Chapter 5

A Note on Generating Functions and Summation Formulae for Meixner Polynomials of One and Two variables

Abstract: The present chapter is a study of Meixner polynomials of one and two variables. This chapter deals with different classes of generating functions and various elegant summation formulae for Meixner polynomials of one and two variables, based on Lagrange’s expansions and some combinatorial identities.

5.1 Introduction

The Lagrange’s expansion is being used in special function to obtain a wide class of new generating functions. Gould (1961), Riordan (1968), Carlitz (1968), Srivastava (1969), Brown(1969), Zeitlin (1970), Pólya & Szegö(1972) showed that the Combinatorial identities are useful in the derivation of mixed generating functions for classical orthogonal polynomials, Which are derivable from certain consequences of Lagrange’s
expansion. Combinatorial identities have brief historical survey which reveals various identities for defining generating function from among these we adopt some standard combinatorial identities due to Gould [121], Riordon [282], Pólya & Szegö [273].

5.2 Classes of generating functions for $m_n(x; \beta, c)$

Let $\lambda$, $\gamma$, and $\mu$ be arbitrary complex numbers. Then the Meixner polynomials $m_n(x; \beta, c)$ satisfy the following generating relation

$$
\sum_{n=0}^{\infty} \frac{\beta(\lambda + \mu n)}{\beta + (1 + \gamma)n} m_n(x; 1 + \beta + \gamma n, c) \frac{t^n}{n!} = (1 + \xi)^\beta \left\{ \lambda + \frac{\mu \xi}{1 - \gamma \xi} \left( \beta + (1 + \xi) \frac{\partial}{\partial \xi} \right) \right\} _2 F_1 \left[ \begin{array}{c} \beta \\ 1 + \gamma \\ \frac{\beta}{1 + \gamma} \end{array} \right| -x; \xi w, \right\}, (5.2.1)
$$

where $c = \frac{1}{w+1}$, $0 < c < 1$, $\xi = t(1 + \xi)^{1+\gamma}$, with the condition $\xi(0) = 0$, and $_2 F_1$ is Gaussian hypergeometric function and is defined by (1.3.15).

Proof of (5.2.1).

From (1.3.80), we can write the following:

$$
\sum_{n=0}^{\infty} \frac{\beta(\lambda + \mu n)}{\beta + (1 + \gamma)n} m_n(x; 1 + \beta + \gamma n, \frac{1}{w+1}) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\beta(\lambda + \mu n)}{\beta + (1 + \gamma)n} \left( \beta + (1 + \gamma)n \right) \frac{(-x)_k (w)_k}{k!} \frac{t^n}{n-k} \frac{t^n}{k!}
$$

$$
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\beta(\lambda + \mu(n+k))}{\beta + (1 + \gamma)(n+k)} \left( \beta + (1 + \gamma)(n+k) \right) \frac{(-x)_k (w)_k}{k!} \frac{t^{n+k}}{k!}
$$

$$
= \sum_{k=0}^{\infty} \frac{\beta}{\beta + (1 + \gamma)k} \frac{(-x)_k (w)_k}{k!} t^k \sum_{n=0}^{\infty} \frac{\beta + (1 + \gamma)k}{\beta + (1 + \gamma)(n+k)} \frac{\beta + (1 + \gamma)(n+k)}{k!} \sum_{n=0}^{\infty} \frac{\beta + (1 + \gamma)k}{\beta + (1 + \gamma)(n+k)} \frac{t^{n+k}}{k!}
$$
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\[ \times \left( \frac{\beta + (1 + \gamma)(n + k)}{n} \right) t^n. \]

Making use of the identity (1.3.130), we find

\[ \sum_{n=0}^{\infty} \frac{\beta(\lambda + \mu n)}{\beta + (1 + \gamma)n} m_n(x; 1 + \beta + \gamma n, \frac{1}{w + 1} \frac{t^n}{n!} \right) \]

\[ = (1 + \xi)^\beta \sum_{k=0}^{\infty} \frac{\beta}{1 + \frac{\beta}{1 + \gamma} + k} \frac{(-x)^k (w)^k \xi^k}{k!} \left\{ \lambda + \frac{\mu \xi}{1 - \gamma \xi} + \frac{\mu(1 + \xi)k}{1 - \gamma \xi} \right\}. \]

It yields

\[ \sum_{n=0}^{\infty} \frac{\beta(\lambda + \mu n)}{\beta + (1 + \gamma)n} m_n(x; 1 + \beta + \gamma n, c) \frac{t^n}{n!} \]

\[ = (1 + \xi)^\beta \left\{ \lambda + \frac{\mu \xi}{1 - \gamma \xi} \left( \beta + (1 + \xi) \frac{\partial}{\partial \xi} \right) \right\} _2 F_1 \begin{bmatrix} \frac{\beta}{1 + \gamma} & -x; \\ 1 + \frac{\delta}{1 + \gamma}; & \xi w \end{bmatrix}. \]

Now, we give special cases of above generating function.

