}(x, y, z; \tau_1, \tau_2)$$

(4.2.1)
and

\[ nC_n(x, y; z; \tau_1, \tau_2) = C_{n-1}(x, y; z; \tau_1, \tau_2) + xC_{n+1}(x, y; z; \tau_1, \tau_2) + 2\tau_1C_{n-2}(x, y; z; \tau_1, \tau_2), \]
\[ + 3\tau_2C_{n-3}(x, y; z; \tau_1, \tau_2) + \frac{3z}{\tau_2}C_{n+3}(x, y; z; \tau_1, \tau_2). \]  

(4.2.2)

The differential equation satisfied by 3V2PTF \( C_n(x, y; z; \tau_1, \tau_2) \) is

\[ \left( -x \frac{\partial^2}{\partial x^2} + \left( 1 + n \right) \frac{\partial}{\partial x} + 2\tau_1 \frac{\partial^2}{\partial x \partial \tau_1} + 3\tau_2 \frac{\partial^2}{\partial x \partial \tau_2} - 1 \right) C_n(x, y; z; \tau_1, \tau_2) = 0. \]  

(4.2.3)

We note the following special cases of 3V2PTF \( C_n(x, y; z; \tau_1, \tau_2) \):

(1) \[ C_n \left( \frac{x^2}{4}, \frac{y^2}{4}, \frac{z^2}{x^2}, \frac{2y \tau_1}{x^2}, \frac{3z \tau_2}{x^3} \right) = \left( \frac{x}{2} \right)^{-n} J_n(x, y; z; \tau_1, \tau_2) \]  

(4.2.4)

where \( J_n(x, y; z; \tau_1, \tau_2) \) is given by Eqs. (4.1.12), (4.1.13).

(2) \[ C_n \left( \frac{x^2}{4}, \frac{y^2}{4}, \frac{z^2}{x^2}, \frac{2y}{x^2}, \frac{4z}{x^3} \right) = \left( \frac{x}{2} \right)^{-n} J_n(x, y, z), \]  

(4.2.5)

where \( J_n(x, y, z) \) denotes 3-variable Bessel function (3VBF) defined by the generating function [37]

\[ \sum_{n=-\infty}^{\infty} J_n(x, y, z)t^n = \exp \left( \frac{x}{2} \left( t - \frac{1}{t} \right) + \frac{y}{2} \left( t^2 - \frac{1}{t^2} \right) + \frac{z}{2} \left( t^3 - \frac{1}{t^3} \right) \right). \]  

(4.2.6)

(3) \[ C_n(x, y, z; 1, 1) = C_n(x, y, z), \]  

(4.2.7)

where \( C_n(x, y, z) \) denotes 3-variable Tricomi function (3VTF) defined by the generating function

\[ \sum_{n=-\infty}^{\infty} C_n(x, y, z)t^n = \exp \left( t - \frac{x}{t} + t^2 - \frac{y}{t^2} + t^3 - \frac{z}{t^3} \right). \]  

(4.2.8)

(4) \[ C_n(x, y, 1; \tau_1 \to \tau, 0) = C_n(x, y; \tau), \]  

(4.2.9)

where \( C_n(x, y; \tau) \) denotes 2-variable 1-parameter Tricomi function (2V1PTF) defined by the generating function ([25]; p.221, Eq.(9)).

\[ \sum_{n=-\infty}^{\infty} C_n(x, y; \tau)t^n = \exp \left( t - \frac{x}{t} + \tau t^2 - \frac{y}{\tau t^2} \right). \]  

(4.2.10)
\begin{align}
(5) \quad & C_n \left( \frac{x^2}{4}, \frac{y^2}{4}, 1; \frac{2yt}{x^2}, 0 \right) = \left( \frac{x}{2} \right)^{-n} J_n(x, y; \tau), \quad (4.2.11) \\
\text{where } & J_n(x, y; \tau) \text{ denotes 2-variable 1-parameter Bessel functions (2V1PBF) defined by the generating function ([37]; p.176 Eq.(1.2))}
\sum_{n=-\infty}^{\infty} J_n(x, y; \tau)t^n = \exp \left( \frac{x}{2} \left( t - \frac{1}{t} \right) + \frac{y}{2} \left( \tau t^2 - \frac{1}{\tau t^2} \right) \right). \quad (4.2.12)
\end{align}

\begin{align}
(6) \quad & C_n \left( \frac{x^2}{4}, \frac{y^2}{4}, 1; \frac{2y}{x^2}, 0 \right) = \left( \frac{x}{2} \right)^{-n} J_n(x, y), \quad (4.2.13) \\
\text{where } & J_n(x, y) \text{ denotes 2-variable Bessel functions (2VBF) defined by the generating function ([41]; p.24 Eq.(1.8(a)))}
\sum_{n=-\infty}^{\infty} J_n(x, y)t^n = \exp \left( \frac{x}{2} \left( t - \frac{1}{t} \right) + \frac{y}{2} \left( t^2 - \frac{1}{t^2} \right) \right). \quad (4.2.14)
\end{align}

