Chapter 4

Generating Functions for Arbitrary Product of Laguerre and Related Polynomials

Abstract: The present chapter deals with the generating function for arbitrary product of Laguerre polynomials and related polynomials in terms of multi-index Tricomi functions. The results established in this chapter are the extensions of the results of a paper of G. Dattoli et.al [107]. The chapter also contains a different expression of Hille-Hardy formula.

4.1 Introduction

The literature contains many examples of the use of fractional derivatives in the theory of hypergeometric functions for obtaining various results. Although other methods are usually available, the fractional derivative approach often give results which are not so obvious by the classical methods. Fractional derivative operator plays the role of augmenting parameters of the hypergeometric functions involved. Applying this operator on identities large number of generating functions for a variety of special functions have been obtained by numerous mathematicians.
In 2007, G. Dattoli, et.al [107] showed that the use of operational methods and of multi-index Tricomi functions allow the derivation of generating functions, involving arbitrary number of simple Laguerre polynomials \( L_n(x) \). The technique is based on the use of fractional derivative operator.

The action of fractional derivative \( D^\mu_z \) on a function \( f(z) \) is given by (1.3.142). In particular, if \( f(z) = z^\lambda \), we get

\[
D^\mu_z z^\lambda = \frac{\Gamma(1 + \lambda)}{\Gamma(1 + \lambda - \mu)} z^{\lambda-\mu}, \quad D_z \equiv \frac{d}{dz},
\]

where \( \text{Re}(\lambda) > -1 \) and \( \text{Re}(\mu) < 0 \).

In 2011, M. A. Khan and M. K. R. Khan [239] defined the operational representation of \( \text{I}_1 \) hypergeometric function and Laguerre polynomials by

\[
\begin{align*}
\text{I}_1[\alpha; \beta; x] &= x^{-\beta+1} (1 - D^{-1}_x)^{-\alpha} \{x^{\beta-1}\} \\
L_n^{(\alpha)}(x) &= \frac{(1 + \alpha)x^{-\alpha}}{n!} (1 - D^{-1}_x)^n \{x^{\alpha}\}
\end{align*}
\]

respectively (see also equation (3.3.4), (3.4.2) of the chapter III).

The relations of Laguerre polynomials with pseudo Laguerre polynomials of Shively \( (R_n(a, x)) \) (see [267]) and modified Laguerre polynomials \( (f_n^{-\alpha}(x)) \) (see [277]) are

\[
R_n(1 + a - n, x) = L_n^{(\alpha)}(x)
\]

and

\[
f_n^{(-\alpha-n)} = (-1)^n \ L_n^{(\alpha)}(x)
\]

respectively.

The \( n^{th} \) order Tricomi functions are defined as

\[
C_n(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r! \ (n+r)!},
\]
which is linked to the cylindrical Bessel functions by

\[ C_n(x) = x^{\frac{n}{2}} J_n(2\sqrt{x}). \]  

(4.1.7)

The Tricomi functions satisfy the generating functions

\[ \sum_{n=-\infty}^{\infty} t^n C_n(x) = \exp\left(t - \frac{x}{t}\right), \]  

(4.1.8)

\[ \sum_{n=\ell}^{\infty} \frac{t^n}{n!} C_n(x) = C_\ell(x - t) \]  

(4.1.9)

and cylindrical Bessel functions satisfy the generating function (see [305], p-427)

\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} J_{\mu+n}(x) = \left(1 - \frac{2t}{x}\right)^{-\frac{\mu}{2}} J_\mu\left(\sqrt{(x^2 - 2xt)}\right). \]  

(4.1.10)

### 4.2 Operational representation of \( C_n(x) \)

The exponential operator \( \exp(a D_x^{-1}) \) plays a central role in the present chapter. Following result can be obtained from (4.1.1) and (4.1.6)

\[ \exp(-a D_x^{-1}) \{x^n\} = \Gamma(1 + n)x^n C_n(a \ x) \]  

(4.2.1)

In the followings section we will demonstrate how the above results extend the results of a paper of G. Dattoli et.al [107].