Setting \( \mu = 0 \) and \( \lambda = 1 \) in generating function (5.2.1), we get the following generating function

\[ \sum_{n=0}^{\infty} \frac{\beta}{\beta + (1 + \gamma)n} m_n(x; 1 + \beta + \gamma n, c) \frac{t^n}{n!} \]

\[ = (1 + \xi)^\beta _2 F_1 \begin{bmatrix} \frac{\beta}{1 + \gamma} & -x; \\ 1 + \frac{\beta}{1 + \gamma}; & \xi w \end{bmatrix}. \] (5.2.2)

Further, if we take \( \mu = \frac{1 + \gamma}{\beta} \) and \( \lambda = 1 \), we get

\[ \sum_{n=0}^{\infty} m_n(x; \beta + \gamma n, c) \frac{t^n}{n!} \]

\[ = (1 + \xi)^{\beta + x} \left( 1 - \frac{\xi}{(1 + \xi)e} \right)^x \frac{1}{1 - \xi \gamma}. \] (5.2.3)

Now, we give a alternate proof of generating function (5.2.3), which is a special case of our generating function
Proof of (5.2.3).

From (1.3.80), we can write the following

\[
\sum_{n=0}^{\infty} m_n(x; \beta + \gamma n, c) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{\beta - 1 + (1 + \gamma) n}{n - k} \right) \frac{(-x)_k (c^{-1} - 1)^k t^n}{k!} \sum_{n=0}^{\infty} \binom{\beta - 1 + (1 + \gamma)(n + k)}{n} t^n,
\]

where \( \binom{\cdot}{\cdot} \) denotes to binomial coefficient, given by (1.3.4).

Now, using the identity (1.3.124), we get

\[
\sum_{n=0}^{\infty} m_n(x; \beta + \gamma n, c) \frac{t^n}{n!} = \frac{(1 + \xi)^\beta}{1 - \xi \gamma} \sum_{k=0}^{\infty} \frac{(-x)_k (c^{-1} - 1)^k}{k!} \frac{\xi^k}{\xi}, \quad (\xi = t(1 + \xi)^{1+\gamma})
\]

\[
= \frac{(1 + \xi)^\beta}{1 - \xi \gamma} \left( 1 + \xi - \frac{\xi}{c} \right)^x.
\]

Hence, we get

\[
\sum_{n=0}^{\infty} m_n(x; \beta + \gamma n, c) \frac{t^n}{n!} = \frac{(1 + \xi)^{\beta+x} \left( 1 - \frac{\xi}{(1+\xi)c} \right)^x}{1 - \xi \gamma}.
\]

Furthermore, If \( \gamma \) and \( \delta \) be arbitrary complex numbers, the Meixner polynomials \( m_n(x; \beta, c) \) hold the following generating relation:

\[
\sum_{n=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)n} m_n(x; 1 + \beta + \gamma n, c) \frac{t^n}{n!} = (1 + \xi)^\beta F_3 \left[ \frac{\delta}{1 + \gamma}, \delta - \beta, -x, 1; 1 + \frac{\delta}{1 + \gamma}; \xi w, \frac{\xi}{1 + \xi} \right], \quad (5.2.4)
\]

where, \( F_3 \) is Appell function of two variables and is defined by (1.3.27).
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Proof of (5.2.4).

From (1.3.80), we can write the following:

\[
\sum_{n=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)n} m_n(x; 1 + \beta + \gamma n, \frac{1}{w + 1}) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\delta}{\delta + (1 + \gamma)n} \left( \frac{\beta + (1 + \gamma)n}{n - k} \right) (-x)_k (w)^k t^n \frac{n!}{k!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)(n+k)} \left( \frac{\beta + (1 + \gamma)(n+k)}{n} \right) (-x)_k (w)^k t^{n+k} \frac{n!}{k!}
\]

\[
= \sum_{k=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)k} \frac{(-x)_k (w)^k}{k!} \sum_{n=0}^{\infty} \frac{\delta + (1 + \gamma)k}{\delta + (1 + \gamma)(n+k)} \left( \frac{\beta + (1 + \gamma)(n+k)}{n} \right) t^n.
\]

Making use of the identity (1.3.129), we have

\[
\sum_{n=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)n} m_n(x; 1 + \beta + \gamma n, \frac{1}{w + 1}) \frac{t^n}{n!} = (1 + \xi)^{\beta} \sum_{k=0}^{\infty} \frac{\delta}{\frac{\delta}{1 + \gamma} + k} \frac{(-x)_k (w)^k \xi^k}{k!} \sum_{n=0}^{\infty} (-1)^n \left( \frac{\beta - \delta}{n} \right) \left( n + \frac{\delta + (1 + \gamma)k}{1 + \gamma} \right)^{-1} \left( \frac{\xi}{1 + \xi} \right)^n
\]

\[
= (1 + \xi)^{\beta} \sum_{k=0}^{\infty} \left( \frac{\frac{\delta}{1 + \gamma}}{1 + \frac{\delta}{1 + \gamma}} \right)_k \frac{(-x)_k (w \xi)^k}{k!} \sum_{n=0}^{\infty} \frac{\delta - \beta, n \left( \frac{1}{1 + \gamma} \right)_k (w \xi) k}{(1 + \frac{\delta}{1 + \gamma})_{n+k}} \left( \frac{\xi}{1 + \xi} \right)^n.
\]