\begin{align}
(7) \quad & C_n \left( x, y, 1; 1, 0 \right) = C_n(x, y), \quad (4.2.15) \\
\text{where } & C_n(x, y) \text{ denotes 2-variable Tricomi functions (2VTF) defined by the generating function}
\sum_{n=-\infty}^{\infty} C_n(x, y)t^n = \exp \left( t - \frac{x}{t} + t^2 - \frac{y}{t^2} \right). \quad (4.2.16)
\end{align}

\begin{align}
(8) \quad & C_n \left( \frac{x^2}{4}, 1, 1; 0, 0 \right) = \left( \frac{x}{2} \right)^{-n} J_n(x), \quad (4.2.17) \\
\text{where } & J_n(x) \text{ denotes ordinary Bessel function defined by the generating function (1.4.19)}
\end{align}

\begin{align}
(9) \quad & C_n \left( x, 1, 1; 0, 0 \right) = C_n(x), \quad (4.2.18) \\
\text{where } & C_n(x) \text{ denotes Tricomi function defined by the generating function (4.1.4)}
\end{align}
4.3. REPRESENTATION $Q(\omega, m_0)$ OF $T_3$ AND GENERATING RELATIONS

Miller [113] have determined realizations of the irreducible representation $Q(\omega, m_0)$ of $T_3$ where $\omega, m_0 \in \mathbb{C}$ such that $\omega \neq 0$ and $0 \leq \text{Re} \ m_0 < 1$. The spectrum $S$ of this representation is the set \{ $m_0 + k : k \text{ an integer}$ \} and the representation space $V$ has a basis $(f_m)_{m \in S}$, such that

\[ J^3 f_m = m f_m, \quad J^+ f_m = \omega f_{m+1}, \quad J^- f_m = \omega f_{m-1}, \]

\[ C_{0,0} f_m = (J^+ J^-) f_m = \omega^2 f_m, \quad \omega \neq 0. \quad (4.3.1) \]

The commutation relations satisfied by the operators $J^3, J^\pm$ are

\[ [J^3, J^+] = J^+, \quad [J^3, J^-] = - J^-, \quad [J^+, J^-] = 0. \quad (4.3.2) \]

In order to find the realizations of this representation on spaces of functions of two complex variables $x$ and $y$, Miller ([113]; pp. 59-60) has taken the functions $f_m(x, y) = Z_m(x)e^{ny}$, such that relations (4.3.1) are satisfied for all $m \in S$, where the differential operators $J^3, J^\pm$ are given by

\[ J^3 = \frac{\partial}{\partial y}, \]

\[ J^+ = e^y \left[ \frac{\partial}{\partial x} - \frac{1}{x} \frac{\partial}{\partial y} \right], \]

\[ J^- = e^{-y} \left[ - \frac{\partial}{\partial x} - \frac{1}{x} \frac{\partial}{\partial y} \right]. \quad (4.3.3) \]

In particular, we look for the functions

\[ f_m(x, y, z; \tau_1, \tau_2) = Z_m(x, y, z; \tau_1, \tau_2) t^m, \quad (4.3.4) \]

such that

\[ K^3 f_m = m f_m, \quad K^+ f_m = \omega f_{m+1}, \quad K^- f_m = \omega f_{m-1}, \]

\[ C_{0,0} f_m = (K^+ K^-) f_m = \omega^2 f_m, \quad (\omega \neq 0; \ m \in S). \quad (4.3.5) \]

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The set of operators \( \{ K^3, K^+, K^- \} \) satisfy the commutation relations identical to (4.3.2).

There are numerous possible solutions of Eq. (4.3.5). We assume that the set of linear differential operators \( \{ K^3, K^+, K^- \} \) takes the form

\[
\begin{align*}
K^3 &= t \frac{\partial}{\partial t}, \\
K^+ &= t \frac{\partial}{\partial x}, \\
K^- &= -\frac{x}{t} \frac{\partial}{\partial x} + \frac{2\tau_1}{t} \frac{\partial}{\partial \tau_1} + \frac{3\tau_2}{t} \frac{\partial}{\partial \tau_2} - \frac{\partial}{\partial t}.
\end{align*}
\]  

(4.3.6)

The operators in Eqs. (4.3.6) satisfy the commutation relations (4.3.2).

