Chapter 8

Application of Z-transform on a New Polynomial Set

Abstract: The present chapter deals with a new polynomial set and application of Z-transform on it. We also define some applications of this polynomial set and $n^{th}$ derivative of $(1 - e^x)^{-1}$, for $n > 0$, in terms of this polynomial set. Some useful recurrence relations, summation formulae, generating functions for this polynomial set and relation of geometric polynomials with this polynomial set are also given.

8.1 Introduction

In present chapter, we discus about a new polynomial set, which is related to geometric polynomials (see [54]). Important feature of this polynomials set is in the theory of related polynomials because Z-transform play an important role with this polynomial set. This polynomial set can be used to find out list of properties of related polynomials. The used techniques is seem to be new and also use to computation of consecutive derivatives.
Geometric polynomials:

Boyadziev K.N. (see [54]) denoted geometric polynomials by \( w_n(x) \) and are defined as:

\[
w_n(x) = \sum_{k=0}^{n} S(n, k) k! x^k, \tag{8.1.1}
\]

where \( S(n, k) \) are Stirling numbers of second kind, given by (1.3.12).

Laguerre polynomials:

Laguerre polynomials \( L_n^{(\alpha)}(x) \) are defined by (1.3.50) and satisfy the following generating functions:

\[
\begin{align*}
\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n &= (1 - t)^{-\alpha - 1} e^{xt}, \tag{8.1.2} \\
\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x) t^n &= (1 + t)^{\alpha} e^{-xt}, \tag{8.1.3} \\
\sum_{n=0}^{\infty} L_n^{(\alpha+n\beta)}(x) t^n &= \frac{(1 + v)^{1+\alpha}}{1 - \beta v} e^{-xv}, \tag{8.1.4}
\end{align*}
\]

where \( v = t(1 + v)^{1+\beta} \).

Jacobi polynomials:

Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \) are defined (6.1.1) and satisfy the following generating functions:

\[
\begin{align*}
\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x) \frac{t^n}{(-\alpha - \beta)_n} &= e^{-\frac{(x-1)t}{2}} {}_1 F_1 \left[ -\alpha; \alpha - \beta; -t \right], \tag{8.1.5} \\
\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x) \frac{t^n}{(-\alpha - \beta)_n} &= e^{-\frac{(x+1)t}{2}} {}_1 F_1 \left[ -\alpha; -\alpha - \beta; -t \right]. \tag{8.1.6}
\end{align*}
\]
8.2 New Polynomial Set and Some Properties

Let $\Upsilon_n(t)$ be a polynomial set defined by

$$\Upsilon_n(t) = \sum_{k=0}^{n} (-1)^{n+k} S(n, k)! t^k (t-1).$$  \hspace{1cm} (8.2.1)

Here $\Upsilon_n(t)$ is a polynomial of degree $n + 1$ and $S(n, k)$ is Stirling number of second kind, defined by (1.3.12).

**Theorem 1:**

The polynomial set $\Upsilon_n(t)$ of degree $n + 1$, defined by (8.2.1) can be written in the following form:

$$\Upsilon_n(t) = \left\{ t(t-1) \frac{d}{dt} \right\}^n t, \hspace{1cm} n > 0.$$

(8.2.2)

**Proof.**

Using (8.2.1) and setting $t = \frac{x}{x-1}$, we have

$$\Upsilon_n\left(\frac{x}{x-1}\right) = \sum_{k=0}^{n} (-1)^{n+k} S(n, k)! \frac{x^k}{(x-1)^{k+1}},$$

$$\Upsilon_n\left(\frac{x}{x-1}\right) = \sum_{k=0}^{n} (-1)^{n} S(n, k)x^k \left( \frac{d}{dx} \right)^k \frac{x}{(x-1)}.$$

From (1.3.13), we get

$$\Upsilon_n\left(\frac{x}{x-1}\right) = \left\{ -x \frac{d}{dx} \right\}^n \frac{x}{(x-1)}.$$  \hspace{1cm} (8.2.3)

Again, by setting $t = \frac{x}{(x-1)}$ in any function $f$, we have

$$\frac{d}{dx} f = \frac{d}{dt} f \frac{dt}{dx}$$

$$\frac{d}{dx} f = \frac{d}{dt} f \frac{-1}{(x-1)^2}.$$  \hspace{1cm} (8.2.4)

From (8.2.3) and (8.2.4), we get

$$\Upsilon_n(t) = \left\{ t(t-1) \frac{d}{dt} \right\}^n t, \hspace{1cm} n > 0.$$  \hspace{1cm} (8.2.5)
From (8.1.1), we can establish the following relation with geometric polynomials:

\[(t - 1)^{-1} \Upsilon_n(t) = (-1)^n w_n(t). \quad (8.2.6)\]

It is clear that the polynomial set \( \Upsilon_n(t) \) is a special geometric polynomials which emerge from the coefficients \( S(n, k) \) and \( S(n, k + 1) \). Now we show that this special geometric polynomials play an important role in the theory of geometric polynomials and in the computation of consecutive derivatives.