Hence, we obtain

\[
\sum_{n=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)n} m_n(x; 1 + \beta + \gamma n, c) \frac{t^n}{n!} = (1 + \xi)^{\beta} \sum_{n=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)n} m_n(x; 1 + \beta + \gamma n, c) \frac{t^n}{n!} = (1 + \xi)^{\beta} F_3 \left[ \frac{\delta}{1 + \gamma}, \delta - \beta, -x, 1; 1 + \frac{\beta}{1 + \gamma}, \xi w, \frac{\xi}{1 + \xi} \right].
\]

Particular cases:

(i) Taking \(\gamma = 0\) in (5.2.3), (5.2.2), (5.2.4) and (5.2.1), we get

\[
\sum_{n=0}^{\infty} m_n(x; \beta, c) \frac{t^n}{n!} = \left( 1 - \frac{t}{c} \right)^x (1-t)^{-x-\beta},
\]

(5.2.5)
\[
\sum_{n=0}^{\infty} \frac{\beta}{\beta + n} m_n(x; 1 + \beta, c) \frac{t^n}{n!} = (1 - t)^{-\beta} \binom{2}{\beta - x; \frac{t(c - 1)}{c(t - 1)}}, \quad (5.2.6)
\]

\[
\sum_{n=0}^{\infty} \frac{\delta}{\delta + n} m_n(x; 1 + \beta, c) \frac{t^n}{n!} = (1 - t)^{-\beta} \binom{3}{\delta, \delta - \beta, -x, 1 + \delta; \frac{t(c - 1)}{c(t - 1)}, t}, \quad (5.2.7)
\]

\[
\sum_{n=0}^{\infty} \frac{\beta(\lambda + \mu n)}{\beta + n} m_n(x; 1 + \beta, c) \frac{t^n}{n!} = (1 - t)^{-\beta} \left\{ \lambda + \mu \frac{t}{1 - t} \left( \beta + (1 - t) \frac{\partial}{\partial t} \right) \right\} \binom{2}{\beta - x; \frac{t(c - 1)}{c(t - 1)}}, \quad (5.2.8)
\]

respectively. Here (5.2.5) is nothing but the generating function (1.3.81), given by Agarwal and Manocha [2]

(ii) Taking \( \gamma = -1 \) and \( c = \frac{1}{w+1} \) in (5.2.3), we find

\[
\sum_{n=0}^{\infty} m_n(x; \beta - n, c) \frac{t^n}{n!} = (1 + t)^{\beta-1} (1 - tw)^x \quad (5.2.9)
\]

(iii) Taking \( \gamma = -\frac{1}{2} \) in (5.2.3), (5.2.2), (5.2.4), (5.2.1), we get

\[
\sum_{n=0}^{\infty} m_n(x; \beta - n, c) \frac{t^n}{n!} = (1 + u(t))^{\beta} (1 - w u(t))^x \left( 1 + \frac{u(t)}{2} \right), \quad (5.2.10)
\]
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5.3 Summation formulae for Meixner polynomials

Now, we establish some interesting summation formulae for the meixner polynomials, as given below:

\[
m_n(x; \beta_1 + \beta_2 + \gamma n, c) = \sum_{k=0}^{n} \frac{k! \beta_1}{\beta_1 + (1 + \gamma)k} \binom{\beta_1 + (1 + \gamma)k}{k} \left( \binom{n}{k} m_{n-k}(x; \beta_2 + \gamma(n-k), c), \right)
\]  

(5.3.1)
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\[ m_n(x; \beta + \gamma n, c) = \sum_{k=0}^{n} \frac{k!}{\beta - \delta + (1 + \gamma)k} \left( \frac{\beta - \delta + (1 + \gamma)k}{k} \right) \binom{n}{k} m_{n-k}(x; \delta + \gamma(n-k), c), \]

(5.3.2)

\[ m_n(x+y; \beta + \gamma n, c) = \sum_{k=0}^{n} \frac{k!}{y + (1 + \gamma)k} \left( \frac{y + (1 + \gamma)k}{k} \right) \binom{n}{k} m_{n-k}(x; \beta + \gamma(n-k), c), \]

(5.3.3)

\[ m_n(x + y; \beta_1 + \beta_2 + \gamma n, c) = \sum_{k=0}^{n} \frac{m!}{1 + (1 + \gamma)m} \left( \frac{1 + (1 + \gamma)m}{m} \right) \times \left( \frac{n}{m + k} \right) \left( \frac{m + k}{k} \right) \left( \frac{\gamma m}{m_k(y; \beta_2 + \gamma k, c)m_{n-k-m}(x; \beta_1 + \gamma(n-k-m), c)}, \right) \]

(5.3.4)

where \( \binom{n}{k} \) is a binomial coefficient, given by (1.3.4).

**Proof of** (5.3.1).

It is easy to write the following from (5.2.3)

\[ \sum_{n=0}^{\infty} m_n(x; \beta_1 + \beta_2 + \gamma n, c) \frac{t^n}{n!} = \frac{(1 + \xi)^{\beta_1 + \beta_2 + x} (1 - \xi(1+\xi)c)^x}{1 - \xi \gamma} \]

\[ = (1 + \xi)^{\beta_1} \sum_{n=0}^{\infty} m_n(x; \beta_2 + \gamma n, c) \frac{t^n}{n!}. \]

Now using the identity (1.3.128), we get

\[ \sum_{n=0}^{\infty} m_n(x; \beta_1 + \beta_2 + \gamma n, c) \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\beta_1}{\beta_1 + (1 + \gamma)k} \left( \frac{\beta_1 + (1 + \gamma)k}{k} \right) m_{n-k}(x; \beta_2 + \gamma(n-k), c) \frac{t^n}{(n-k)!}. \]
Equating the coefficient of $\frac{t^n}{n!}$ from both sides, we find

$$m_n(x; \beta_1+\beta_2+\gamma n, c) = \sum_{k=0}^{n} \frac{k! \beta_1}{\beta_1 + (1 + \gamma)k} \binom{n}{k} \binom{\beta_1 + (1 + \gamma)k}{k} m_{n-k}(x; \beta_2+\gamma(n-k), c).$$

Similarly, we can prove (5.3.2).