In terms of the functions \( Z_m(x, y, z; \tau_1, \tau_2) \) and using operators (4.3.6), relations (4.3.5) reduce to

(i) \[
\frac{\partial}{\partial x} Z_m(x, y, z; \tau_1, \tau_2) = \omega Z_{m+1}(x, y, z; \tau_1, \tau_2),
\]

(ii) \[
\left[ -x \frac{\partial}{\partial x} + 2\tau_1 \frac{\partial}{\partial \tau_1} + 3\tau_2 \frac{\partial}{\partial \tau_2} - m \right] Z_m(x, y, z; \tau_1, \tau_2) = \omega Z_{m-1}(x, y, z; \tau_1, \tau_2),
\]

(iii) \[
\left[ -x \frac{\partial^2}{\partial x^2} + 2\tau_1 \frac{\partial^2}{\partial x \partial \tau_1} + 3\tau_2 \frac{\partial^2}{\partial x \partial \tau_2} - (m + 1) \frac{\partial}{\partial x} \right] Z_m(x, y, z; \tau_1, \tau_2)
= \omega^2 Z_m(x, y, z; \tau_1, \tau_2).
\]

(4.3.7)

We can take \( \omega = -1 \), without any loss of generality. For this choice of \( \omega \) and in terms of the functions \( Z_m(x) \), relations (4.3.1) become ([113]; p.60 (3.25))

(i) \[
\left[ \frac{d}{dx} - \frac{m}{x} \right] Z_m(x) = -Z_{m+1}(x),
\]

(ii) \[
\left[ \frac{d}{dx} + \frac{m}{x} \right] Z_m(x) = Z_{m-1}(x),
\]

(iii) \[
\left[ -\frac{d^2}{dx^2} - \frac{1}{x} \frac{d}{dx} + \frac{m^2}{x^2} \right] Z_m(x) = Z_m(x).
\]

(4.3.8)

We observe that (i) and (ii) of Eqs. (4.3.8) agree with the conventional recurrence relations for Bessel functions \( J_m(x) \) and (iii) coincides with the differential equation for \( J_m(x) \). Thus we see that \( Z_m(x) = J_m(x) \) is a solution of Eqs. (4.3.8) for all \( m \in S \).
Similarly, we see that for $\omega = -1$, (iii) of Eqs. (4.3.7) coincides with the differential equation (4.2.3) of 3V2PTF $C_m(x, y, z; \tau_1, \tau_2)$. In fact, for all $m \in S$ the choice for $Z_m(x, y, z; \tau_1, \tau_2) = C_m(x, y, z; \tau_1, \tau_2)$ satisfy Eqs. (4.3.7). Thus we conclude that the functions $f_m(x, y, z, t; \tau_1, \tau_2) = C_m(x, y, z; \tau_1, \tau_2)t^m$, $m \in S$ form a basis for a realization of the representation $Q(-1, m_0)$ of $T_3$. By using Theorem 1.7.1, this representation of $T_3$ can be extended to a local multiplier representation of $T_3$. Using operators (4.3.6), the local multiplier representation $T(g)$, $g \in T_3$ defined on $\mathcal{F}$, the space of all functions analytic in a neighbourhood of the point $(x^0, y^0, z^0, t^0, \tau^0_1, \tau^0_2) = (1, 0, 0, 1, 1, 1)$, takes the form

$$
[T(\exp b\mathcal{J}^+)f](x, y, z, t; \tau_1, \tau_2) = f\left( x\left(1 + \frac{bt}{x}\right), y, z, t; \tau_1, \tau_2 \right).
$$

$$
[T(\exp c\mathcal{J}^-)f](x, y, z, t; \tau_1, \tau_2)
= f\left( x\left(1 - \frac{c}{t}\right), y, z, t\left(1 - \frac{c}{t}\right)^{-1}; \tau_1\left(1 - \frac{c}{t}\right)^{-2}, \tau_2\left(1 - \frac{c}{t}\right)^{-3} \right),
$$

$$
[T(\exp a\mathcal{J}^3)f](x, y, z, t; \tau_1, \tau_2) = f(x, y, z, te^a; \tau_1, \tau_2),
$$

(4.3.9)

If $g \in T_3$ is given by Eq. (1.6.25), we find

$$
T(g) = T(\exp b\mathcal{J}^+)T(\exp c\mathcal{J}^-)T(\exp a\mathcal{J}^3)
$$

and therefore we obtain

$$
[T(g)f](x, y, z, t; \tau_1, \tau_2) = f\left( x\left(1 + \frac{bt}{x}\right), y, z, te^a\left(1 - \frac{c}{t}\right); \tau_1\left(1 - \frac{c}{t}\right)^{-2}, \tau_2\left(1 - \frac{c}{t}\right)^{-3} \right), \quad \frac{bt}{x} < 1, \quad \left|\frac{c}{t}\right| < 1.
$$

(4.3.10)