First we define recurrence relations for this polynomial set. From (8.2.2), we can write

\[
\Upsilon_{n+m}(t) = \left\{ t(t - 1) \frac{d}{dt} \right\}^{n+m} t, \quad n > 0. \quad (8.2.7)
\]

If we take \( m = 1 \) in (8.2.7), we get

\[
\Upsilon_{n+1}(t) = t(t - 1) \Upsilon'_n(t). \quad (8.2.8)
\]

Again, if we take \( m = 2 \), we get

\[
\Upsilon_{n+2}(t) = (2t - 1) \Upsilon_{n+1}(t) + t(t - 1) \Upsilon''_n(t). \quad (8.2.9)
\]

**Theorem 2:**

Let \( \Upsilon_n(t) \) be a polynomial set defined by (8.2.1), then for any positive integer \( m \);

\[
\Upsilon_n(t) = \frac{1-t}{t} \sum_{k=0}^{m} \sum_{s=0}^{k} \binom{m}{k} \binom{k}{s} \Upsilon_{m-k}(x) \Upsilon_{n+m-s}(x), \quad n \geq m > 0; \quad (8.2.10)
\]

\[
\Upsilon_n(t) = \frac{1-t}{t} \sum_{k=0}^{n-m} \sum_{s=0}^{k} \binom{n-m}{k} \binom{k}{s} \Upsilon_{n-m}(x) \Upsilon_{m+s}(x), \quad n \geq m > 0. \quad (8.2.11)
\]

**Proof.**

For any positive integer \( m \), (8.2.2) can be written in the following form:

\[
\Upsilon_n(t) = \left\{ t(t - 1) \frac{d}{dt} \right\}^{n-m+m} t, \quad n \geq m > 0
\]
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\[
\{ t(t-1) \frac{d}{dt} \}^m t \cdot \Upsilon_{n-m}(t) \frac{1}{t} \\
= \sum_{k=0}^{m} \binom{m}{k} \Upsilon_{m-k}(t) \left\{ t(t-1) \frac{d}{dt} \right\}^k \left\{ \Upsilon_{n-m}(t) \frac{1}{t} \right\} \quad \text{(by (1.3.14))}
\]

\[
= \frac{1}{t} \sum_{k=0}^{m} \binom{m}{k} \Upsilon_{m-k}(t) \sum_{s=0}^{k} \binom{k}{s} \Upsilon_{n+s-m}(t) \left\{ t(t-1) \frac{d}{dt} \right\}^s \frac{1}{t}
\]

\[
= \frac{(1-t)}{t} \sum_{k=0}^{m} \sum_{s=0}^{k} \binom{m}{k} \left( \binom{k}{s} \Upsilon_{m-k}(t) \Upsilon_{n+s-m}(t). \right)
\]

The proof of (8.2.11) is similar to (8.2.10).

**Theorem 3:**

The \( n^{th} \) derivative of \((1 - e^x)^{-1}\), for \( n > 0 \) will be

\[
\left( \frac{d}{dx} \right)^n \frac{1}{1 - e^x} = \sum_{k=0}^{n} (-1)^{n+k} S(n, k)! \frac{e^x}{(1 - e^x)^{k+1}}. \tag{8.2.12}
\]

**Proof.**

Let

\[
t = \frac{e^{-x}}{(e^{-x} - 1)} \quad \Rightarrow \quad x = \log(t - 1) - \log t.
\]

Thus for any function \( f \);

\[
\frac{d}{dt} f = \frac{1}{t(t-1)} \frac{d}{dx} f.
\]

Now, setting \( t = \frac{e^{-x}}{(e^{-x} - 1)} \) in (8.2.2), we get

\[
\Upsilon_n \left( \frac{e^{-x}}{e^{-x} - 1} \right) = \left\{ \frac{d}{dx} \right\}^n \frac{e^{-x}}{e^{-x} - 1}.
\]