**Proof of (5.3.2).**

From (5.2.3), we can write the following:

$$\sum_{n=0}^{\infty} m_n(x+y; \beta_1+\beta_2+\gamma n, c) \frac{t^n}{n!} = \frac{(1 + \xi)^{\beta_1+\beta_2+x+y} (1 - \frac{\xi}{1+\gamma})^{x+y}}{1 - \xi \gamma}$$

$$= (1 - \gamma \xi) \sum_{n=0}^{\infty} m_n(x; \beta_1+\gamma n, c) t^n \sum_{k=0}^{\infty} \frac{m_k(y; \beta_2+\gamma k, c) t^k}{k!}, \quad (5.3.5)$$

as we know that

$$\xi = t (1 + \xi)^{1+\gamma} \Rightarrow -\gamma \xi = -\gamma t (1 + \xi)^{1+\gamma}. \quad (5.3.6)$$

Now, using the identities (1.3.128) and (5.3.6) in (5.3.5), we get

$$\sum_{n=0}^{\infty} m_n(x+y; \beta_1+\beta_2+\gamma n, c) \frac{t^n}{n!}$$

$$= \sum_{n,m,k=0}^{\infty} \frac{1}{1 + (1 + \gamma)m} \binom{1 + (1 + \gamma)m}{m} m_n(x; \beta_1+\gamma n, c) m_k(y; \beta_2+\gamma k, c) \frac{t^{n+m+k}(-\gamma)^m}{n! k!}$$

Equating the coefficient of $\frac{t^n}{n!}$ from both sides, we get

$$m_n(x+y; \beta_1+\beta_2+\gamma n, c) = \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{m!}{1 + (1 + \gamma)m} \frac{1 + (1 + \gamma)m}{m} \binom{n}{m} \binom{m+k}{k} (\gamma)^m m_k(y; \beta_2+\gamma k, c) m_{n-k-m}(x; \beta_1+\gamma(n-k-m), c).$$

Similarly, we can prove (5.3.3).
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Particular cases:

(i) Taking $\gamma = 0$ in (5.3.1), (5.3.2), (5.3.3) and (5.3.4), we obtain

$$m_n(x; \beta_1 + \beta_2, c) = \sum_{k=0}^{n} (\beta_1)_k \binom{n}{k} m_{n-k}(x; \beta_2, c), \quad (5.3.7)$$

$$m_n(x; \beta, c) = \sum_{k=0}^{n} \binom{n}{k} (\beta - \delta)_k m_{n-k}(x; \delta, c), \quad (5.3.8)$$

$$m_n(x + y; \beta_1 + \beta_2, c) = \sum_{k=0}^{n} \binom{n}{k} m_{n-k}(x_1; \beta_1, c) m_k(x_2; \beta_2, c), \quad (5.3.9)$$

$$m_n(x + y; \beta, c) = \sum_{k=0}^{n} \binom{n}{k} m_{n-k}(x; \beta_2, c) m_k(y; \beta_2, c) \quad (5.3.10)$$

respectively. All these summation formulae was given by M.A. Khan and M. Akhlaq (see [216], p. 10).

(ii) Taking $\gamma = -1$ in (5.3.1), (5.3.2), (5.3.3) and (5.3.4), we get

$$m_n(x; \beta_1 + \beta_2 - n, c) = \sum_{k=0}^{n} k! \binom{\beta_1}{k} \binom{n}{k} m_{n-k}(x; \beta_2 - n + k, c), \quad (5.3.11)$$

$$m_n(x; \beta - n, c) = \sum_{k=0}^{n} k! \binom{\beta - \delta}{k} \binom{n}{k} m_{n-k}(x; \delta - n + k, c), \quad (5.3.12)$$

$$m_n(x+y; \beta - n, c) = \sum_{k=0}^{n} k!(1 - \frac{1}{c})^k \binom{y}{k} \binom{n}{k} m_{n-k}(x; \beta - n + k, c), \quad (5.3.13)$$

$$m_n(x + y; \beta_1 + \beta_2 - n, c) = \sum_{k=0}^{n} \binom{n}{k} m_k(y; \beta_2 - k, c) m_{n-k}(x; 1 + \beta_1 - n + k, c) \quad (5.3.14)$$

respectively.
5.4 Further Generating functions for $m_n(x; \beta, c)$

Consider the generating function (1.3.81)

$$
\sum_{n=0}^{\infty} m_n(x; \beta, c) \frac{t^n}{n!} = \left(1 - \frac{t}{c}\right)^x (1 - t)^{-x - \beta}, \quad \{|t| < \text{min}(1, |c|)\}
$$

$$
= \sum_{k=0}^{\infty} m_k(1; x - 1 + \beta, c) \frac{t^k}{k!} \left(1 - \frac{t}{c}\right)^x
$$

$$
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} m_k(1; x - 1 + \beta, c) \frac{t^k}{k!} \left(\frac{x - 1}{n}\right) (-c^{-1})^n t^n.
$$

