Chapter 1

Introduction

1.1 Special Functions and Its Growth

A special function is a real or complex valued function of one or more real or complex variables which is specified so completely that its numerical values could in principle be tabulated. Besides elementary functions such as $x^n$, $e^x$, $\log x$, and $\sin x$, ‘higher’ functions, both transcendental (such as Bessel functions) and algebraic (such as various polynomials) come under the category of special functions. In fact special functions are solutions of a wide class of mathematically and physically relevant functional equations. As far as the origin of special functions is concerned the special function of mathematical physics arises in the solution of partial differential equations governing the behavior of certain physical quantities. Probably the most frequently occurring equation of this type in all physics is Laplace’s equation

$$\nabla^2 \psi = 0 \quad (1.1.1)$$

satisfied by a certain function $\psi$ describing the physical situation under discussion. The mathematical problem consists of finding those functions which satisfy equation (1.1.1) and also satisfy certain prescribed conditions on the surfaces bounding the region being considered. For example, if $\psi$ denotes the electrostatic potential of a system, $\psi$ will be constant over any conducting surface. The shape of these boundaries often makes it desirable to work in curvilinear coordinates $q_1, q_2, q_3$ instead of in rectangular Cartesian coordinates $x, y, z$. In this
case we have relations

\[ x = x(q_1, q_2, q_3), \quad y = y(q_1, q_2, q_3), \quad z = z(q_1, q_2, q_3). \]  

(1.1.2)

expressing the Cartesian coordinates in terms of the curvilinear coordinates. If equations (1.1.2) are such that

\[
\frac{\partial x}{\partial q_i} \frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i} \frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i} \frac{\partial z}{\partial q_j} = 0,
\]

when \( i \neq j \) we say that the coordinates \( q_1, q_2, q_3 \) are orthogonal curvilinear coordinates. The element of length \( dl \) is then given by

\[
dl^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2.\]

(1.1.3)

where

\[
h_i^2 = \left( \frac{\partial x}{\partial q_i} \right)^2 + \left( \frac{\partial y}{\partial q_i} \right)^2 + \left( \frac{\partial z}{\partial q_i} \right)^2.
\]

(1.1.4)

and it can easily be shown that

\[
\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left( h_2 h_3 \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( h_3 h_1 \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( h_1 h_2 \frac{\partial \psi}{\partial q_3} \right) \right\}. \tag{1.1.5}
\]

One method of solving Laplace’s equation consists of finding solutions of the type

\[
\psi = Q_1(q_1) Q_2(q_2) Q_3(q_3),
\]

by substituting from (1.1.5) into (1.1.1). We then find that

\[
\frac{1}{Q_1} \left\{ \frac{\partial}{\partial q_1} \left( h_2 h_3 \frac{\partial Q_1}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( h_3 h_1 \frac{\partial Q_2}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( h_1 h_2 \frac{\partial Q_3}{\partial q_3} \right) \right\} = 0.
\]

If further, it so happens that

\[
h_2 h_3 \quad h_1 = f_1(q_1) F_1(q_2, q_3),
\]

etc., then this last equation reduces to the form

\[
F_1(q_2, q_3) \frac{d}{d q_1} \left\{ f_1(q_1) \frac{d Q_1}{d q_1} \right\} + F_2(q_1, q_3) \frac{d}{d q_2} \left\{ f_2(q_2) \frac{d Q_2}{d q_2} \right\} + \]

\[
+ F_3(q_1, q_2) \frac{d}{d q_3} \left\{ f_3(q_3) \frac{d Q_3}{d q_3} \right\} = 0.
\]
Now, in certain circumstances, it is possible to find three functions \( g_1(q_1), g_2(q_2), g_3(q_3) \) with the property that

\[
F_1(q_2, q_3)g_1(q_1) + F_2(q_3, q_1)g_2(q_2) + F_3(q_1, q_2)g_3(q_3) = 0.
\]

When this is so, it follows immediately that the solution of Laplace’s equation (1.1.1) reduces to the solution of three self-adjoint ordinary linear differential equations

\[
\frac{d}{dq_i} \left\{ f_i \frac{dQ_i}{dq_i} \right\} g_i Q_i = 0, \quad (i = 1, 2, 3). \tag{1.1.6}
\]

It is the study of differential equations of this kind which leads to the special functions of mathematical physics. The adjective “special” is used in this connection because here we are not, as in analysis, concerned with the general properties of functions, but only with the properties of functions which arise in the solution of special problems.

To take a particular case, consider the cylindrical polar coordinates \((Q, \varphi, z)\) defined by the equations

\[
x = Q \cos \varphi, \quad y = Q \sin \varphi, \quad z = z
\]

for which \( h_1 = 1, h_2 = Q, h_3 = 1. \)

From equation (1.1.5) we see that, for these coordinates, Laplace’s equation is of the form

\[
\frac{\partial^2 \psi}{\partial Q^2} + \frac{1}{Q} \frac{\partial \psi}{\partial Q} + \frac{1}{Q^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2} = 0. \tag{1.1.7}
\]

If we now make the substitution

\[
\psi = R(Q) \Phi(\varphi) Z(z), \tag{1.1.8}
\]

we find that equation (1.1.7) may be written in the form

\[
\frac{1}{R} \left( \frac{\partial^2 R}{\partial Q^2} + \frac{1}{Q} \frac{\partial R}{\partial Q} \right) + \frac{1}{Q^2 \Phi} \frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0. \tag{1.1.9}
\]
This shows that if $\Phi$, $Z$, $R$ satisfy the equations

$$\frac{\partial^2 \Phi}{\partial \varphi^2} + n^2 \Phi = 0, \quad (1.1.9a)$$

$$\frac{\partial^2 Z}{\partial z^2} + m^2 Z = 0, \quad (1.1.9b)$$

$$\frac{\partial^2 R}{\partial Q^2} + \frac{1}{Q} \frac{\partial R}{\partial Q} + \left( m^2 - \frac{n^2}{Q^2} \right) R = 0, \quad (1.1.9c)$$