### 4.3 Generating functions

We use the operational representations of Laguerre polynomials \( L_n^{(\alpha)}(x) \) and Tricomi function \( C_n(x) \) in terms of fractional derivative operator. From (4.1.3) and (4.2.1), we get the following
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\[ \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1+k+n)} L_n^{(k)}(x) = e^t C_k(tx). \]  

(4.3.1)

Let us now come to the specific problem, by considering the product of two Laguerre polynomials, we get the following

\[ \sum_{n=0}^{\infty} \frac{t^n n!}{\Gamma(1+k+n)\Gamma(1+m+n)} L_n^{(k)}(x) L_n^{(m)}(y) = \exp[t(1-D_x^{-1})(1-D_y^{-1})]\{x^k y^m\} \]

\[ = e^t C_{k,m}(tx, ty, txy), \]  

(4.3.2)

where \( C_{k,m}(x, y, t) \) is given by (1.6.4) (see also [107]). Above expression can be recognized as a different way of formulating the Hille-Hardy formula ([107], [288]).

Now, we further generalize the above result as follows:

\[ \sum_{n=0}^{\infty} \frac{t^n (n!)^2}{\Gamma(1+k+n)\Gamma(1+m+n)\Gamma(1+\ell+n)} L_n^{(k)}(x) L_n^{(m)}(y) L_n^{(\ell)}(z) \]

\[ = \exp[t(1-D_x^{-1})(1-D_y^{-1})(1-D_z^{-1})]\{x^k y^m z^\ell\} \]

\[ = e^t C_{k,m,\ell}(tx, ty, tz, txy, txyz), \]  

(4.3.3)

where \( C_{k,m,\ell}(x, y, z, t, u, v, w) \) is given by (1.6.6)(see also [107]). If we take \( k = 0, m = 0, \ell = 0 \) in (4.3.1), (4.3.2), (4.3.3), these results reduce to the results given by G. Dattoli et.al [107] as follows:

\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = e^t C_0(tx), \]  

(4.3.4)

\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x)L_n(y) = e^t C_{0,0}(tx, ty, txy), \]  

(4.3.5)

\[ \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} L_n(x)L_n(y)L_n(z) = e^t C_{0,0,0}(tx, ty, tz, txy, txyz), \]  

(4.3.6)
It is noted that three-index and seven variables Tricomi functions \( C_{k,m,\ell}(x, y, z, t, u, v, w) \) satisfy the following generating function, given by G. Dattoli et.al \[107\]

\[
G(x, y, z; u, v, w) = \sum_{m,n,p=\infty}^{\infty} u_1^m u_2^n u_3^p C_{m,n,p}(x, y, z) = \exp \left\{ 3 \sum_{\alpha=1}^{\infty} (u_j - \frac{x_\alpha}{u_\alpha} + \frac{x_\alpha}{u_\alpha u_{\alpha+1}}) - \frac{z}{u_1 u_2 u_3} \right\}, \quad u_4 = u_1, \quad (4.3.7)
\]

where \( C_{m,n,p}(x, y, z) \), \( \alpha = 1, 2, 3 \), is given by (1.6.6).

Further, we extend the result (4.3.3) to the product of \( m \) Laguerre polynomials as follows:

\[
\sum_{n=0}^{\infty} t^n (n!)^{m-1} \prod_{i=1}^{m} \frac{L_n^{(\ell_i)}(x_i)}{\Gamma(1 + \ell_i + n)} = \exp \left( t \prod_{i=1}^{m} (1 - D_{x_i}^{-1}) \right) \prod_{i=1}^{m} \{x_i^{\ell_i}\}, \quad (4.3.8)
\]

where \( i, j \) and \( k \) running from 0 to \( m \) and

\[
[x_it] = x_1t, x_2t, \ldots, x_mt,
\]

\[
[x_ix_jt]_{i \leq j} = (x_1x_2t, x_2x_3t, \ldots, mC_2 \text{ variables }),
\]

\[
[x_ix_jx_kt]_{i \leq j \leq k} = (x_1x_2x_3t, x_2x_3x_4t, \ldots, mC_3 \text{ variables }).
\]