Equating the coefficient of $\frac{t^n}{n!}$ from both sides, we get

$$
m_n(x; \beta, c) = \sum_{k=0}^{n} \left(\frac{x - 1}{n - k}\right) (-c^{-1})^{n-k} \frac{m_k(1; x - 1 + \beta, c)}{k!}.
$$

(5.4.1)

Thus, we can write the following:

$$
m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) = \sum_{k=0}^{n} \left(\frac{x - 1 + (1 + \gamma)n}{n - k}\right) (-c^{-1})^{n-k} \frac{m_k(1; x - 1 + \beta, c)}{k!}.
$$

(5.4.2)

Let $\gamma$ and $\beta$ be complex constant and independent from $n$, then from (5.4.2), we get

$$
\sum_{n=0}^{\infty} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} = \sum_{k=0}^{\infty} m_k(1; x - 1 + \beta, c) \frac{t^k}{k!} \sum_{n=0}^{\infty} \left(\frac{x - 1 + (1 + \gamma)(n + k)}{n}\right) (-c^{-1})^n t^n.
$$

Now, we apply the identity (1.3.124) to get

$$
\sum_{n=0}^{\infty} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} = \frac{(1 + \xi x)}{1 - \xi \gamma} \sum_{k=0}^{\infty} m_k(1; x - 1 + \beta, c) \frac{(1 + \xi)^{(1+\gamma)k} t^k}{k!},
$$

where $\xi = -\frac{t}{c}(1 + \xi)^{1+\gamma}$.  

\[ \sum_{n=0}^{\infty} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} = \frac{(1 + \xi)^x}{1 - \xi \gamma} \sum_{k=0}^{\infty} m_k(1; x - 1 + \beta, c) \frac{(-c\xi)^k}{k!}. \]

Thus, we get the following
\[ \sum_{n=0}^{\infty} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} = \frac{(1 + \xi)^{x+1} (1 + c\xi)^{-x-\beta}}{1 - \xi \gamma}. \tag{5.4.3} \]

If \( x \) and \( y \) are the parameter and independent from \( t \) and \( n \), then for arbitrary constant \( \beta, \lambda, \gamma, \mu \) and \( \delta \) independent from \( n \), we can obtain the following generating functions on the similar lines of (5.2.2), (5.2.4) and (5.2.1)
\[ \sum_{n=0}^{\infty} \frac{x}{x + (1 + \gamma)n} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} = (1 + \xi)^x \begin{array}{c} x \\ \begin{array}{c} x + \beta; \\ 1 + \frac{x}{1 + \gamma}; \\ -c\xi \end{array} \end{array}_{2F1} \tag{5.4.4} \]
\[ \sum_{n=0}^{\infty} \frac{y}{y + (1 + \gamma)n} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} = \frac{(1 + \xi)^x}{1+\gamma} \begin{array}{c} y \\ \begin{array}{c} y - x; \\ x + \beta, 1; \begin{array}{c} 1 + \gamma \\ -c\xi, \frac{\xi}{1 + \xi} \end{array} \end{array} \end{array}_{F3} \tag{5.4.5} \]
\[ \sum_{n=0}^{\infty} \frac{x(\lambda + \mu n)}{x + (1 + \gamma)n} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} = \frac{(1 + \xi)^x}{1+\gamma} \begin{array}{c} \lambda + \frac{\mu\xi}{1 - \gamma\xi} (x + (1 + \xi) \frac{\partial}{\partial\xi}) \\ \begin{array}{c} x + \beta; \\ 1 + \frac{x}{1 + \gamma}; \\ -c\xi \end{array} \end{array}_{2F1} \tag{5.4.6} \]

respectively, where \( 0 < c < 1 \) and \( \xi = -\frac{x}{c}(1 + \xi)^{1+\gamma} \), with the condition \( \xi(0) = 0 \).

Again if we take \( \xi = -\frac{y}{c} \), then \( \eta = t(1 - \frac{y}{c})^{1+\gamma} \), we can write the above results in the following form.
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\[ \sum_{n=0}^{\infty} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} \]

\[ = \frac{(1 - \frac{t}{c})^{x+1} (1 - \eta)^{-x-\beta}}{1 + \frac{x}{c}}, \quad (5.4.7) \]

\[ \sum_{n=0}^{\infty} \frac{x}{x + (1 + \gamma)n} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} \]

\[ = \left(1 - \frac{\eta}{c}\right)^x \binom{2}{\begin{array}{cc} \frac{x}{1 + \gamma} : x + \beta; \\ 1 + \frac{x}{1 + \gamma}; \end{array}} \quad (5.4.8) \]

\[ \sum_{n=0}^{\infty} \frac{y}{y + (1 + \gamma)n} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} \]

\[ = \left(1 - \frac{\eta}{c}\right)^y \binom{3}{y}{\begin{array}{cc} \frac{y}{1 + \gamma}, y - x; x + \beta, 1; 1 + \frac{y}{1 + \gamma}: \eta, -\frac{\eta}{c} - \eta \end{array}}, \quad (5.4.9) \]