The matrix elements of $T(g)$ with respect to the analytic basis $(f_m)_{m \in S}$ are the functions $A_{lk}(g)$ uniquely determined by $Q(-1, m_0)$ of $T_3$ and are defined by

$$
[T(g)f_m(x, y, z, t; \tau_1, \tau_2)] = \sum_{l = -\infty}^{\infty} A_{lk}(g)f_{m+l}(x, y, z, t; \tau_1, \tau_2), \quad k = 0, \pm 1, \pm 2, \pm 3 \ldots.
$$

(4.3.11)

Therefore, we prove the following result:
Theorem 4.3.1. The following generating equation holds
\[
\left(1 - \frac{c}{t}\right)^m C_m \left(x \left(1 + \frac{bt}{x}\right) \left(1 - \frac{c}{t}\right), y, z; \tau_1 \left(1 - \frac{c}{t}\right)^{-2}, \tau_2 \left(1 - \frac{c}{t}\right)^{-3}\right)
= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} b^{(p+|p|)/2} c^{(-p+|p|)/2} {}_0F_1[-; |p| + 1; bc] C_{m+p}(x, y, z; \tau_1, \tau_2)t^p,
\]
\[
\left|\frac{bt}{x}\right| < 1, \quad \left|\frac{c}{t}\right| < 1. \quad (4.3.12)
\]
Proof. Using (4.3.10), we obtain
\[
\exp(m\tau) \left(1 - \frac{c}{t}\right)^m C_m \left(x \left(1 + \frac{bt}{x}\right) \left(1 - \frac{c}{t}\right), y, z; \tau_1 \left(1 - \frac{c}{t}\right)^{-2}, \tau_2 \left(1 - \frac{c}{t}\right)^{-3}\right)
= \sum_{l=-\infty}^{\infty} A_{l,m-m_0}(g)C_{m_0+l}(x, y, z; \tau_1, \tau_2)t^{m_0+l-m} \quad (4.3.13)
\]
and the matrix elements \(A_{lk}(g)\) are given by ([113]; p.56(3.12)'),
\[
A_{lk}(g) = \exp((m_0 + k)a) \frac{(-1)^{|k-l|}}{|k-l|!} b^{(l-k+|k-l|)/2} c^{(k-l+|k-l|)/2} {}_0F_1[-; |k-l| + 1; bc], \quad (4.3.14)
\]
valid for all integral values of \(l, k\) and where \(0F_1\) denotes confluent hypergeometric function [2].

Substituting the value of \(A_{lk}(g)\) given by (4.3.14) into (4.3.13) and simplifying we obtain result (4.3.12).

Corollary 4.3.1. The following generating equation holds
\[
\left(1 + \frac{r}{2\nu t}\right)^m C_m \left(x \left(1 + \frac{\nu vt}{2x}\right) \left(1 + \frac{r}{2\nu t}\right), y, z; \tau_1 \left(1 + \frac{r}{2\nu t}\right)^{-2}, \tau_2 \left(1 + \frac{r}{2\nu t}\right)^{-3}\right)
= \sum_{p=-\infty}^{\infty} (-\nu)^p J_p(r) C_{m+p}(x, y, z; \tau_1, \tau_2)t^p, \quad \left|\frac{\nu vt}{2x}\right| < 1, \quad \left|\frac{r}{2\nu t}\right| < 1. \quad (4.3.15)
\]
Proof. If \(bc \neq 0\), we can introduce the co-ordinates \(r, \nu\) such that \(b = \frac{\nu v}{2}\) and \(c = -\left(\frac{r}{2\nu}\right)^2\), with these new co-ordinates the matrix elements (4.3.14) can be expressed as
\[
A_{lk}(g) = \exp((m_0 + k)a) (-\nu)^{l-k} J_{l-k}(r), \quad k = 0, \pm 1, \pm 2 \cdots \quad (4.3.16)
\]
and generating relation (4.3.12) yields (4.3.15).
4.4. APPLICATIONS

We discuss some applications of the generating relations obtained in the preceding section.

I. Taking \( c = 0 \) and \( t = 1 \) in generating relation (4.3.12), we get

\[
C_m((x + b), y, z; \tau_1, \tau_2) = \sum_{p=0}^{\infty} \frac{(-b)^p}{p!} C_{m+p}(x, y, z; \tau_1, \tau_2), \quad \left| \frac{b}{x} \right| < 1. \tag{4.4.1}
\]

Again, taking \( b = 0 \) and \( t = 1 \) in generating relation (4.3.12), we get

\[
(1-c)^m C_m(x(1-c), y, z; \tau_1(1-c)^{-2}, \tau_2(1-c)^{-3}) = \sum_{p=0}^{\infty} \frac{c^p}{p!} C_{m-p}(x, y, z; \tau_1, \tau_2), \quad |c| < 1. \tag{4.4.2}
\]