respectively, then the function (1.1.8) is a solution of Laplace’s equation (1.1.7). The study of these ordinary differential equations will lead us to the special functions appropriate to this coordinate system. For instance, equation (1.1.9a) may be taken as the equation defining the circular functions. In this context $\sin(n\varphi)$ is defined as that solution of (1.1.9a) which has value 0 when $\varphi = 0$ and $\cos(n\varphi)$ as that which has value 1 when $\varphi = 0$. Similarly equation (1.1.9b) defines the exponential functions. In actual practice we do not proceed in this way merely because we have already encountered these functions in another context and from their familiar properties studied their relation to equations (1.1.9a) and (1.1.9b). The situation with respect to equation (1.1.9c) is different; we cannot express its solution in terms of the elementary functions of analysis, as we were able to do with the other two equations. In this case we define new functions in terms of the solutions of this equation and by investigating the series solutions of the equations derive the properties of the functions so defined. Equation (1.1.9c) is called Bessel’s equation and solutions of it are called Bessel functions. Bessel functions are of great importance in theoretical physics. The study of special functions grew up with the calculus and is consequently one of the oldest branches of analysis. It flourished in the nineteenth century as part of the theory of complex variables. In the second half of the twentieth century it has received a new impetus from a connection with Lie groups and a connection with averages of elementary functions. The history of special functions is closely tied to the problem of terrestrial and celestial mechanics that were solved in the eighteenth and nineteenth centuries, the boundary-value problems of electromagnetism and heat in
the nineteenth, and the eigenvalue problems of quantum mechanics in the twentieth.

Seventeenth-century England was the birthplace of special functions. John Wallis at Oxford took two first steps towards the theory of the gamma function long before Euler reached it. Wallis had also the first encounter with elliptic integrals while using Cavalieri’s primitive forerunner of the calculus. [It is curious that two kinds of special functions encountered in the seventeenth century, Wallis’ elliptic integral and Newton’s elementary symmetric functions, belongs to the class of hypergeometric functions of several variables, which was not studied systematically nor even defined formally until the end of the nineteenth century]. A more sophisticated calculus, which made possible the real flowering of special functions, was developed by Newton at Cambridge and by Leibnitz in Germany during the period 1665-1685. Taylor’s theorem was found by Scottish mathematician Gregory in 1670, although it was not published until 1715 after rediscovery by Taylor.

In 1703 James Bernoulli solved a differential equation by an infinite series which would now be called the series representation of a Bessel function. Although Bessel functions were met by Euler and others in various mechanics problems, no systematic study of the functions was made until 1824, and the principal achievements in the eighteenth century were the gamma function and the theory of elliptic integrals. Euler found most of the major properties of the gamma functions around 1730. In 1772 Euler evaluated the Beta-function integral in terms of the gamma function. Only the duplication and multiplication theorems remained to be discovered by Legendre and Gauss, respectively, early in the next century. Other significant developments were the discovery of Vandermonde’s theorem in 1722 and the definition of Legendre polynomials and the discovery of their addition theorem by Laplace and Legendre during 1782-1785. In a slightly different form the polynomials had already been met by Liouville in 1722.

The golden age of special functions, which was centered in nineteenth century German and France, was the result of developments in both mathematics and physics:
the theory of analytic functions of a complex variable on one hand, and on the other hand, the field theories of physics (e.g. heat and electromagnetism) and their property of double periodicity was published by Abel in 1827. Elliptic functions grew up in symbiosis with the general theory of analytic functions and flourished throughout the nineteenth century, specially in the hands of Jacobi and Weierstrass.

Another major development was the theory of hypergeometric series which began in a systematic way (although some important results had been found by Euler and Pfaff) with Gauss’s memoir on the $2F_1$ series in 1812, a memoir which was a landmark also on the path towards rigour in mathematics. The $3F_2$ series was studied by Clausen (1928) and the $1F_1$ series by Kummer (1836). The functions which Bessel considered in his memoir of 1824 are $0F_1$ series; Bessel started from a problem in orbital mechanics, but the functions have found a place in every branch of mathematical physics, near the end of the century Appell (1880) introduced hypergeometric functions of two variables, and Lauricella generalized them to several variables in 1893.

The subject was considered to be part of pure mathematics in 1900, applied mathematics in 1950. In physical science special functions gained added importance as solutions of the Schrodinger equation of quantum mechanics, but there were important developments of a purely mathematical nature also. In 1907 Barnes used gamma function to develop a new theory of Gauss’s hypergeometric function $2F_1$. Various generalizations of $2F_1$ were introduced by Horn, Kampé de Fériet, MacRobert, and Mijer. From another new view point, that of a differential difference equation discussed much earlier for polynomials by Appell (1880), Truesdell (1948) made a partly successful effort at unification by fitting a number of special functions into a single framework.

### 1.2 Definitions, Notations and Results Used

Frequently occurring definitions, notations and miscellaneous results used in this thesis are as given below:
The Gamma Function

The Gamma function has several equivalent definitions, most of which are due to Euler,

\[ \Gamma(z) = \int_0^\infty t^{z-1}e^{-t} \, dt, \quad \text{Re}(z) > 0. \quad (1.2.1) \]

Upon integrating by parts, equation (1.2.1) yields the recurrence relation

\[ \Gamma(z+1) = z\Gamma(z). \quad (1.2.2) \]

From the relations (1.2.1) and (1.2.2) it follows that

\[ \Gamma(z) = \begin{cases} \int_0^\infty t^{z-1}e^{-t} \, dt, & \text{Re}(z) > 0 \\ \frac{\Gamma(z+1)}{z}, & \text{Re}(z) < 0, \ z \neq 0, -1, -2, \ldots \end{cases} \quad (1.2.3) \]

The relation (1.2.2) yields the useful result

\[ \Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots \]

which shows that the gamma function is the generalization of factorial function.