From (8.2.1), we have

\[
\left( \frac{d}{dx} \right)^n \frac{1}{1 - e^x} = \sum_{k=0}^{n} (-1)^{n+k} S(n, k)! \frac{e^x}{(1 - e^x)^{k+1}}.
\]
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The $n^{th}$ derivative of $(1 - e^x)^{-1}$ can also be given by replacing $x$ by $e^x$ in a result (See [54], equation (3.8)) of K. N. Boyadziev. Now, we consider a series $\Phi_n(x)$ with Mellin derivative and is defined as:

$$\Phi_n(x) = (-xD_x)^n \frac{x}{(x-1)}, \quad (8.2.13)$$

where $D_x \equiv \frac{d}{dx}$ and $n$ is non-negative integer. From (1.3.13), it is clear that

$$\Phi_n(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} k! x^k D_x^k \left( \frac{-x}{(x-1)} \right), \quad n \geq 0. \quad (8.2.14)$$

Further, $\Phi_n(x)$ satisfies the following properties:

$$\Phi_{n+1}(x) = -x\Phi_n'(x), \quad (8.2.15)$$

$$\Phi_{n+2}(x) = \Phi_{n+1}(x) + x^2 \Phi_n''(x), \quad (8.2.16)$$

$$\Phi_{n+m}(x) = (-xD_x)^{n+m} \frac{x}{(x-1)}, \quad (8.2.17)$$

$$\Phi_{n+m}(x) = (-xD_x)^n \Phi_m(x), \quad (8.2.18)$$

$$\Phi_{n}(x) = \sum_{k=0}^{m} \sum_{s=0}^{k} \binom{m}{k} \binom{k}{s} \Phi_{m-k}(x) \Phi_{n+s-m}(x), \quad (8.2.19)$$

The $n^{th}$ derivative of $(1 - e^{-t})^{-1}$ can also be given in terms of $\Phi_n(x)$ by taking $x = e^t$,

then for any differentiable function $f$,

$$\frac{d}{dx} f = \left( \frac{d}{dt} f \right) \frac{1}{e^t}. \quad (8.2.20)$$

From (8.2.13), we get

$$\Phi_n(e^t) = (-D_t)^n (1 - e^{-t})^{-1}. \quad (8.2.20)$$
Theorem 4:

The relations between $\Phi_n(x)$ and $\Upsilon_n(t)$ are defined as

$$\lim_{x \to \frac{t}{t-1}} \Phi_n(x) = \Upsilon_n(t), \quad n > 0, \quad (8.2.21)$$

$$\lim_{t \to \frac{x}{x-1}} \Upsilon_n(t) = \Phi_n(x), \quad n > 0. \quad (8.2.22)$$

Proof.

If we take $x = \frac{t}{t-1}$ in (8.2.13), we find

$$\Phi_n\left(\frac{t}{t-1}\right) = \Upsilon_n(t).$$

Thus, we can write

$$\lim_{x \to \frac{t}{t-1}} \Phi_n(x) = \Upsilon_n(t), \quad n > 0.$$  

Similarly, if we take $t = \frac{x}{x-1}$ in (8.2.2), we get (8.2.22).

Theorem 5:

The series $\Phi_n(x)$, defined by (8.2.13) has the form

$$\Phi_n(x) = \sum_{k=0}^{\infty} k^n x^{-k}, \quad (8.2.23)$$

where $|x| > 1$.

Proof.

Consider (8.2.13)

$$\Phi_n(x) = (-xD_x)^n \frac{x}{(x-1)},$$

$$\Phi_n(x) = (-xD_x)^n \left(1 - \frac{1}{x}\right)^{-1}.$$  

For $|x| > 1$, we get

$$\Phi_n(x) = (-xD_x)^n \sum_{k=0}^{\infty} x^{-k},$$
\[ \Phi_n(x) = (-xD_x)^{n-1} \sum_{k=0}^{\infty} k \ x^{-k}. \]  
(8.2.24)

By the induction, we can write

\[ \Phi_n(x) = \sum_{k=0}^{\infty} k^n \ x^{-k}. \]

For the convergence condition of series like (8.2.23), we take a function of the form

\[ f(t) = \sum_{n=0}^{\infty} (a)_n \ t^n, \]

where \((a)_n\) is Pochhammer symbol, defined by (1.3.2). For convenience, we take

\[ f(k) = \sum_{n=0}^{\infty} (a)_n \ k^n. \]

Let \(g(x)\) be another function, defined as

\[ g(x) = \sum_{k=0}^{\infty} x^{-k}, \]

where \(|x| > 1\), then

\[ f(k)g(x) = \sum_{k=0}^{\infty} f(k)x^{-k}, \]
(8.2.25)

provided that the series on the right side converges. When \(f(k)\) is a polynomial, the idea of writing (8.2.25) was exploited by Schwatt [285] and more recently by Boyadzhiev [54] to evaluate the series like (8.2.25).