II. Replacing \( x \) by \( \frac{x^2}{4} \), \( y \) by \( \frac{y^2}{4} \), \( z \) by \( \frac{z^2}{4} \), \( \tau_1 \) by \( \frac{2y}{x^2} \), \( \tau_2 \) by \( \frac{3z}{x^3} \) and \( t \) by \( \frac{xt}{2} \) in generating relation (4.3.12) and using Eq. (4.2.4), we get

\[
\left( \frac{1-(2c/xt)}{1+(2bt/x)} \right)^{m/2} J_m \left( x \left( 1+\frac{2bt}{x} \right)^{1/2} \left( 1-\frac{2c}{xt} \right)^{1/2} ; y, z ; \tau_1 \left( 1+\frac{2bt}{x} \right)^{-1} \left( 1-\frac{2c}{xt} \right)^{-1} \right),
\]

\[
\tau_2 \left( 1+\frac{2bt}{x} \right)^{3/2} \left( 1-\frac{2c}{xt} \right)^{-3/2} = \sum_{p=\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} \left[ \frac{(p+|p|)!}{|p|!} \right] \times c^{(-p+|p|)/2} \left[ |p| + 1 + bc \right] J_{m+p}(x, y, z; \tau_1, \tau_2) t^p, \quad \left| \frac{2bt}{x} \right| < 1, \quad \left| \frac{2c}{xt} \right| < 1, \tag{4.4.3}
\]

where \( J_m(x, y, z; \tau_1, \tau_2) \) denotes the 3V2PBF defined by Eqs. (4.1.12) and (4.1.13). Similarly replacing \( x \) by \( \frac{x^2}{4} \), \( y \) by \( \frac{y^2}{4} \), \( z \) by \( \frac{z^2}{4} \), \( \tau_1 \) by \( \frac{2y}{x^2} \), \( \tau_2 \) by \( \frac{3z}{x^3} \) and \( t \) by \( \frac{xt}{2} \) in generating relation (4.3.15) and using Eq. (4.2.4), we get

\[
\left( 1+\frac{(r/vxt)}{1+(rvt/x)} \right)^{m/2} J_m \left( x \left( 1+\frac{rvt}{x} \right)^{1/2} \left( 1+\frac{r}{vxt} \right)^{1/2} ; y, z ; \tau_1 \left( 1+\frac{rvt}{x} \right)^{-1} \left( 1+\frac{r}{vxt} \right)^{-1} \right),
\]

\[
\tau_2 \left( 1+\frac{rvt}{x} \right)^{3/2} \left( 1+\frac{r}{vxt} \right)^{-3/2} = \sum_{p=-\infty}^{\infty} (-v)^p J_p(r) J_{m+p}(x, y, z; \tau_1, \tau_2) t^p, \quad \left| \frac{rvt}{x} \right| < 1, \quad \left| \frac{r}{vxt} \right| < 1. \tag{4.4.4}
\]
Further, taking \( \tau_1 = \tau_2 = 1 \) in generating relation (4.4.3), we get

\[
\left(\frac{1 - (2c/xt)}{1 + (2b/tx)}\right)^{m/2} J_m\left(x\left(1 + \frac{2bt}{x}\right)^{1/2} \left(1 - \frac{2c}{xt}\right)^{1/2} , y, z\right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} b^{(p+|p|)/2} \Gamma(p+1) c^{(-p+|p|)/2} 0F_1[-; |p| + 1; bc] J_{m+p}(x, y, z) t^p, \quad \left|\frac{2bt}{x}\right| < 1, \quad \left|\frac{2c}{xt}\right| < 1,
\]

(4.4.5)

where \( J_m(x, y, z) \) denotes 3VBF given by Eq. (4.4.2).

Similar result can be obtained from generating relation (4.4.4).

III. Taking \( \tau_1 = \tau_2 = 1 \) in generating relation (4.3.12) and using Eq. (4.2.7), we get

\[
\left(1 - \frac{c}{t}\right)^{m} C_m\left(x\left(1 + \frac{bt}{x}\right) \left(1 - \frac{c}{t}\right)^2 , y, \tau \left(1 - \frac{c}{t}\right)^{-2}\right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} b^{(p+|p|)/2} \Gamma(p+1) c^{(-p+|p|)/2} 0F_1[-; |p| + 1; bc] C_{m+p}(x, y, z) t^p, \quad \left|\frac{bt}{x}\right| < 1, \quad \left|\frac{c}{t}\right| < 1,
\]

(4.4.6)

where \( C_m(x, y, z) \) denotes 3VTF given by Eq. (4.2.8). Similar results can be obtained from generating relations (4.3.15).