The Beta Function

The Beta function \( B(p, q) \) is a function of two complex variables \( p \) and \( q \), defined by

\[ B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} \, dx, \quad \text{Re}(p) > 0, \ \text{Re}(q) > 0. \quad (1.2.4) \]

The Gamma function and the Beta function are related by the following relation

\[ B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p, q \neq 0, -1, -2, \ldots \quad (1.2.5) \]

From the relations (1.2.4) and (1.2.5) it follows that

\[ B(p, q) = \begin{cases} \int_0^1 x^{p-1}(1-x)^{q-1} \, dx, & \text{Re}(p) > 0, \ \text{Re}(q) > 0, \\ \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, & \text{Re}(p) < 0, \ \text{Re}(q) < 0, \ p, q \neq -1, -2, \ldots \end{cases} \quad (1.2.6) \]
The Pochhammer Symbol

The Pochhammer symbol \((\lambda)_n\) is defined by

\[
(\lambda)_n = \begin{cases} 
1 & \text{if } n = 0 \\ 
\lambda(\lambda + 1) \cdots (\lambda + n - 1) & \text{if } n = 1, 2, 3, \cdots . 
\end{cases}
\]  (1.2.7)

In terms of Gamma functions, we have

\[
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}, \quad \lambda \neq 0, -1, -2, \cdots .
\]  (1.2.8)

Further,

\[
(\lambda)_{m+n} = (\lambda)_m (\lambda + m)_n. \quad (1.2.9)
\]

\[
(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n}, \quad n = 1, 2, 3, \cdots; \quad \lambda \neq 0, \pm 1, \pm 2, \cdots .
\]  (1.2.10)

\[
(\lambda)_{n-m} = \frac{(-1)^m (\lambda)_n}{(1-\lambda-n)_m}, \quad 0 \leq m \leq n. \quad (1.2.11)
\]

For \(\lambda = 1\), equation (1.2.11) reduces to

\[
(n-m)! = \frac{(-1)^m n!}{(-n)_m}, \quad 0 \leq m \leq n. \quad (1.2.12)
\]

Another useful relation of Pochhammer’s symbol \((\lambda)_n\) is included in Gauss’s multiplication theorem:

\[
(\lambda)_{mn} = (m)^m \prod_{j=1}^{m} \left( \frac{\lambda + j - 1}{m} \right)_n, \quad n = 0, 1, 2, \cdots , \quad (1.2.13)
\]

where \(m\) is positive integer.

For \(m = 2\) the equation (1.2.13) reduces to Legendre’s duplication formula

\[
(\lambda)_{2n} = 2^{2n} \left( \frac{\lambda}{2} \right)_n \left( \frac{\lambda}{2} + \frac{1}{2} \right)_n, \quad n = 0, 1, 2, \cdots . \quad (1.2.14)
\]

In particular, one has

\[
(2n)! = 2^{2n} \left( \frac{1}{2} \right)_n n! \quad \text{and} \quad (2n+1)! = 2^{2n} \left( \frac{3}{2} \right)_n n!. \quad (1.2.15)
\]
Also, the binomial coefficient \( \binom{\lambda}{n} \) is defined by (see [5])

\[
\binom{\lambda}{n} = \frac{(-1)^n(-\lambda)_n}{n!}
\]  

(1.2.16)

where

\[
(-\lambda)_n = (-\lambda)(-\lambda + 1)(-\lambda + 2) \cdots (-\lambda + n - 1), \\
(-\lambda)_0 = 1.
\]  

(1.2.17)

**Gaussian Hypergeometric Series**

The hypergeometric series is given by

\[
_2 F_1(a, b; c; z) = 1 + \frac{a.b}{1.c} z + \frac{a(a+1)b(b+1)}{1.2.c(c+1)} z^2 + ... \\
= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \quad c \neq 0, -1, -2, ...
\]  

(1.2.18)

was introduced by a German mathematician Gauss, C. F. (1777-1855). Who in the year (1812) introduced this series into analysis and give the \( F \)-notation for it.

The special case \( a = c, \ b = 1 \) or \( b = c, \ a = 1 \) yields the elementary geometric series.

\[
\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + ... + z^n + ...
\]  

(1.2.19)

Hence (1.2.18) is called the hypergeometric series or more precisely, Gauss hypergeometric series.

In (1.2.18), \( (a)_n \) denotes the pochhammer’s symbol defined by (1.2.7), \( z \) is real or complex variable, \( a, \ b \) and \( c \) are parameters which can take arbitrary real or complex values and \( c \neq 0, -1, -2, ... \).

If \( c \) is zero or negative integer, the series (1.2.18) does not exist and hence the function \( _2 F_1(a, b; c; z) \) is not defined unless one of the parameters \( a \) or \( b \) is negative integer such that \(-c < -a \) is also negative integer. If either of the parameters \( a \) or
b is a negative integer, then in this case, equation (1.2.18) reduces to hypergeometric polynomials.

The hypergeometric series (1.2.18), converges absolutely within the unit circle $|z| < 1$, provided that $\text{Re}(c - a - b) > 0$ for $z = 1$ and $\text{Re}(c - a - b) > -1$ for $z = -1$.

**Generalized Hypergeometric Function**

The hypergeometric function $2F_1$ defined in equation (1.2.18) can be generalized with notation $pF_q$ in an obvious way:

$$
\begin{align*}
\phantom{=} & \,_{p}F_{q}\left[ \begin{array}{c}
\alpha_1, \alpha_2, \cdots, \alpha_p; \\
\beta_1, \beta_2, \cdots, \beta_q;
\end{array} \right] \nonumber \\
= & \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\
= & \,_{p}F_{q}(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z), \quad (1.2.20)
\end{align*}
$$

where $p \leq q$ are positive integers or zero. The numerator parameter $\alpha_1, \cdots, \alpha_p$ and the denominator parameter $\beta_1, \cdots, \beta_q$ take on complex values, provided that

$$
\beta_j \neq 0, -1, -2, \cdots; \quad j = 1, 2, \cdots, q.
$$

**Convergence of $pF_q$**

The series $pF_q$ defined by (1.2.20)

(i) converges for all $|z| < \infty$ if $p \leq q$,

(ii) converges for $|z| < 1$ if $p = q + 1$, and

(iii) diverges for all $z$, $z \neq 0$, if $p > q + 1$.

Furthermore, if we set

$$
\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j,
$$

it is known that the $pF_q$ series, with $p = q + 1$, is

(i) absolutely convergent for $|z| = 1$ if $\text{Re}(\omega) > 0$, ...
(ii) conditionally convergent for \( |z| = 1, z \neq 1 \), if \(-1 < \text{Re}(\omega) \leq 0\), and

(iii) divergent for \( |z| = 1 \), if \( \text{Re}(\omega) \leq -1 \).