### 8.3 Application of Z-transform to \(\Phi_n(x)\) and \(\Upsilon_n(x)\)

The Z-transform of a function \(y^k\) is defined as follows:

\[ Z\{y^k\} = \sum_{k=0}^{\infty} (Tk)^n \ x^{-k}, \]
(8.3.1)

where \(T\) is a fixed positive number usually referred to as the sampling period. If we take \(T = 1\) in (8.3.1), we get

\[ Z\{y^k\} = \sum_{k=0}^{\infty} (k)^n \ x^{-k} = \Phi_n(x). \]
(8.3.2)
8.3.1 Generating Functions

The $\Phi_n(x)$ and $\Upsilon_n(x)$ satisfy the following generating functions:

\[ \sum_{n=0}^{\infty} \frac{\Phi_n(x)t^n}{n!} = x(x - e^{-t})^{-1}, \quad (8.3.3) \]

\[ \sum_{n=0}^{\infty} \frac{\Upsilon_n(x)t^n}{n!} = x(x - (x - 1)e^{-t})^{-1}, \quad (8.3.4) \]

\[ \sum_{n=0}^{\infty} \frac{\Phi_{n+1}(x)t^n}{n!} = xe'^{x - e'}, \quad (8.3.5) \]

\[ \sum_{n=0}^{\infty} \frac{\Upsilon_{n+1}(x)t^n}{n!} = x(x - 1)^{-3}(x - (x - 1)e'^t)^2, \quad (8.3.6) \]

\[ \sum_{n=0}^{\infty} \frac{\Phi_{n+2}(x)t^n}{n!} = x(x + e')(x - e')^{-3}, \quad (8.3.7) \]

\[ \sum_{n=0}^{\infty} \frac{\Upsilon_{n+2}(x)t^n}{n!} = x(x - 1)(x + (x - 1)e') (x - (x - 1)e'^t)^{-3}, \quad (8.3.8) \]

\[ \sum_{n=0}^{\infty} \frac{\Phi_{m+n}(x)t^n}{n!} = \Phi_m(xe^{-t}), \quad (8.3.9) \]

\[ \sum_{n=0}^{\infty} \frac{\Upsilon_{m+n}(x)t^n}{n!} = \Phi_m \left( \frac{xe^{-t}}{x - 1} \right), \quad (8.3.10) \]

\[ \sum_{n=0}^{\infty} \frac{\Phi_{2n}(x)(-1)^nt^{2n}}{2n!} = \frac{x(x - \cos t)}{x^2 - 2x \cos t + 1}, \quad (8.3.11) \]

\[ \sum_{n=0}^{\infty} \frac{\Upsilon_{2n}(x)(-1)^nt^{2n}}{2n!} = \frac{x(x - (x - 1) \cos t)}{x^2 - 2x(x - 1) \cos t + (x - 1)^2}, \quad (8.3.12) \]

\[ \sum_{n=0}^{\infty} \frac{\Phi_{2n+1}(x)(-1)^nt^{2n+1}}{(2n + 1)!} = \frac{x \sin t}{x^2 - 2x \cos t + 1}, \quad (8.3.13) \]

\[ \sum_{n=0}^{\infty} \frac{\Upsilon_{2n+1}(x)(-1)^nt^{2n+1}}{(2n + 1)!} = \frac{x(x - 1) \sin t}{x^2 - 2x(x - 1) \cos t + (x - 1)^2}. \quad (8.3.14) \]
Proof of (8.3.3).

Taking the left hand side of (8.3.3), and applying the result (8.3.2), we get
\[
\sum_{n=0}^{\infty} \frac{\Phi_n(x)t^n}{n!} = \sum_{n=0}^{\infty} \frac{Z\{y^n\}t^n}{n!}. 
\]

As $Z$-transform is a linear operator (see (1.3.132)), we get
\[
\sum_{n=0}^{\infty} \frac{\Phi_n(x)t^n}{n!} = Z \sum_{n=0}^{\infty} \frac{y^n t^n}{n!} = Z \{e^{yt}\}. 
\]

Again, from the shifting theorem (see property (iii) of subsection (1.3.8)) of $Z$-transform
\[
\sum_{n=0}^{\infty} \frac{\Phi_n(x)t^n}{n!} = \lim_{x \to xe^{-t}} Z\{1\}
= \lim_{x \to xe^{-t}} \frac{x}{x - 1}
= \frac{xe^{-t}}{xe^{-t} - 1}
= x(1 - xe^{-t})^{-1}. 
\]

If we apply Theorem 4 on (8.3.3), we get the generating function (8.3.4).