IV. Taking \( z = 1, \tau_2 = 0 \) and replacing \( \tau_1 \) by \( \tau \) in generating relation (4.3.12) and using Eq. (4.2.9), we get

\[
\left(1 - \frac{c}{t}\right)^{m} C_m\left(x\left(1 + \frac{bt}{x}\right) \left(1 - \frac{c}{t}\right), y, \tau \left(1 - \frac{c}{t}\right)^{-2}\right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} b^{(p+|p|)/2} \Gamma(p+1) c^{(-p+|p|)/2} 0F_1[-; |p| + 1; bc] C_{m+p}(x, y, \tau) t^p, \quad \left|\frac{bt}{x}\right| < 1, \quad \left|\frac{c}{t}\right| < 1,
\]

(4.4.7)

where \( C_m(x, y; \tau) \) denotes 2V1PTF given by Eq. (4.2.10).

Further, replacing \( x \) by \( \frac{x^2}{4} \), \( y \) by \( \frac{y^2}{4} \), \( \tau \) by \( \frac{2yt}{x^2} \) and \( t \) by \( \frac{xt}{2} \) in generating relation (4.4.7), we get

\[
\left(1 - \frac{(2c/xt)}{1 + (2b/tx)}\right)^{m/2} J_m\left(x\left(1 + \frac{2bt}{x}\right)^{1/2} \left(1 - \frac{2c}{xt}\right)^{1/2} , y, \tau \left(1 + \frac{2bt}{x}\right) \left(1 - \frac{2c}{xt}\right)^{-1}\right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} b^{(p+|p|)/2} \Gamma(p+1) c^{(-p+|p|)/2} 0F_1[-; |p| + 1; bc] J_{m+p}(x, y; \tau) t^p, \quad \left|\frac{2bt}{x}\right| < 1, \quad \left|\frac{2c}{xt}\right| < 1,
\]

(4.4.8)
where \( J_m(x,y;\tau) \) denotes 2V1PBF given by Eq. (4.2.12).

Further, taking \( \tau = 1 \) in generating relation (4.4.8), we get

\[
\left( \frac{1 - (2c/xt)}{1 + (2bt/x)} \right)^{m/2} J_m \left( x \left( 1 + \frac{2bt}{x} \right)^{1/2} \left( 1 - \frac{2c}{xt} \right)^{1/2}, y \right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{\left| p \right|}}{|p|!} b^{(p+|p|)/2} \times c^{-(p+|p|)/2} F_1[-;|p|+1;bc] J_{m+p}(x,y) t^p, \quad \left| \frac{2bt}{x} \right| < 1, \quad \left| \frac{2c}{xt} \right| < 1
\]  

(4.4.9)

where \( J_m(x,y) \) denotes 2VBF given by Eq. (4.2.14).

Similar results can be obtained from generating relation (4.3.15).

V. Taking \( z = \tau_1 = 1, \tau_2 = 0 \) in generating relation (4.3.12) and using Eq. (4.2.15), we get

\[
\left( 1 - \frac{c}{t} \right)^m C_m \left( x \left( 1 + \frac{bt}{x} \right) \left( 1 - \frac{c}{t} \right), y \right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{\left| p \right|}}{|p|!} b^{(p+|p|)/2} \times c^{-(p+|p|)/2} F_1[-;|p|+1;bc] C_{m+p}(x,y) t^p, \quad \left| \frac{bt}{x} \right| < 1, \quad \left| \frac{c}{t} \right| < 1
\]  

(4.4.10)

where \( C_m(x,y) \) denotes 2VTB given by Eq. (4.2.16). Similar results can be obtained from generating relation (4.3.15).

VI. Taking \( y = z = 1 \) and \( \tau_1 = \tau_2 = 0 \) in generating relations (4.3.12), (4.3.15) and using Eq. (4.2.18), we get

\[
\left( 1 - \frac{c}{t} \right)^m C_m \left( x \left( 1 + \frac{bt}{x} \right) \left( 1 - \frac{c}{t} \right) \right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{\left| p \right|}}{|p|!} b^{(p+|p|)/2} \times c^{-(p+|p|)/2} F_1[-;|p|+1;bc] C_{m+p}(x,y) t^p, \quad \left| \frac{bt}{x} \right| < 1, \quad \left| \frac{c}{t} \right| < 1
\]  

(4.4.11)

\[
\left( 1 + \frac{r}{2\nu t} \right)^m C_m \left( x \left( 1 + \frac{r\nu t}{2x} \right) \left( 1 + \frac{r}{2\nu t} \right) \right) = \sum_{p=-\infty}^{\infty} (-\nu)^p J_p(r) C_{m+p}(x,y) t^p, \quad \left| \frac{r\nu t}{2x} \right| < 1, \quad \left| \frac{r}{2\nu t} \right| < 1
\]  