An important special case when \( p = q = 1 \), the equation (1.2.20) reduces to the confluent hypergeometric series \( _1F_1 \) named as Kummer's series [181], (see also Slater [207]) and is given by

\[
_1F_1(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}.
\]  

(1.2.21)

when \( p = 2 & q = 1 \), equation (1.2.20) reduces to ordinary hypergeometric function \( _2F_1 \) of second order given by (1.2.18).

Also in terms of hypergeometric function, we have

\[
(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} = _1F_0 \left[ a; -; z \right].
\]  

(1.2.22)

\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = _0F_0 \left[ -; -; z \right].
\]  

(1.2.23)

**A Further Generalization of \( _pF_q \)**

An interesting further generalization of the series \( _pF_q \) is due to Fox [22] and Wright [234, 235] who studied the asymptotic expansion of the generalized hypergeometric function defined by

\[
_p\Psi_q \left[ (\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z^n}{n!},
\]  

(1.2.24)

where the coefficients \( A_1, \ldots, A_p \) and \( B_1, \ldots, B_q \) are positive real numbers such that

\[
1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0.
\]
By comparing the definitions (1.2.20) and (1.2.24), we have

\[ p \Psi_q \left[ \left( \alpha_1, 1 \right), \ldots, \left( \alpha_p, 1 \right); (\beta_1, 1), \ldots, (\beta_q, 1); z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} pF_q \left[ \frac{\alpha_1, \ldots, \alpha_p; z}{\beta_1, \ldots, \beta_q}; z \right]. \]  

(1.2.25)

Some useful linear transformations of the hypergeometric function, known as Euler’s transformations, may be recalled here as follows:

\[ 2F_1 \left[ \frac{a, b; c; z}{z} \right] = (1 - z)^{-a} 2F_1 \left[ \frac{a, c - b; c; z}{z - 1} \right], \]  

\[ c \neq 0, -1, -2, \ldots, |arg(1 - z)| < \pi; \]  

(1.2.26)

\[ 2F_1 \left[ \frac{a, b; c; z}{z} \right] = (1 - z)^{-b} 2F_1 \left[ \frac{c - a, b; c; z}{z - 1} \right], \]  

\[ c \neq 0, -1, -2, \ldots, |arg(1 - z)| < \pi; \]  

(1.2.27)

\[ 2F_1 \left[ \frac{a, b; c; z}{z} \right] = (1 - z)^{c - a - b} 2F_1 \left[ \frac{c - a, c - b; c; z}{z} \right], \]  

\[ c \neq 0, -1, -2, \ldots, |arg(1 - z)| < \pi. \]  

(1.2.28)

1.3 Hypergeometric Functions of Two and Several Variables

Appell Function

In 1880, Appell [4] introduced four hypergeometric series which are generalization of Gauss hypergeometric function \( 2F_1 \) and are given below:

\[ F_1[a, b, b'; c, x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m}(b')_{n}}{(c)_{m+n}} \frac{x^m y^n}{m! n!}, \]  

\[ \max\{|x|, |y|\} < 1; \]  

(1.3.1)

\[ F_2[a, b, b'; c, c', x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m}(b')_{n}}{(c)_{m}(c')_{n}} \frac{x^m y^n}{m! n!}, \]  

\[ |x| + |y| < 1; \]  

(1.3.2)
\[ F_3[a, a', b, b'; c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n x^m y^n}{(c)_{m+n} m! n!}, \quad (1.3.3) \]

\[ \max\{|x|, |y|\} < 1; \]

\[ F_4[a, b; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n} x^m y^n}{(c)_{m(c')_n} m! n!}, \quad (1.3.4) \]

\[ \sqrt{|x|} + \sqrt{|y|} < 1. \]

The standard work on the theory of Appell series is the monograph by Appell and Kampé de Fériet [5]. See Erdelyi et al. [20] for a review of the subsequent work; see also Slater [(207), Chapter 8] and Exton [(21), p. 23-28]

**Generalization of Appell’s Functions of Two Variables**

In 2002, M. A. Khan and G. S. Abukhammash {cf. [89], p. 61-83} introduced two variables ten Appell’s type generalized functions \( M_i, i = 1, 2, ..., 10 \) by considering the product of two \( F_2 \) functions instead of product of two Gauss hypergeometric function \( _2F_1 \) taken by Appell to define \( F_1, F_2, F_3 \) and \( F_4 \) functions. These generalized functions are defined as follows:

\[ M_1[a, a', b, b', c, c'; d, e, e'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)(a')_n(b)(b')_n(c)(c')_n x^m y^n}{(d)(e)(e')_n m! n!}, \quad (1.3.5) \]

\[ M_2[a, a', b, b', c, c'; d, e; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)(a')_n(b)(b')_n(c)(c')_n x^m y^n}{(d)(e)(e')_n m! n!}, \quad (1.3.6) \]

\[ M_3[a, b, b', c, c'; d', e, e'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)(b)(b')_n(c)(c')_n x^m y^n}{(d')(e')(e')_n m! n!}, \quad (1.3.7) \]

\[ M_4[a, b, b', c, c'; d, e', e'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)(b)(b')_n(c)(c')_n x^m y^n}{(d)(e)(e')_n m! n!}, \quad (1.3.8) \]

\[ M_5[a, b, b', c, c'; d, e, x, y] = \sum_{m,n=0}^{\infty} \frac{(a)(b)(b')_n(c)(c')_n x^m y^n}{(d)(e)(e')_n m! n!}, \quad (1.3.9) \]

\[ M_6[a, b, c, c'; d, e', e'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)(b)(b')_n(c)(c')_n x^m y^n}{(d)(d')(e)(e')_n m! n!}, \quad (1.3.10) \]
Horn Functions

In the year 1931 J. Horn [25] defined the following ten hypergeometric functions of two variables and denoted by $G_1, G_2, G_3, H_1, ..., H_7$; he thus completed the set of all possible second-order (complete) hypergeometric functions of two variables in the terminology given in Appell and Kampé de Fériet [5]. {See also Erdelyi et al. ([20], Vol. 1, pp. 224-228).}

\[
G_1[\alpha, \beta, \beta'; x, y] = \sum_{m,n=0}^{\infty} (\alpha)_{m+n} (\beta)_{m-n} x^m y^n \frac{m! n!}{m! n!}, \tag{1.3.15}
\]