\[ \sum_{n=0}^{\infty} \frac{x(\lambda + \mu n)}{x + (1 + \gamma)n} m_n(x + (1 + \gamma)n; \beta - (1 + \gamma)n, c) \frac{t^n}{n!} \]

\[ = \left(1 - \frac{\eta}{c}\right)^x \left\{ \lambda + \frac{\mu \eta}{c + \eta \gamma} \left(x + (c - \eta) \frac{\partial}{\partial \eta}\right) \right\} \binom{4}{\begin{array}{cc} \frac{x}{1 + \gamma} : x + \beta; \\ 1 + \frac{x}{1 + \gamma}; \end{array}} \quad (5.4.10) \]

where \( \eta = t(1 - \frac{\eta}{c})^{1+\gamma}. \) \( (5.4.11) \)

**Particular cases:**

(i) Taking \( \gamma = 0 \) in (5.4.7), (5.4.8), (5.4.9) and (5.4.10), we get

\[ \sum_{n=0}^{\infty} m_n(x + n; \beta - n, c) \frac{t^n}{n!} \]

\[ = \left(1 + \frac{t}{c}\right)^{\beta-1} \left(1 - t + \frac{t}{c}\right)^{-x-\beta}, \quad (5.4.12) \]
\[ \sum_{n=0}^{\infty} \frac{x}{x+n} m_n(x+n; \beta - n, c) t^n \frac{x}{n!} = \left(1 + \frac{t}{c}\right)^{-x} \binom{2}{x} \left[ \begin{array}{c} x : x + \beta; \\ c t \\
 1 + x; \end{array} \right], \quad (5.4.13) \]

\[ \sum_{n=0}^{\infty} \frac{y}{y+n} m_n(x+n; \beta - n, c) t^n \frac{y}{n!} = \left(1 + \frac{t}{c}\right)^{-x} \binom{3}{y} \left[ \begin{array}{c} y, y - x; x + \beta, 1 + y : c t \frac{c}{c + t} - \frac{t}{c} \end{array} \right], \quad (5.4.14) \]

and
\[ \sum_{n=0}^{\infty} \frac{x(\lambda + \mu n)}{x+n} m_n(x+n; \beta - n, c) t^n \frac{x}{n!} = \left(1 + \frac{t}{c}\right)^{-x} \left\{ \lambda + \frac{\mu t}{c + 2t} \left( x + (c + t) \frac{2}{\partial t} \right) \right\} \binom{2}{x} \left[ \begin{array}{c} x : x + \beta; \\ c t \\
 1 + x; \end{array} \right], \quad (5.4.15) \]

respectively.

(ii) Let \( \gamma = -\frac{1}{2} \), then we find that
\[ \eta = \frac{t}{2c} \left( -t + (t^2 + 4c^2)^{\frac{1}{2}} \right) = u(t). \quad (5.4.16) \]

If we replace \( \gamma \) by \( -\frac{1}{2} \) and \( \eta \) by \( u(t) \) in (5.4.7), (5.4.8), (5.4.9) and (5.4.10), then we get special generating functions for Meixner polynomials

5.5 Classes of Generating Functions for \( m_n(x, y; \beta, c, d) \)

In 2010, M.A. Khan and M. Akhlaq [216] defined two variables Meixner polynomials by (1.3.83). If \( \beta \) be arbitrary parameters and independent from \( t \), then for arbitrary
constant \( \lambda, \gamma, \mu \) and \( \delta \) (independent from \( n \) and \( t \)) the polynomials \( m_n(x, y; \beta, c, d) \) defined by (1.3.83) and satisfy the following generating relations:

\[
\sum_{n=0}^{\infty} m_n(x, y; \beta + \gamma n, c, d) \frac{t^n}{n!} = \frac{(1 + \xi) \beta (1 - \xi w)^x (1 - \xi v)^y}{1 - \xi \gamma},
\]

(5.5.1)

\[
\sum_{n=0}^{\infty} \frac{\beta}{\beta + (1 + \gamma) n} m_n(x, y; 1 + \beta + \gamma n, c, d) \frac{t^n}{n!} = (1 + \xi)^\beta \sum_{n=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma) n} m_n(x, y; 1 + \beta + \gamma n, c, d) \frac{t^n}{n!} = (1 + \xi)^\beta \sum_{n=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma) n} m_n(x, y; 1 + \beta + \gamma n, c, d) \frac{t^n}{n!} \]

\[
= (1 + \xi)^\beta F_1 \left[ \frac{\beta}{1 + \gamma}, -x, -y; 1 + \beta + \gamma n, c, d \right],
\]

(5.5.2)

\[
= (1 + \xi)^\beta \left[ \lambda + \frac{\mu \xi}{1 - \gamma \xi} \left( \beta + (1 + \xi) \frac{\partial}{\partial \xi} \right) \right] F_1 \left[ \frac{\beta}{1 + \gamma}, -x, -y; 1 + \beta + \gamma n, c, d \right],
\]

(5.5.3)

where \( c = \frac{1}{w+1}, \ d = \frac{1}{v+1}, \ 0 < c < 1, \ 0 < d < 1, \ \xi = t(1 + \xi)^{1+\gamma} \), with the condition \( \xi(0) = 0 \), and \( F^{(3)}[x, y, z] \) denotes a general triple hypergeometric series, defined by (1.3.36).