Proof of (8.3.5).

Taking the left hand side of (8.3.5) and applying the result (8.3.2),
\[
\sum_{n=0}^{\infty} \frac{\Phi_n(x)t^n}{n!} = \sum_{n=0}^{\infty} \frac{Z\{y^n\}t^n}{n!}. 
\]

Again on the similar line of proof of (8.3.3), we get
\[
\sum_{n=0}^{\infty} \frac{\Phi_n(x)t^n}{n!} = \lim_{x \to xe^{-t}} Z\{t\}
= \lim_{x \to xe^{-t}} \frac{x}{(x - 1)^2}
= xe^t(x - e^t)^2. 
\]

(8.3.6) can be obtained by using Theorem 4 on (8.3.5).

The proofs of (8.3.8) and (8.3.10) are similar to (8.3.3).
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Proof of (8.3.9).

Taking the left hand side of (8.3.9), and applying the result (8.3.2), we get
\[ \sum_{n=0}^{\infty} \frac{\Phi_n(x)t^n}{n!} = \sum_{n=0}^{\infty} \frac{Z\{y^{n+m}\}t^n}{n!}. \]

As Z-transform is a linear operator (see (1.3.132)), we get
\[ \sum_{n=0}^{\infty} \frac{\Phi_n(x)t^n}{n!} = Z\{y^m\} \sum_{n=0}^{\infty} \frac{y^nt^n}{n!} = Z\{y^m_1\}. \]

Again, from the shifting theorem (see property (iii) of subsection {1.3.8}) of Z-transform
\[ \sum_{n=0}^{\infty} \frac{\Phi_n(x)t^n}{n!} = \lim_{x \to xe^{-t}} Z\{y^m\} = \lim_{x \to xe^{-t}} \phi_m(x) = \phi_m(xe^{-t}). \]

Again, we apply theorem 4 on (8.3.9) to get the generating function (8.3.10).

Proof of (8.3.11) and (8.3.13).

We take \( t = -it \) in (8.3.3) to get
\[ \sum_{n=0}^{\infty} \frac{\Phi_n(x)(-it)^n}{n!} = Z\{e^{-it}\}. \]

Again, from the shifting theorem (see property (iii) of subsection {1.3.8}) of Z-transform
\[ \lim_{x \to xe^{it}} Z\{1\} = \lim_{x \to xe^{it}} \frac{x}{x - 1} = \frac{xe^{it}}{(xe^{it} - 1)}. \]
= \frac{x(x - e^{it})}{(x - e^{-it})(x - e^{-it})}
= \frac{x(x - \cos t - i \sin t)}{\{x - x(e^{it} + e^{-it}) + 1\}}
= \frac{x(x - \cos t)}{\{x - 2x \cos t + 1\}} + \frac{-ix \sin t}{\{x - 2x \cos t + 1\}}. \quad (8.3.15)

Further, we take left hand side of (8.3.3)
\sum_{n=0}^{\infty} \frac{\Phi_n(x)(-it)^n}{n!} = \sum_{n=0}^{\infty} \frac{\Phi_{2n}(x)(-1)^n t^{2n}}{2n!} + i \sum_{n=0}^{\infty} \frac{\Phi_{2n+1}(x)(-1)^n t^{2n+1}}{(2n+1)!}. \quad (8.3.16)

Equating real and imaginary parts from (8.3.15) and (8.3.16), we get
\sum_{n=0}^{\infty} \frac{\Phi_{2n}(x)(-1)^n t^{2n}}{2n!} = \frac{x(x - \cos t)}{x^2 - 2x \cos t + 1},
\sum_{n=0}^{\infty} \frac{\Phi_{2n+1}(x)(-1)^n t^{2n+1}}{(2n+1)!} = \frac{x \sin t}{x^2 - 2x \cos t + 1}.

By theorem 4, we can prove (8.3.12) and (8.3.14).

8.3.2 Generating Function’s Theorem

Theorem 7:
For series \Phi_n(x) (n > 0) given by (8.2.13), define new one \Psi_n(x) by
\Psi_n(x) = \sum_{k=0}^{\infty} \frac{\alpha + (\beta + 1)n}{n-k} \frac{\Phi_k(x)}{k!}, \quad n \in \{0, 1, 2, \ldots\}, \quad (8.3.17)

where \alpha and \beta are complex parameter independent of n. Then
\sum_{n=0}^{\infty} \Psi_n(x)t^n = \frac{(1 + \xi)^{1+\alpha}}{1 - \beta \xi} x(x - e^\xi)^{-1}, \quad (8.3.18)

where \xi is a function of t, defined by the relation
\xi = t(1 + \xi)^{1+\beta}, \quad \xi(0) = 0.
Proof.