\(|x| < r, \quad |y| < s, \quad r + s = 1;\)

\[
G_2[\alpha', \beta; \delta, \gamma; x, y] = \sum_{m,n=0}^{\infty} (\alpha')_m (\beta)_{m-n} x^m y^n \frac{m! n!}{m! n!}, \tag{1.3.16}
\]

\(|x| < 1, \quad |y| < 1;\)

\[
G_3[\alpha', \gamma; \delta; x, y] = \sum_{m,n=0}^{\infty} (\alpha')_{2m-n} x^m y^n \frac{m! n!}{m! n!}, \tag{1.3.17}
\]

\(|x| < r, \quad |y| < s, \quad 27r^2s^2 + 18rs \pm 4(r - s) - 1 = 0;\)

\[
H_1[\alpha, \beta, \gamma; \delta, \epsilon; x, y] = \sum_{m,n=0}^{\infty} (\alpha)_{m-n} (\beta)_{m+n} (\gamma)_{m-n} x^m y^n \frac{m! n!}{m! n!}, \tag{1.3.18}
\]

\(|x| < r, \quad |y| < s, \quad 4rs = (s - 1)^2;\)
Humbert Function

In 1920, P. Humbert [27] has studied seven confluent forms of the four Appell functions and denoted by $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$. These confluent hypergeometric functions of two variables are defined as follows:

$$\phi_1 [\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m x^m y^n}{(\gamma)_m m! n!} ; \quad |x| < 1, |y| < \infty ; \quad (1.3.25)$$

$$\phi_2 [\beta, \beta'; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_n x^m y^n}{(\gamma)_m m! n!} ; \quad |x| < \infty, |y| < \infty ; \quad (1.3.26)$$
\[
\phi_3[\beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!}, \quad |x| < \infty, |y| < \infty; 
\]
(1.3.27)

\[
\Psi_1[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m x^m y^n}{(\gamma)_{m+n} m! n!}, \quad |x| < 1, |y| < \infty; 
\]
(1.3.28)

\[
\Psi_2[\alpha; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} x^m y^n}{(\gamma')_{m+n} m! n!}, \quad |x| < \infty, |y| < \infty; 
\]
(1.3.29)

\[
\Xi_1[\alpha, \alpha', \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m x^m y^n}{(\gamma)_{m+n} m! n!}, \quad |x| < 1, |y| < \infty; 
\]
(1.3.30)

\[
\Xi_2[\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_m x^m y^n}{(\gamma)_{m+n} m! n!}, \quad |x| < 1, |y| < \infty. 
\]
(1.3.31)

**Kampé de Fériet Function**

Appell’s four double hypergeometric functions \(F_1, F_2, F_3\) and \(F_4\), were unified and generalized by Kampé de Fériet [29], (see also [5], p. 150,) who defined a general hypergeometric function of two variables. The notation introduced by Kampe’ de Fériet for his double hypergeometric function of superior order was subsequently abbreviated in 1941 by Burchnall and Chaundy [12]. We recall here definition of a more general double hypergeometric function (than the one defined by Kampe de Fériet) in a slightly modified notation [see, for example, [224], p. 423. Eq. (26)]:

\[
\begin{aligned}
{F}_{pq;kl;m:n}^{\alpha ; \beta ; c} & : \{(a_p): (b_q); (c_k); (a_l): (b_m); (c_l); x, y\} = \\
& \sum_{r,s=0}^{\infty} \prod_{j=1}^{p} (a_j)_{r+j} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s \frac{x^r y^s}{r! s!}.
\end{aligned} 
\]
(1.3.32)

where, for convergence,

(i) \(p + q < l + m + 1, \ p + k < l + n + 1, \ |x| < \infty, \ |y| < \infty, \) or

(ii) \(p + q = l + m + 1, \ p + k = l + n + 1, \ \) and

\[
\begin{cases}
|x|^{1/(p-l)} + |y|^{1/(p-l)} < 1, & \text{if} \ p > 1, \\
\max \{|x|, |y|\} < 1, & \text{if} \ p \leq l.
\end{cases}
\]
(1.3.33)
Although the double hypergeometric function defined by (1.3.32) reduces to the Kampé de Fériet function in the special case:

\[ q = k \quad \text{and} \quad m = n, \]

yet it is usually referred to in the literature as Kampé de Fériet function.

**Lauricella Functions of \( n \) Variables**

Lauricella [182] generalized the four Appell double hypergeometric functions \( F_1, \ldots, F_4 \) (cf. [20], p. 224) to functions of \( n \) variables. These Lauricella functions, viz. \( F^{(n)}_A, F^{(n)}_B, F^{(n)}_C \) and \( F^{(n)}_D \) are defined by

\[
F^{(n)}_A[a, b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \quad (1.3.34)
\]

\[
\left| x_1 \right| + \cdots + \left| x_n \right| < 1;
\]

\[
F^{(n)}_B[a_1, \ldots, a_n, b_1, \ldots, b_n; c; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a_1)_{m_1} \cdots (a_n)_{m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \quad (1.3.35)
\]

\[
\max \{\left| x_1 \right|, \ldots, \left| x_n \right| \} < 1;
\]

\[
F^{(n)}_C[a, b; c_1, \ldots, c_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b)_{m_1+\cdots+m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \quad (1.3.36)
\]

\[
\sqrt{|x_1|} + \cdots + \sqrt{|x_n|} < 1;
\]

\[
F^{(n)}_D[a, b_1, \ldots, b_n; c; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \quad (1.3.37)
\]

\[
\max \{\left| x_1 \right|, \ldots, \left| x_n \right| \} < 1.
\]

Clearly, we have \( F^{(2)}_A = F_2, F^{(2)}_B = F_3, F^{(2)}_C = F_4 \) and \( F^{(2)}_D = F_1 \).
Generalization of Lauricella functions of n variables