(4.4.12)

respectively, where \( C_m(x) \) denotes Tricomi function given by Eqs. (4.1.1) and (4.1.4).
Further, replacing $x$ by $z^2/4$, $t$ by $zt/2$ in generating relation (4.4.11) and using Eq. (4.1.3), we obtain ([113]; p.62(3.29)), for $Z_m = J_m$

\[
\left( \frac{1 - \frac{2c}{zt}}{1 + \frac{2bt}{z}} \right)^{m/2} J_m \left( z \left( 1 + \frac{2bt}{z} \right)^{1/2} \left( 1 - \frac{2c}{zt} \right)^{1/2} \right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} \frac{b^{|p|}}{|p|^{p+|p|/2}} \times c^{(-p+|p|)/2} \, _0F_1(-|p|; 1; bc) J_{m+p}(z) t^p,
\]
\[
\quad \left| \frac{2bt}{z} \right| < 1, \quad \left| \frac{2c}{zt} \right| < 1.
\]  

(4.4.13)

Several of the fundamental identities for cylindrical functions are special cases of generating relation (4.4.13). Also, for $c = 0$, $t = 1$ and $b = 0$, $t = 1$, relation (4.4.13) gives the formulas of Lommel ([113]; p.62(3.30) and (3.31), for $Z_m = J_m$).

Again, replacing $x$ by $z^2/4$, $t$ by $z/2$ in generating relation (4.4.12) and using Eq. (4.1.3), we obtain a generalization of Graf's addition theorem ([113]; p.63(3.32), for $Z_m = J_m$

\[
\left( \frac{1 + \frac{r}{\nu z}}{1 + \frac{r}{\nu z}} \right)^{m/2} J_m \left( z \left( 1 + \frac{r}{\nu z} \right)^{1/2} \left( 1 + \frac{r}{\nu z} \right)^{1/2} \right) = \sum_{p=-\infty}^{\infty} (-\nu)^p \frac{r^p}{p!} J_{p+r}(z),
\]
\[
\quad \left| \frac{r}{\nu z} \right| < 1, \quad \left| \frac{r}{\nu z} \right| < 1.
\]  

(4.4.14)

Further, taking $c = 0$, $t = 1$ and replacing $x$ by $x^2/4$ and $b$ by $-xt/2$ in relation (4.4.11) and using Eq. (4.1.3), we get a well known generating relation ([108]; p.42 Eq.8.4(56)).

\[
\sum_{p=0}^{\infty} \frac{t^p}{p!} J_{m+p}(x) = \left( \frac{x}{x - 2t} \right)^{m/2} J_m \left( \sqrt{x^2 - 2xt} \right), \quad m \in \mathbb{C}
\]  

(4.4.15)

4.5. CONCLUDING REMARKS

We note that the expressions (4.3.11) are valid only for group elements $g$ in a sufficiently small neighbourhood of the identity element of the Lie group $T_3$. However,
we can also use operators (4.3.6) to derive generating relations for 3V2PTF and related functions with group elements bounded away from the identity.

If \( f(x, y, z, t; \tau_1, \tau_2) \) is a solution of the equation \( C_{0,0}f = \omega^2 f \), i.e.,

\[
\left( -x \frac{\partial^2}{\partial x^2} + 2\tau_1 \frac{\partial}{\partial x \partial \tau_1} + 3\tau_2 \frac{\partial^2}{\partial x \partial \tau_2} - (m + 1) \frac{\partial}{\partial x} \right) f(x, y, z, t; \tau_1, \tau_2) = \omega^2 f(x, y, z, t; \tau_1, \tau_2),
\]

then the function \( T(g)f \) given by (4.3.10) satisfies the equation

\[
C_{0,0}(T(g)f) = \omega^2 (T(g)f).
\]

This follows from the fact that \( C_{0,0} \) commutes with the operators \( K^+, K^- \) and \( K^3 \).

Now if \( f \) is a solution of the equation

\[
(x_1 K^+ + x_2 K^- + x_3 K^3)f(x, y, z, t; \tau_1, \tau_2) = \lambda f(x, y, z, t; \tau_1, \tau_2),
\]

for constants \( x_1, x_2, x_3 \) and \( \lambda \), then \( T(g)f \) is a solution of the equation

\[
[T(g)(x_1 K^+ + x_2 K^- + x_3 K^3)T(g^{-1})][T(g)f] = \lambda[T(g)f].
\]

The inner automorphism \( \mu_g \) of Lie group \( T_3 \) defined by

\[
\mu_g(h) = ghg^{-1}, \quad h \in T_3,
\]

induces an automorphism \( \mu_g^* \) of Lie algebra \( T_3 \) where

\[
\mu_g^*(\alpha) = g\alpha g^{-1}, \quad \alpha \in T_3.
\]