The proof of theorem is based on combinational identities. In view of the definition (8.3.23), we get

\[
\sum_{n=0}^{\infty} \Psi_n(x) t^n = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \left( \frac{\alpha + (\beta + 1)n}{n-k} \right) \frac{\Phi_k(x)}{k!}
\]

\[
= Z \left\{ \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \left( \frac{\alpha + (\beta + 1)n}{n-k} \right) \frac{y^k}{k!} \right\}
\]

\[
= Z \left\{ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{\alpha + (\beta + 1)(n+k)}{n} \right) \frac{y^k t^n+k}{k!} \right\}
\]

\[
= Z \left\{ \sum_{k=0}^{\infty} \frac{y^k}{k!} t^k \sum_{n=0}^{\infty} \left( \frac{\alpha + (\beta + 1)(n+k)}{n} \right) \frac{y^k t^n+k}{k!} \right\}.
\]

Now, using the identity (1.3.124) [see Pólya and Szegö [273], p.349]

\[
\sum_{n=0}^{\infty} \left( \frac{\alpha + (\beta + 1)n}{n} \right) t^n = \frac{(1 + \xi)^{1+\alpha}}{1 - \beta \xi},
\] (8.3.19)

where \(\alpha\) and \(\beta\) are complex parameter independent of \(n\) and \(\xi\) is a function of \(t\) defined implicitly as:

\[
\xi = t(1 + \xi)^{1+\beta}, \quad \xi(0) = 0,
\] (8.3.20)

we get

\[
\sum_{n=0}^{\infty} \Psi_n(x) t^n = Z \left\{ \frac{(1 + \xi)^{1+\alpha}}{1 - \beta \xi} \sum_{k=0}^{\infty} \frac{y^k(1 + \xi)^{(1+\beta)k}}{k!} t^k \right\}
\]

\[
= \frac{(1 + \xi)^{1+\alpha}}{1 - \beta \xi} Z \sum_{k=0}^{\infty} \frac{y^k \xi^k}{k!}
\]

\[
= \frac{(1 + \xi)^{1+\alpha}}{1 - \beta \xi} \{ e^{\xi y} \}
\]

\[
= \frac{(1 + \xi)^{1+\alpha}}{1 - \beta \xi} x(x - e^\xi)^{-1}.
\]
Theorem 6:

Let $\Gamma_k(x), k \geq 0$ be a polynomial of degree $k$ and is defined as

$$\Gamma_k(x) = \sum_{r=0}^{k} (k - r) x^r, \quad (8.3.21)$$

then $\Gamma_k(x)$ can be written as

$$\Gamma_k(x) = x^{k+1}\Phi_n(x) - \sum_{s=0}^{n} \left( \begin{array}{c} n \\ s \end{array} \right) (k + 1)^{n-s}\Phi_s(x). \quad (8.3.22)$$

Proof.

From the shifting theorem (see properties (iv) in subsection {1.3.8}) of $\mathbf{Z}$-transform, we can write the following:

$$\mathbf{Z} f(y + 1) = x[F(x) - f(0)],$$

where $F(x)$ is the $\mathbf{Z}$-transform of $f(y)$ and $T = 1$.

$$\mathbf{Z}(y + 1)^n = x\Phi_n(x),$$

$$\mathbf{Z}(y + 2)^n = x[\mathbf{Z}(y + 1)^n - 1],$$

$$\mathbf{Z}(y + 2)^n = x^2\Phi_n(x) - x,$$

$$\mathbf{Z}(y + 3)^n = x^3\Phi_n(x) - x^2 - 2x.$$

From the induction, we get the following

$$\mathbf{Z}(y + k + 1)^n = x^{k+1}\Phi_n(x) - \sum_{r=0}^{k} (k - r) x^r, \quad (8.3.23)$$

$$\Gamma_k(x) = x^{k+1}\Phi_n(x) - \sum_{p=0}^{\infty} (p + k + 1)^n x^{-p}, \quad (8.3.24)$$

$$\Gamma_k(x) = x^{k+1}\Phi_n(x) - \sum_{p=0}^{\infty} x^{-p} \sum_{s=0}^{n} \left( \begin{array}{c} n \\ s \end{array} \right) (k + 1)^{n-s} p^s, \quad (8.3.25)$$

$$\Gamma_k(x) = x^{k+1}\Phi_n(x) - \sum_{s=0}^{n} \left( \begin{array}{c} n \\ s \end{array} \right) (k + 1)^{n-s}\Phi_s(x). \quad (8.3.26)$$
8.4 Summation Formulae for $\Phi_n(x)$, $\Upsilon_n(x)$ and $S(n, k)$