Srivastava and Daoust ([218], p. 454), (see also [221], p. 37) considered a multivariable extension of the function $p \Psi _q$ defined by \((1.2.24)\). Their multiple hypergeometric function, known as the generalized Lauricella function in several variables, is defined by (cf. Srivastava and Daoust [218], p. 224)

\[
F^{A; B'; \ldots ; B(n)}_{C; D'; \ldots ; D(n)} \left( \begin{array}{c}
\frac{z_1}{z_n} \\
\vdots \\
\frac{z_{n-1}}{z_n}
\end{array} \right) = \sum_{m_1, \ldots, m_n=0}^{\infty} \Omega(m_1, \ldots, m_n) \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!},
\]

where, for convenience

\[
\Omega(m_1, \ldots, m_n) = \frac{\prod_{j=1}^{A} (a_j)_{m_1 \theta_j^{(k)} + \ldots + m_n \theta_j^{(n)}} \prod_{j=1}^{B'} (b'_j)_{m_1 \phi_j^{(k)} \ldots \phi_j^{(n)}} \prod_{j=1}^{B(n)} (b_j^{(n)})_{m_1 \phi_j^{(n)} \ldots \phi_j^{(n)}}}{\prod_{j=1}^{C} (c_j)_{m_1 \psi_j^{(k)} + \ldots + m_n \psi_j^{(n)}} \prod_{j=1}^{D'} (d'_j)_{m_1 \delta_j^{(k)} \ldots \delta_j^{(n)}} \prod_{j=1}^{D(n)} (d_j^{(n)})_{m_1 \delta_j^{(n)} \ldots \delta_j^{(n)}}},
\]

the coefficients

\[
\begin{aligned}
\theta_j^{(k)}, & \quad j = 1, \ldots, A; \quad \phi_j^{(k)}, \quad j = 1, \ldots, B(k); \quad \psi_j^{(k)}, \quad j = 1, \ldots, C; \\
\delta_j^{(k)}, & \quad j = 1, \ldots, D(k); \quad \forall \ k \in \{1, \ldots, n\}
\end{aligned}
\]

are real and positive, and \((a)\) abbreviates the array of \(A\) parameters \(a_1, \ldots, a_A\), \((b^{(k)})\) abbreviates the array of \(B^{(k)}\) parameters

\[
b_j^{(k)}, \quad j = 1, \ldots, B(k); \quad \forall \ k \in \{1, \ldots, n\},
\]

with similar interpretations for \((c)\) and \((d^{(k)})\), \(k = 1, \ldots, n\); et cetera.

Further, by considering the product of \(n-3F_2\) hypergeometric functions Khan and Nisar [193] gave generalization of Lauricella functions of \(n\) variables as follows:

\[
N_1^{(n)}[a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n; d, e_1, \ldots, e_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a_1)_{m_1} \cdots (a_n)_{m_n} (b_1)_{m_1} \cdots (b_n)_{m_n} (c_1)_{m_1} \cdots (c_n)_{m_n} x_1^{m_1} \cdots x_n^{m_n}}{(d)_{m_1 + \ldots + m_n} (e_1)_{m_1} \cdots (e_n)_{m_n} m_1! \cdots m_n!},
\]

where \(a_i, b_i, c_i, d, e_i, x_i\) are real and positive.
\[ N_2^{(n)}[a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n; d, e; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a_1)_m \cdots (a_n)_m (b_1)_m \cdots (b_n)_m (c_1)_m \cdots (c_n)_m}{(d)_m + \ldots + (e)_m + \ldots + m_n} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \tag{1.3.42} \]

\[ N_3^{(n)}[a, b_1, \ldots, b_n, c_1, \ldots, c_n; d_1, \ldots, d_n, e_1, \ldots, e_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n} (c_1)_{m_1} \cdots (c_n)_{m_n}}{(d)_{m_1} \cdots (e)_{m_1} \cdots (e)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \tag{1.3.43} \]

\[ N_4^{(n)}[a, b_1, \ldots, b_n, c_1, \ldots, c_n; d, e_1, \ldots, e_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n} (c_1)_{m_1} \cdots (c_n)_{m_n}}{(d)_{m_1 + \ldots + m_n} (e)_{m_1 + \ldots + m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \tag{1.3.44} \]

\[ N_5^{(n)}[a, b_1, \ldots, b_n, c_1, \ldots, c_n; d, e; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n} (c_1)_{m_1} \cdots (c_n)_{m_n}}{(d)_{m_1 + \ldots + m_n} (e)_{m_1} \cdots (e)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \tag{1.3.45} \]

\[ N_6^{(n)}[a, b, c_1, \ldots, c_n; d_1, \ldots, d_n, e_1, \ldots, e_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b)_{m_1 + \ldots + m_n} (c_1)_{m_1} \cdots (c_n)_{m_n}}{(d)_{m_1} \cdots (e)_{m_1} \cdots (e)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \tag{1.3.46} \]

\[ N_7^{(n)}[a, b_1, \ldots, c_n; d, e_1, \ldots, e_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b)_{m_1 + \ldots + m_n} (c_1)_{m_1} \cdots (c_n)_{m_n}}{(d)_{m_1 + \ldots + m_n} (e)_{m_1} \cdots (e)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \tag{1.3.47} \]

\[ N_8^{(n)}[a, b_1, \ldots, c_n; d, e; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b)_{m_1 + \ldots + m_n} (c_1)_{m_1} \cdots (c_n)_{m_n}}{(d)_{m_1 + \ldots + m_n} (e)_{m_1} \cdots (e)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \tag{1.3.48} \]

\[ N_9^{(n)}[a, b, c; d_1, \ldots, d_n, e_1, \ldots, e_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b)_{m_1 + \ldots + m_n} (c)_{m_1 + \ldots + m_n}}{(d)_{m_1} \cdots (e)_{m_1} \cdots (e)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \tag{1.3.49} \]
Hypergeometric Functions of Three Variables