If \( \alpha = x_1 \mathcal{J}^+ + x_2 \mathcal{J}^- + x_3 \mathcal{J}^3 \) where \( \mathcal{J}^+, \mathcal{J}^- \) and \( \mathcal{J}^3 \) are given by Eq. (1.6.28) and \( g \) is given by Eq. (1.6.25), then we have

\[
\mu_g^*(\alpha) = (x_1 e^a - bx_3) \mathcal{J}^+ + (x_2 e^{-a} + cx_3) \mathcal{J}^- + x_3 \mathcal{J}^3,
\]

as a consequence of which, we can write

\[
T(g)(x_1 K^+ + x_2 K^- + x_3 K^3)T(g^{-1}) = (x_1 e^a - bx_3) K^+ + (x_2 e^{-a} + cx_3) K^- + x_3 K^3.
\]
To give an example of the application of these remarks, we consider the function
\[ f(x, y, z, t; \tau_1, \tau_2) = C_m(x, y, z; \tau_1, \tau_2)t^m, \quad m \in \mathbb{C}. \] Since \( C_{0,0}f = f \) and \( K^3f = mf \), so the function
\[ [T(g)f](x, y, z, t; \tau_1, \tau_2) = e^{ma(t-c)}mC_m \left( \left( x + \frac{x}{t} \right), y, z; \tau_1 \left( 1 - \frac{c}{t} \right)^{-2}, \tau_2 \left( 1 - \frac{c}{t} \right)^{-3} \right) \]
(4.5.7)
satisfies the equations
\[ C_{0,0}[T(g)f] = [T(g)f], \] (4.5.8)
\[ (-bK^3 + cK^- + K^3)[T(g)f] = m[T(g)f]. \] (4.5.9)

For \( a = b = 0 \) and \( c = -1 \), we can express the function (4.5.7) in the form
\[ h(x, y, z, t; \tau_1, \tau_2) = (t+1)^m C_m \left( \left( x + \frac{x}{t} \right), y, z; \tau_1 \left( 1 + \frac{1}{t} \right)^{-2}, \tau_2 \left( 1 + \frac{1}{t} \right)^{-3} \right), \quad |t| < 1. \] (4.5.10)

Now using the Laurent expansion
\[ h(x, y, z, t; \tau_1, \tau_2) = \sum_{k=-\infty}^{\infty} h_k(x, y, z; \tau_1, \tau_2)t^k, \quad |t| < 1, \]
in Eq. (4.5.8), we note that \( h_k(x, y, z; \tau_1, \tau_2) \) is a solution of differential equation (4.2.3) for each integer \( k \). Since the function \( h(x, y, z, t; \tau_1, \tau_2) \) is bounded for \( x = y = z = 0 \), we have
\[ h_k(x, y, z; \tau_1, \tau_2) = c_k C_k(x, y, z; \tau_1, \tau_2), \quad c_k \in \mathbb{C}. \]
Thus
\[ h_k(x, y, z, t; \tau_1, \tau_2) = \sum_{k=-\infty}^{\infty} c_k C_k(x, y, z; \tau_1, \tau_2)t^k. \] (4.5.11)

Now from Eq. (4.5.9), we have
\[ (-K^- + K^3)h(x, y, z, t; \tau_1, \tau_2) = mh(x, y, z, t; \tau_1, \tau_2) \]
and therefore it follows that
\[ c_{k+1} = (m - k)c_k. \]
Further taking $x = y = z = 0$ in (4.5.10) and using (4.5.11), we get $c_0 = 1/\Gamma(m+1)$ and hence $c_k = 1/\Gamma(m - k + 1)$. Thus we obtain the following result:

$$(t+1)^m C_m \left( \left( x + \frac{z}{t} \right) y, z; \tau_1 \left( 1 + \frac{1}{t} \right)^{-2}, \tau_2 \left( 1 + \frac{1}{t} \right)^{-3} \right)$$

$$= \sum_{k=-\infty}^{\infty} \frac{C_k(x,y,z;\tau_1,\tau_2)t^k}{\Gamma(m - k + 1)}, \quad |t| < 1,$$

(4.5.12)

which is obviously not a special case of generating relation (4.3.12).

Several other examples of generating relations can be derived by this method see for example Weisner [151].

We have considered $3V2PTF \ C_m(x,y,z;\tau_1,\tau_2)$ within the group representation formalism. These functions appeared as basis functions for a realization of the representation $Q(-1,m_0)$ of the Lie algebra $T_3$. The analysis presented in this Chapter confirms the possibility of extending this approach to other useful forms of generalized Tricomi functions as well as to their Bessel counter parts.