The summation formulae for $\Phi_n(x)$, $\Upsilon_n(x)$ and $S(n, k)$ are as follows

$$\sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (\beta \gamma)_s}{(\beta + \gamma)_{r+s} r! s!} \Phi(x) = \frac{(-1)^n (\gamma)_n x^{\beta + \gamma}}{x(\beta + \gamma)_n}, \quad (8.4.1)$$

$$\sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (\beta \gamma)_s}{(\beta + \gamma)_{r+s} r! s!} \Upsilon(x) = \frac{(-1)^n (\gamma)_n x^{\beta + \gamma}}{x(\beta + \gamma)_n}, \quad (8.4.2)$$

$$\sum_{k=0}^{n} \sum_{s=0}^{k} 2^s (-1)^k S(k, s) s! = 0, \quad (8.4.5)$$

$$\frac{(x - 1)}{x} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-\alpha)_k (\alpha)_r}{k! r!} \Phi_{k+r}(x) = 1, \quad (8.4.6)$$

$$\frac{1}{x} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-\alpha)_k (\alpha)_r}{k! r!} \Upsilon_{k+r}(x) = 1, \quad (8.4.7)$$

$$\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-\alpha)_k (\alpha)_r}{k! r!} \frac{2^s}{k+r} S(k + r, s) s! = 4, \quad (8.4.8)$$

where $S(n, k)$ is Stirling number of second kind, defined by (1.3.12).

**Proof.**

Taking the left hand side of (8.4.1) and using (8.3.2), we get

$$Z \left\{ \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (\beta \gamma)_s}{(\beta + \gamma)_{r+s} r! s!} y^s \right\}$$

$$= Z \{ F_1[-n, \beta, \gamma; \beta + \gamma; 1, y] \}.$$
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Now, using the result [see Erdelyi et al., p. 238]

\[ F_1[\alpha, \beta, \gamma; \beta + \gamma; x, y] = (1 - y)^{-\alpha} F_1 \left[ \frac{\alpha, \beta; x - y}{\beta + \gamma; 1 - y} \right], \quad (8.4.9) \]

we get

\[ Z \left\{ (1 - y)^n F_1 \left[ \begin{array}{c} -n, \beta; \\ \beta \end{array} \right] \right\}. \]

Again, using the result (1.3.17) in above expression, we find

\[
\frac{(\gamma)_n}{(\beta + \gamma)_n} \sum_{r=0}^{n} \frac{(-n)_r}{r!} \sum_{k=0}^{\infty} (k)^r x^{-k} \\
= \frac{(\gamma)_n}{(\beta + \gamma)_n} \sum_{k=0}^{\infty} x^{-k} \sum_{r=0}^{n} \frac{(-n)_r}{r!} (k)^r \\
= \frac{(\gamma)_n}{(\beta + \gamma)_n} \sum_{k=0}^{\infty} x^{-k} (1 - k)^n \\
= \frac{(-1)^n(\gamma)_n}{x(\beta + \gamma)_n} \Phi_n(x).
\]

Similarly,

**Proofs of (8.4.4), (8.4.3) and (8.4.5).**

We know that

\[
(xD_x)^n \{1\} = 0, \quad (xD_x)^n \frac{x}{x-1} (x-1) = 0.
\]

Now, applying Leibniz rule (1.3.14), we get

\[
\sum_{k=0}^{n} \binom{n}{k} \Phi_k(x)(xD_x)^{n-k} \left(1 - \frac{1}{x}\right) = 0,
\]

\[
\frac{1}{x} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k+1} \Phi_k(x) = 0.
\]
If we set \( x = \frac{t}{t-1} \) in (8.4.3), we get

\[
\frac{t - 1}{t} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Upsilon_k(x) = 0.
\]

This completes the proof of (8.4.4).

From (8.4.4) and (8.2.1), we get

\[
\frac{t - 1}{t} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{s=0}^{k} (-1)^{s+k} S(k,s)! \ t^s(t-1) = 0,
\]

\[
\frac{t - 1}{t} \sum_{k=0}^{n} \sum_{s=0}^{k} (-1)^{n+s} \binom{n}{k} S(k,s)! t^s(t-1) = 0.
\]

Set \( t = 2 \) to obtain

\[
\sum_{k=0}^{n} \sum_{s=0}^{k} (-1)^{n+s} 2^s S(k,s) s! = 0.
\]

**Proofs of (8.4.6), (8.4.7) and (8.4.8).**

Consider the identity

\[
(1 - y)^{-\alpha}(1 - y)^{\alpha} = 1.
\]