Lauricella ([182], p.114) introduced fourteen complete hypergeometric functions of three variables and of the second order. He denoted his triple hypergeometric functions by the symbols, $F_1$, $F_2$, $F_3,...$, $F_{14}$ of which $F_1$, $F_2$, $F_3$ and $F_9$ correspond, respectively, to the three variables Lauricella functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$ defined by (1.3.34) through (1.3.37) with $n = 3$. The remaining ten functions $F_3$, $F_4$, $F_5$, $F_7$, $F_8$, $F_{10},...,F_{14}$, of Lauricella’s set apparently fell into oblivion [except that there is an isolated appearance of the triple hypergeometric function $F_8$ in a paper by Mayr ([191], p. 365)]. Saran [198] initiated a systematic study of these ten triple hypergeometric functions of Lauricella’s set. We give below the definitions of these functions using Saran’s notations $F_E$, $F_F,...,F_T$ and also indicating Lauricella’s notations:

$$F_4 : F_E \left[ \alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z \right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_n(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p}{m! n! p!} \frac{x^m y^n z^p}{m! n! p!}, \quad |x| < r, \ |y| < s, \ |z| < t, \ r + \left(\sqrt{s} + \sqrt{t}\right)^2 = 1; \quad (1.3.51)$$

$$F_{14} : F_F \left[ \alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z \right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\gamma_1)_m(\gamma_2)_n(\gamma_2)_p}{m! n! p!} \frac{x^m y^n z^p}{m! n! p!}, \quad |x| < r, \ |y| < s, \ |z| < t, \ (1-s)(s-t) = rs; \quad (1.3.52)$$

$$F_8 : F_G \left[ \alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z \right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{m! n! p!} \frac{x^m y^n z^p}{m! n! p!}, \quad |x| < r, \ |y| < s, \ |z| < t, \ r + s = 1 = r + t; \quad (1.3.53)$$

$$N_{10}^{(n)} \left[ a, b, c; d, e_1, ..., e_n; x_1, ..., x_n \right]$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a)_{m_1+...+m_n}(b)_{m_1+...+m_n} c_{m_1+...+m_n}}{(d)_{m_1+...+m_n}(e_1)_{m_1}...(e_n)_{m_n}} \frac{x_1^{m_1} x_2^{m_2} ... x_n^{m_n}}{m_1! ... m_n!}. \quad (1.3.50)$$
\[ F_3 : F_K \left[ \alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z \right] \\
= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_{n+p} (\beta_1)_{m+p} (\beta_2)_{n+p} x^m y^n z^p}{(\gamma_1)_m (\gamma_2)_{n+p} (\gamma_3)_{p}} \frac{m! n! p!}{}, \]

\[ |x| < r, \ |y| < s, \ |z| < t, \ (1 - r)(1 - s) = t; \]

\[ F_{11} : F_M \left[ \alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z \right] \\
= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_{n+p} (\beta_1)_{m+p} (\beta_2)_{n+p} x^m y^n z^p}{(\gamma_1)_m (\gamma_2)_{n+p}} \frac{m! n! p!}{}, \]

\[ |x| < r, \ |y| < s, \ |z| < t, \ r + t = 1 = s; \]

\[ F_6 : F_N \left[ \alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_1, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z \right] \\
= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_{n+p} (\beta_1)^2_{m+p} (\beta_2)_{n+p} x^m y^n z^p}{(\gamma_1)_m (\gamma_2)_{n+p}} \frac{m! n! p!}{}, \]

\[ |x| < r, \ |y| < s, \ |z| < t, \ (1 - r)s + (1 - s)t = 0; \]

\[ F_{12} : F_P \left[ \alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z \right] \\
= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+p} (\alpha_2)_{n} (\beta_1)_{m+n} (\beta_2)_{p} x^m y^n z^p}{(\gamma_1)_m (\gamma_2)_{n+p}} \frac{m! n! p!}{}, \]

\[ |x| < r, \ |y| < s, \ |z| < t, \ (st - s - t)^2 = 4rst; \]

\[ F_{10} : F_R \left[ \alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z \right] \\
= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+p} (\alpha_2)_{n} (\beta_1)_{m+n} (\beta_2)_{p} x^m y^n z^p}{(\gamma_1)_m (\gamma_2)_{n+p}} \frac{m! n! p!}{}, \]

\[ |x| < r, \ |y| < s, \ |z| < t, \ s (1 - r)^2 + (1 - s)t = 0; \]

\[ F_7 : F_S \left[ \alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_1, \gamma_1; x, y, z \right] \\
= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_{n+p} (\beta_1)_{m} (\beta_2)_{p} x^m y^n z^p}{(\gamma_1)_{m+n+p}} \frac{m! n! p!}{}, \]

\[ |x| < r, \ |y| < s, \ |z| < t, \ r + s = rs, \ s = t; \]
\[ F_{13} : F_T \left[ \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1; x, y, z \right] \]
\[ = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)m+p(\beta_2)n}{{(\gamma_1)m+n+p}} \frac{x^m y^n z^p}{m! n! p!} , \]  
\[ |x| < r, \quad |y| < s, \quad |z| < t, \quad r - rs + s = t. \]  

In the course of a further investigation of Lauricella’s fourteen hypergeometric functions of three variables, H. M. Srivastava [208] noticed the existence of three additional complete hypergeometric functions of the second order; these three functions \( H_A, H_B \) and \( H_C \) had not been included in Lauricella’s conjecture. Their series definitions are given below:

\[ H_A [\alpha, \beta, \beta', \gamma, \gamma'; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta)_{m+n}(\beta')_{n+p}}{(\gamma)_{m+n+p}} \frac{x^m y^n z^p}{m! n! p!} , \]  \[ |x| < r, \quad |y| < s, \quad |z| < t, \quad r + s + t = 1 + st; \]  

\[ H_B [\alpha, \beta, \beta'; \gamma_1, \gamma_2, \gamma_3; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta)_{m+n}(\beta')_{n+p}}{(\gamma_1)_{m+n}(\gamma_2)_n(\gamma_3)_p} \frac{x^m y^n z^p}{m! n! p!} , \]  \[ |x| < r, \quad |y| < s, \quad |z| < t, \quad r + s + t + 2\sqrt{rst} = 1; \]  

\[ H_C [\alpha, \beta, \beta'; \gamma; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta)_{m+n}(\beta')_{n+p}}{(\gamma)_{m+n+p}} \frac{x^m y^n z^p}{m! n! p!} , \]  \[ |x| < 1, \quad |y| < 1, \quad |z| < 1. \]  