The above \( C \) is \( m \)-index and \( 2^m - 1 \) variables Tricomi functions, if we take \( \ell_1 = 0, \ell_2 = 0, \ldots, \ell_{m-1} = 0 \) and \( \ell_m = 0 \) in (4.3.8), it reduces to the result given by G. Dattoli et.al \[107\].
\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} \prod_{i=1}^{m} L_n(x_i) = \exp(t) \ C_{0,0,...,0} \left\{ [x_i t], [x_i x_j t]_{i \leq j}, [x_i x_j x_k t]_{i \leq j \leq k}, \ldots, \prod_{s=1}^{m} x_s t \right\}. \]  (4.3.9)

From the relations (4.1.4) and (4.1.5), we can write the generating function involving arbitrary product for Shively’s pseudo Laguerre polynomials \( R_n(a, x) \) and modified Laguerre polynomials \( f_{-n}(x) \) as follows:

\[ \sum_{n=0}^{\infty} \frac{t^n (n!)^{m-1}}{\prod_{i=1}^{m} \Gamma(1 + \ell_i + n)} \prod_{i=1}^{m} \frac{R_n(1 + \ell_i - n, x_i)}{\Gamma(1 + \ell_i + n)} \]

\[ = \exp(t) \ C_{\ell_1, \ell_2, \ldots, \ell_m} \left\{ [x_i t], [x_i x_j t]_{i \leq j}, [x_i x_j x_k t]_{i \leq j \leq k}, \ldots, \prod_{s=1}^{m} x_s t \right\}, \quad (4.3.10) \]

\[ \sum_{n=0}^{\infty} \frac{t^n (-1)^{mn} (n!)^{m-1}}{\prod_{i=1}^{m} \Gamma(1 + \ell_i + n)} \prod_{i=1}^{m} f_{n}^{-\ell_i - n}(x_i) \]

\[ = \exp(t) \ C_{\ell_1, \ell_2, \ldots, \ell_m} \left\{ [x_i t], [x_i x_j t]_{i \leq j}, [x_i x_j x_k t]_{i \leq j \leq k}, \ldots, \prod_{s=1}^{m} x_s t \right\} \quad (4.3.11) \]

respectively.

Before discussing the problem of further generalizations of the results, we consider the Tricomi functions and changing the index with arbitrary complex parameter as follows:

\[ \phi_{\alpha}(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r! \Gamma(1 + \alpha + r)}. \quad (4.3.12) \]

Thus, we can write the following:

\[ \phi_{\alpha}(a x) = (a x)^{-\frac{\alpha}{2}} J_{\alpha}(2 \sqrt{a x}), \quad (4.3.13) \]
if $\alpha = n$, then

$$\phi_n(x) = C_n(x).$$  \hspace{1cm} (4.3.14)

Along with the one-index one variable functions, we introduce one-index two variable functions as follows:

$$\phi_\alpha(x, y) = \sum_{s=0}^{\infty} \frac{(-1)^s \phi_{\alpha+s}(x)(y)^s}{s!}. \hspace{1cm} (4.3.15)$$

Again, if we take $\alpha = n$ in above, we get the one-index two variable Tricomi functions, given by G. Dattoli et.al [107].

Now, we introduce two-index three variable function as follows:

$$\phi_{\alpha,\beta}(x, y, z) = \sum_{s=0}^{\infty} \frac{\phi_{\alpha+s}(x)\phi_{\beta+s}(y)(z)^s}{s!}. \hspace{1cm} (4.3.16)$$

It is easy to show that

$$\phi_\alpha(x, y) = [x + y]^{-\frac{\alpha}{2}} J_\alpha (\sqrt{(4x + 2y)}), \hspace{1cm} (4.3.17)$$

$$\phi_{\alpha,\beta}(x, y, z) = x^{-\alpha} y^{-\beta} \exp \left( \frac{z}{x y} \right) J_\alpha (\sqrt{x}) J_\beta (\sqrt{y}). \hspace{1cm} (4.3.18)$$

As $\phi_n = C_n, \phi_{n,m} = C_{n,m}$, then the relevant generating functions will be

$$\sum_{n=\infty}^{\infty} u^n \phi_n(x, y) = \exp \left( u - \frac{x}{u} - \frac{y}{u} \right), \hspace{1cm} (4.3.19)$$

$$\sum_{n,m=\infty}^{\infty} u^n v^m \phi_{n,m}(x, y, z) = \exp \left( u + v - \frac{x}{u} - \frac{y}{v} + \frac{z}{uv} \right). \hspace{1cm} (4.3.20)$$