Proof of (5.5.1).

From (1.3.86), we can write the following

\[
\sum_{n=0}^{\infty} m_n(x, y; \beta + \gamma n, c) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \binom{\beta - 1 + (1 + \gamma) n}{n - k - m} \frac{(-x)_k (c^{-1} - 1)_k (-y)_m (d^{-1} - 1)_m t^{n+k}}{k! m!}
\]
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\begin{equation}
\sum_{k,m=0}^{\infty} \frac{(-x)_k (wt)_k (-y)_m (vt)_m}{k!m!} \sum_{n=0}^{\infty} \left( \frac{\beta - 1 + (1 + \gamma)(n + k + m)}{n} \right) t^n,
\end{equation}

where \( \binom{\cdot}{\cdot} \) denotes a binomial coefficient, given by (1.3.4).

Now, using the identity (1.3.124), we get
\begin{equation}
\sum_{n=0}^{\infty} m_n(x, y; \beta + \gamma n, c, d) \frac{t^n}{n!} = \frac{(1 + \xi)^\beta}{1 - \xi \gamma} (1 - \xi w)^x (1 - \xi v)^y.
\end{equation}

Hence, we get
\begin{equation}
\sum_{n=0}^{\infty} m_n(x, y; \beta + \gamma n, c) \frac{t^n}{n!} = \frac{(1 + \xi)^\beta (1 - \xi w)^x (1 - \xi v)^y}{1 - \xi \gamma}.
\end{equation}

**Proof of** (5.5.2).

Consider the series
\begin{equation}
\sum_{n=0}^{\infty} \frac{\beta}{\beta + (1 + \gamma)n} m_n(x, y; 1 + \beta + \gamma n, c, d) \frac{t^n}{n!}
\end{equation}

\begin{align*}
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{\beta}{\beta + (1 + \gamma)n} \binom{\beta + (1 + \gamma)n}{n - k - m} (-x)_k (w)_k (-y)_m (v)_m t^k (-y)_m (v)_m t^m \frac{1}{k!m!} \\
&= \sum_{k,m=0}^{\infty} \frac{\beta}{\beta + (1 + \gamma)(k + m)} \left( -x \right)_k (w)_k (-y)_m (v)_m t^{k+m} \frac{1}{k!m!} \\
&\times \left( \frac{\beta + (1 + \gamma)(n + k + m)}{n} \right) t^n.
\end{align*}
Now using the identity (1.3.128), we get
\[
\sum_{n=0}^{\infty} \frac{\beta}{\beta + (1 + \gamma)n} m_n(x, y; 1 + \beta + \gamma n, c, d) \frac{t^n}{n!}
\]
\[
= (1 + \xi)^{\beta} \sum_{k,m=0}^{\infty} \frac{\beta}{1 + \gamma + k + m} \frac{(-x)_k (w\xi)^k (-y)_m (w\xi)^m}{k!m!}
\]
It yields
\[
\sum_{n=0}^{\infty} \frac{\beta}{\beta + (1 + \gamma)n} m_n(x, y; 1 + \beta + \gamma n, c, d) \frac{t^n}{n!}
\]
\[
= (1 + \xi)^{\beta} F_1 \left[ \frac{\beta}{1 + \gamma}, -x, -y; 1 + \frac{\beta}{1 + \gamma}; \xi w, \xi v \right].
\]
**Proof of (5.5.5).**

Consider the series
\[
\sum_{n=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)n} m_n(x, y; 1 + \beta + \gamma n, c, d) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{\delta}{\delta + (1 + \gamma)n} \left( \frac{\beta + (1 + \gamma)n}{n - k - m} \right) \frac{(-x)_k (w)^k (-x)_m (w)^m t^n}{k!m!},
\]
\[
= \sum_{n=0}^{\infty} \sum_{m,k=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)(n + k + m)} \left( \frac{\beta + (1 + \gamma)(n + k + m)}{n} \right) \frac{(-x)_k (wt)^k (-y)_m (vt)^m t^n}{k!m!}
\]
\[
= \sum_{m,k=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)(k + m)} \frac{(-x)_k (wt)^k (-y)_m (vt)^m}{k!m!}
\]
\[
\times \sum_{n=0}^{\infty} \frac{\delta + (1 + \gamma)(k + m)}{\delta + (1 + \gamma)(n + k + m)} \left( \frac{\beta + (1 + \gamma)(n + k + m)}{n} \right) t^n.
\]
Making use of the identity (1.3.129), we have
\[
\sum_{n=0}^{\infty} \frac{\delta}{\delta + (1 + \gamma)n} m_n(x, y; 1 + \beta + \gamma n, c, d) \frac{t^n}{n!}
\]
\[ = (1 + \xi)^{\beta} \sum_{m,k=0}^{\infty} \frac{\delta}{1 + \gamma + k + m} (-x)^{k} (w)^{k} (y)^{m} \frac{(vt)^{m}}{k!m!} \]

\[ \times \sum_{n=0}^{\infty} (-1)^{n} \left( \frac{\beta - \delta}{n} \right) \left( \frac{n + \delta + (1 + \gamma)(k + m)}{1 + \gamma} \right)^{-1} \left( \frac{\xi}{1 + \xi} \right)^{n}. \]

\[ \text{Hence, we obtain} \]
\[ \sum_{n=0}^{\infty} \frac{\delta}{n} \frac{m_{n}(x,y; 1 + \beta + \gamma n, c,d)}{n!} t^{n} \]
\[ = (1 + \xi)^{\beta} F^{(3)} \left[ \begin{array}{cccc} \vdots & -; & \frac{\delta}{1 + \gamma}; & - \\ 1 + \frac{\delta}{1 + \gamma} & \vdots & - & -; \end{array} \right] \left( \frac{\xi}{1 + \xi}, \xi w, \xi v \right) \tag{5.5.5} \]