Now, we operate Z-transform on both side, we get

\[
\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-\alpha)_k(\alpha)_r}{k! \ r!} Z \{y^{k+r}\} = Z\{1\}
\]

\[
\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-\alpha)_k(\alpha)_r}{k! \ r!} \Phi_{k+r}(x) = \frac{x}{(x-1)},
\]

where \( |x| > 1 \). Taking \( x = \frac{t}{t-1} \)

\[
\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-\alpha)_k(\alpha)_r}{k! \ r!} \Upsilon_{k+r}(t) = t.
\]

Using (8.2.1) and taking \( t = 2 \), we have

\[
\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-\alpha)_k(\alpha)_r}{k! \ r!} \sum_{s=0}^{k+r} (-1)^{k+r+s} 2^s S(k+r,s) s! = 4.
\]
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8.5 Generating Function’s Theorem

Let \( G(\lambda, x) \) and \( G(\lambda, \Phi) \) be any series, defined as

\[
G(\lambda, x) = \sum_{r=0}^{\infty} \lambda(n, r, \alpha)x^r, \quad n = 0, 1, 2, 3, 4, \ldots, \tag{8.5.1}
\]

\[
G(\lambda, \Phi) = \sum_{r=0}^{\infty} \lambda(n, r, \alpha)\phi_r(y), \quad n = 0, 1, 2, 3, 4, \ldots, \tag{8.5.2}
\]

where \( \lambda(n, r, \alpha) \) be any sequence and \( \alpha \) is complex parameter.

Theorem 8:

If \( G(\lambda, x) \) be polynomial set having generating function of the form

\[
\sum_{n=0}^{\infty} \gamma_n G(\lambda, x)t^n = A(t) e^{xB(t)}, \tag{8.5.3}
\]

then

\[
\sum_{n=0}^{\infty} \gamma_n G(\lambda, \Phi)t^n = A(t) \frac{y}{y - e^{B(t)}}. \tag{8.5.4}
\]

Proof.

By operating \( Z \)-transform on both sides of (8.5.3), we get

\[
\sum_{n=0}^{\infty} \gamma_n G(\lambda, \Phi)t^n = A(t) \mathcal{Z}\{e^{xB(t)}\}.
\]

From the shifting theorem (see properties (iii) in subsection \{1.3.8\}) of \( Z \)-transform, we get

\[
\sum_{n=0}^{\infty} \gamma_n G(\lambda, \Phi)t^n = A(t) \frac{y}{y - e^{B(t)}}.
\]

8.5.1 Applications of theorem

From the generating functions (8.1.2), (8.1.3), (8.1.4), (??) and (??), we get

\[
\sum_{n=0}^{\infty} \frac{(1 + \alpha)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(1 + \alpha)_k} k! \Phi_k(y)t^n = (1 - t)^{-\alpha - 1} \frac{y}{(y - e^{B(t)})}, \tag{8.5.5}
\]
\[
\sum_{n=0}^{\infty} \frac{(1 + \alpha - n)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(1 + \alpha - n)_k} \Phi_k(y) t^n = (1 - t)^{\alpha} \frac{y}{(y - e^{-t})}, \quad (8.5.6)
\]
\[
\sum_{n=0}^{\infty} \frac{(1 + \alpha + n\beta)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(1 + \alpha + n\beta)_k} \Phi_k(y) t^n = \frac{(1 + v)^{1+\alpha}}{1 - \beta v} \frac{y}{(y - e^{-t})}, \quad (8.5.7)
\]
\[
\sum_{n=0}^{\infty} \frac{(1 + \alpha - n)_n}{(-\alpha - \beta)_n \ n!} \sum_{k=0}^{n} \frac{(-n)_k (1 + \alpha - n + \beta)_k}{(1 + \alpha - n)_k} \Phi_k(y) t^n = _1F_1 \left[ \begin{array}{c} -\alpha; \\ \alpha - \beta; \\ \end{array} \right] \frac{y}{(y - e^{-t})}, \quad (8.5.8)
\]
\[
\sum_{n=0}^{\infty} \frac{(1 + \alpha - n)_n}{(-\alpha - \beta)_n \ n!} \sum_{k=0}^{n} \frac{(-n)_k (1 + \alpha - n + \beta)_k}{(1 + \alpha - n)_k} \Phi_k(y) t^n = e^t _1F_1 \left[ \begin{array}{c} -\alpha; \\ -\alpha - \beta; \\ \end{array} \right] \frac{y}{(y - e^{-t})}, \quad (8.5.9)
\]
respectively.