From (4.3.15) and (4.3.16), we conclude that

$$\sum_{s=0}^{\infty} \frac{u^s}{s!} \phi_{\alpha+s}(x, y) = \phi_\alpha(x - u, y), \hspace{1cm} (4.3.21)$$

$$\sum_{s=0}^{\infty} \frac{v^s}{s!} \phi_{\alpha+s}(x, y) = \phi_\alpha(x, y + v). \hspace{1cm} (4.3.22)$$
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\[\sum_{s=0}^{\infty} \frac{u^s}{s!} \phi_{\alpha+s,\beta}(x, y, z) = \phi_{\alpha,\beta}(x - u, y, z), \quad (4.3.23)\]

\[\sum_{s=0}^{\infty} \frac{v^s}{s!} \phi_{\alpha,\beta+s}(x, y, z) = \phi_{\alpha,\beta}(x, y - v, z), \quad (4.3.24)\]

\[\sum_{s=0}^{\infty} \frac{w^s}{s!} \phi_{\alpha+s,\beta+s}(x, y, z) = \phi_{\alpha,\beta}(x, y, z + w). \quad (4.3.25)\]

Further, if we replace all index with arbitrary complex parameters of multi-index multi variable tricomi function, we get multi-index multi variable function \(\phi\).

### 4.4 Operational representation of \(\phi_{\alpha}(x)\)

Following results are obtained from (4.1.1) and (4.3.12)

\[\exp(-a \, D_x^{-1}) \{x^\alpha\} = \Gamma(1 + \alpha)x^\alpha \phi_{\alpha}(a \, x), \quad (4.4.1)\]

\[\exp(-a \, D_x^{-1}) \{x^\alpha\} = \Gamma(1 + \alpha)\left(\frac{x}{a}\right)^{\frac{\alpha}{2}} J_{\alpha}(2\sqrt(ax)). \quad (4.4.2)\]

### 4.5 Generating Functions

Now, we use the operational representations of Laguerre polynomials and function \(\phi_{\alpha}\) in terms of fractional derivative operator. From (4.1.3) and (4.4.1), we get the followings

\[\sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1 + \alpha + n)} L_n^{(\alpha)}(x) = e^t \, \phi_{\alpha}(tx). \quad (4.5.1)\]

Further, by considering the product of two Laguerre polynomials, we obtain

\[\sum_{n=0}^{\infty} \frac{t^n \, n!}{\Gamma(1 + \alpha + n)\Gamma(1 + \beta + n)} L_n^{(\alpha)}(x)L_n^{(\beta)}(y)\]

\[= \exp[t(1 - D_x^{-1})(1 - D_y^{-1})]\{x^\alpha y^\beta\}\]

\[= e^t \, \phi_{\alpha,\beta}(tx, ty, txy), \quad (4.5.2)\]
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Thus, the expression (4.5.2) can be recognized as a different way of formulating the Hille-Hardy formula ([107], [288]).

Now, we further generalized the above result as follows:

\[
\sum_{n=0}^{\infty} \frac{t^n (n!)^2}{\Gamma(1 + \alpha + n)\Gamma(1 + \beta + n)\Gamma(1 + \gamma + n)} L_n^{(\alpha)}(x) L_n^{(\beta)}(y) L_n^{(\gamma)}(z) = \exp[t(1 - D_x^{-1})(1 - D_y^{-1})(1 - D_z^{-1})]\{x^\alpha y^\beta z^\gamma\}
\]

\[
= e^t \phi_{\alpha,\beta,\gamma}(tx, ty, zt, txy, txz, tyz, txyz).
\]