\textbf{Proof of (5.2.1).}

From (1.3.86), we can write the following:

\[ \sum_{n=0}^{\infty} \frac{\beta(\lambda + \mu n)}{n} m_{n}(x,y; 1 + \beta + \gamma n, \frac{1}{w + 1}, \frac{1}{v + 1}) \frac{t^{n}}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{\beta(\lambda + \mu n)}{\beta + (1 + \gamma)n} \binom{\beta + (1 + \gamma)n}{n - k - m} \frac{(-x)^{k} (w)^{k} (-y)^{m} (v)^{m} t^{n}}{k!m!} \]

\[ \times \frac{(-x)^{k} (w)^{k} t^{n} (-y)^{m} (v)^{m}}{k!m!} \]

\[ = \sum_{k=0}^{\infty} \frac{\beta}{\beta + (1 + \gamma)(k + m)} (-x)^{k} (w)^{k} (-y)^{m} (v)^{m} \frac{t^{n}}{k!m!} \]

\[ \times \sum_{n=0}^{\infty} \frac{\beta + (1 + \gamma)(k + m)}{\beta + (1 + \gamma)(n + k + m)} \left( \frac{\beta + (1 + \gamma)(n + k + m)}{n} \right)^{n}. \]

Making use of the identity (1.3.130), we find that
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\[ \sum_{n=0}^{\infty} \frac{\beta(\lambda + \mu n)}{\beta + (1 + \gamma)n} m_n(x, y; 1 + \beta + \gamma n, \frac{1}{w + 1}, \frac{1}{v + 1}) \frac{t^n}{n!} \]

\[ = (1 + \xi)^\beta \sum_{m,k=0}^{\infty} \frac{\beta}{1 + \gamma + k + m} \frac{(-x)_k (w \xi)^k (-y)_m (v \xi)^m}{k!m!} \left\{ \lambda + \frac{\mu \beta \xi}{1 - \gamma \xi} + \frac{\mu(1 + \xi)(k + m)}{1 - \gamma \xi} \right\} \]

Thus, we have

\[ \sum_{n=0}^{\infty} \frac{\beta(\lambda + \mu n)}{\beta + (1 + \gamma)n} m_n(x, y; 1 + \beta + \gamma n, c, d) \frac{t^n}{n!} \]

\[ = (1 + \xi)^\beta \left\{ \lambda + \frac{\mu \xi}{1 - \gamma \xi} \left( \beta + (1 + \xi) \frac{\partial}{\partial \xi} \right) \right\} F_1 \left[ \frac{\beta}{1 + \gamma}, -x, -y; 1 + \frac{\beta}{1 + \gamma}; \xi w, \xi v \right]. \tag{5.5.6} \]

If \( x \) and \( y \) are the parameter and independent from \( t \) and \( n \), then for arbitrary constant \( \beta \) and \( \gamma \) independent from \( n \), we can obtained the following generating functions;

\[ \sum_{n=0}^{\infty} m_n(x + (1 + \gamma)n, y; \beta - (1 + \gamma)n, c, d) \frac{t^n}{n!} \]

\[ = (1 + \xi)^x (1 + c \xi)^{-x-\beta-y} (1 + \frac{\xi c}{\xi d})^y \frac{1}{1 - \xi \gamma} \tag{5.5.7} \]

\[ \sum_{n=0}^{\infty} m_n(x, y + (1 + \gamma)n; \beta - (1 + \gamma)n, c, d) \frac{t^n}{n!} \]

\[ = (1 + \xi)^y (1 + d \xi)^{-x-\beta-y} (1 + \frac{\xi d}{\xi c})^x \frac{1}{1 - \xi \gamma} \tag{5.5.8} \]

\[ \sum_{n=0}^{\infty} \frac{x}{n} m_n(x + (1 + \gamma)n, y; \beta - (1 + \gamma)n, c, d) \frac{t^n}{n!} \]

\[ = (1 + \xi)^x F^2 \left[ \frac{x}{1 + \gamma}, -x - \beta : -y; -; \frac{1 - d}{d}, -c \xi \right] \tag{5.5.9} \]

\[ \sum_{n=0}^{\infty} \frac{y}{n} m_n(x, y + (1 + \gamma)n; \beta - (1 + \gamma)n, c, d) \frac{t^n}{n!} \]

\[ = (1 + \xi)^y F^2 \left[ \frac{y}{1 + \gamma}, -y - \beta : -x; -; \frac{1 - c}{c}, -d \xi \right] \tag{5.5.10} \]