If we set \(\alpha = m, \beta = p\) and \(\gamma = q\) in (4.5.1) (4.5.2) and (4.5.3), we get

\[
\sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1 + m + n)} L_n^{(m)}(x) = e^t C_m(tx),
\]

\[
\sum_{n=0}^{\infty} \frac{t^n n!}{\Gamma(1 + m + n)\Gamma(1 + p + n)} L_n^{(m)}(x) L_n^{(p)}(y) = e^t C_{m,p}(tx, ty, txy),
\]

\[
\sum_{n=0}^{\infty} \frac{t^n (n!)^2}{\Gamma(1 + m + n)\Gamma(1 + p + n)\Gamma(1 + q + n)} L_n^{(m)}(x) L_n^{(p)}(y) L_n^{(q)}(z) = e^t C_{m,p,q}(tx, ty, zt, txy, txz, tyz, txyz).
\]

Further, if we take \(\alpha = 0, \beta = 0\) and \(\gamma = 0\) in (4.5.1), (4.5.2) and (4.5.3), these results reduce to the results given by G. Dattoli et.al [107].

Now, we extend the result (4.5.3) to the product of \(m\) Laguerre polynomials as follows:

\[
\sum_{n=0}^{\infty} t^n (n!)^{m-1} \prod_{i=1}^{m} \frac{L_n^{(\alpha_i)}(x_i)}{\Gamma(1 + \alpha_i + n)}
\]
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\[= \exp \left[ t \prod_{i=1}^{m} (1 - D_{x_i}^{-1}) \right] \prod_{i=1}^{m} \left\{ x_i^{\alpha_i} \right\} \]

\[= \exp(t) \left\{ [x_1 t], [x_2 x_1 t]_{i \leq j}, [x_3 x_2 x_1 t]_{i \leq j \leq k}, \ldots, \prod_{s=1}^{m} x_s t \right\}, \quad (4.5.7)\]

where \(i, j\) and \(k\) are running from 0 to \(m\) and

\[=[x_i t] = x_1 t, x_2 t, \ldots, x_m t, \]

\[[x_i x_j t]_{i \leq j} = (x_1 x_2 t, x_2 x_3 t, \ldots, \quad mC_2 \text{ variables }), \]

\[[x_i x_j x_k t]_{i \leq j \leq k} = (x_1 x_2 x_3 t, x_2 x_3 x_4 t, \ldots, \quad mC_3 \text{ variables }). \]

The above \(C\) is \(m\)-index and \(2^m - 1\) variables function, if we take \(\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_{m-1} = 0\) and \(\alpha_m = 0\) in (4.5.7), it reduces to the results given by G. Dattoli et.al [107] as follows:

\[= \exp(t) \phi_{\alpha_1, \alpha_2, \ldots, \alpha_m} \left\{ [x_1 t], [x_2 x_1 t]_{i \leq j}, [x_3 x_2 x_1 t]_{i \leq j \leq k}, \ldots, \prod_{s=1}^{m} x_s t \right\} \]

\[= \exp(t) \phi_{\alpha_1, \alpha_2, \ldots, \alpha_m} \left\{ [x_i t], [x_i x_j t]_{i \leq j}, [x_i x_j x_k t]_{i \leq j \leq k}, \ldots, \prod_{s=1}^{m} x_s t \right\}, \quad (4.5.8)\]

From the relations (4.1.4) and (4.1.5), we can write the generating function involving arbitrary product for Shively’s pseudo Laguerre polynomials \((R_n(a, x))\) and modified Laguerre polynomials \((f_n^{-a}(x))\) as follows:

\[= \exp(t) \phi_{\alpha_1, \alpha_2, \ldots, \alpha_m} \left\{ [x_i t], [x_i x_j t]_{i \leq j}, [x_i x_j x_k t]_{i \leq j \leq k}, \ldots, \prod_{s=1}^{m} x_s t \right\}, \quad (4.5.9)\]
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\[
\sum_{n=0}^{\infty} t^n (n!)^{m-1} \prod_{i=1}^{m} \frac{f_n^{-\alpha_i - n}(x_i)}{\Gamma(1 + \alpha_i + n)}
\]

\[
= \exp(t) \phi_{\alpha_1, \alpha_2, \ldots, \alpha_m} \left\{ [x_i t], [x_i x_j t]_{i \leq j}, [x_i x_j x_k t]_{i \leq j \leq k}, \ldots, \prod_{s=1}^{m} x_s t \right\} \quad (4.5.10)
\]

respectively.