In 1963, Pandey [194] established two interesting Horn’s type hypergeometric functions of three variables, while transforming Pochhammer’s double-loop contour integrals associated with the Lauricella’s functions \( F_G \) and \( F_F \) are given below:

\[ G_A [\alpha, \beta, \beta'; \gamma; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta)_{m+n}(\beta')_n}{(\gamma)_{n+p-m}} \frac{x^m y^n z^p}{m! n! p!} , \]  \[ G_B [\alpha, \beta_1, \beta_2, \beta_3; \gamma; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma)_{n+p-m}} \frac{x^m y^n z^p}{m! n! p!} , \]
A similar investigation of the system of partial differential equations associated with the triple hypergeometric function $H_C$ defined by (1.3.63) led to the new function (cf. Srivastava [215], p. 105):

$$G_C[\alpha, \beta, \beta'; \gamma; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{\alpha m + p \beta m + n \beta' n - p \gamma m + n - p}{\gamma_{m+n-p}} x^m y^n z^p \left( \frac{x}{m!} \frac{y}{n!} \frac{z}{p!} \right),$$  

which evidently furnishes a generalization of Appell’s function $F_1$ and Horn’s functions $G_2$ and $H_1$.

**General Triple Hypergeometric Series**

A unification of Lauricella’s fourteen hypergeometric functions $F_1, \ldots, F_{14}$ ([182], p. 114) and the additional functions $H_A$, $H_B$, $H_C$ [210] was introduced by Srivastava (see, e.g. ([209], p. 428) and ([223], p. 69)) in the form of triple hypergeometric series $F^{(3)}[x, y, z]$ defined as

$$F^{(3)}[x, y, z] \equiv F^{(3)} \begin{pmatrix} (a) :: (b) ; (b') : (c) ; (c') ; (c'') : (x, y, z) \\ (e) :: (g) ; (g') ; (g'') : (h) ; (h') ; (h'') \end{pmatrix},$$

where, for convenience

$$\Lambda(m, n, p) = \frac{\prod_{j=1}^{A} (a_j)_{m+n+p} \prod_{j=1}^{B} (b_j)_{m+n} \prod_{j=1}^{B'} (b'_j)_{n+p} \prod_{j=1}^{B''} (b''_j)_{p+m}}{\prod_{j=1}^{E} (e_j)_{m+n+p} \prod_{j=1}^{G} (g_j)_{m+n} \prod_{j=1}^{G'} (g'_j)_{n+p} \prod_{j=1}^{G''} (g''_j)_{p+m}} \times \frac{\prod_{j=1}^{C} (c_j)_m \prod_{j=1}^{C'} (c'_j)_n \prod_{j=1}^{C''} (c''_j)_p}{\prod_{j=1}^{H} (h_j)_m \prod_{j=1}^{H'} (h'_j)_n \prod_{j=1}^{H''} (h''_j)_p},$$

where $(a)$ abbreviates, the array of $A$ parameters $a_1, a_2, \ldots a_A$, with similar interpretations for $(b)$, $(b')$, $(b'')$, etc. The triple hypergeometric series in (1.3.67) converges absolutely when

$$\begin{align*}
1 + E + G + G'' + H - A - B - B'' - C &\geq 0, \\
1 + E + G + G' + H' - A - B - B' - C' &\geq 0, \\
1 + E + G' + G'' + H'' - A - B' - B'' - C'' &\geq 0,
\end{align*}$$

(1.3.69)
where the equalities hold true for suitable constrained values of \(|x|, |y|\) and \(|z|\).

1.4 The Classical Orthogonal Polynomials

Orthogonal polynomials constitute an important class of special functions in general and hypergeometric functions in particular. Some of the orthogonal polynomials and their connection with hypergeometric functions used in our work are as given below:

Hermite Polynomials

Hermite Polynomials \(H_n(x)\) are defined by means of generating relation

\[
\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!},
\]

valid for all finite \(x\) and \(t\) and one can easily obtain

\[
H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n! (2x)^{n-2k}}{k! (n-2k)!}.
\]

The hypergeometric form of (1.4.2) can be written as

\[
H_n(x) = (2x)^n \, _2F_0 \left[ -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}; -\frac{1}{x^2} \right].
\]

Laguerre Polynomials

The generalized Laguerre Polynomial \(L_n^{(\alpha)}(x)\) is defined by means of generating relation

\[
\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1 - t)^{-(\alpha+1)} \exp \left( \frac{xt}{t-1} \right).
\]

For \(\alpha = 0\), the above equation (1.4.4) yield the generating function for simple Laguerre Polynomial \(L_n(x)\).

\[
L_n(x) = L_n^{(0)}(x) = _1F_1 [ -n; 1; x ].
\]

A series representation of \(L_n^{(\alpha)}(x)\) for non negative integers \(n\), is given by

\[
L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1 + \alpha)_n x^k}{k! (n-k)! (1 + \alpha)_k}.
\]
The hypergeometric form of generalized Laguerre polynomials are defined by (see [196], p. 200)

\[ L_n^{(\alpha)}(x) = \frac{(1 + \alpha)^n}{n!} {}_1F_1 \left[ -n; 1 + \alpha; x \right]. \quad (1.4.7) \]

**Jacobi Polynomials**

The Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \) are defined as

\[ P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha)^n}{n!} {}_2F_1 \left[ -n, 1 + \alpha + \beta + n; 1 + \alpha; \frac{1 - x}{2} \right]. \quad (1.4.8) \]

**Special cases**

(i) When \( \alpha = \beta = 0 \), the polynomial in (1.4.8) becomes the Legendre polynomials \( P_n(x) \).

(ii) If \( x = 1 \) in (1.4.8) then \( P_n^{(\alpha, \beta)}(1) = \frac{(1+\alpha)_n}{n!} \).

(iii) If \( \beta = \alpha \), the Jacobi polynomials in (1.4.8) becomes the Ultraspherical polynomials \( P_n^{(\alpha, \alpha)}(x) \).

The other \( {}_2F_1 \) forms for \( P_n^{(\alpha, \beta)}(x) \) can also be written as

\[ P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha)_n}{n!} \left( \frac{x+1}{2} \right)^n {}_2F_1 \left[ -n, -\beta - n; 1 + \alpha; \frac{x-1}{2} \right]. \quad (1.4.9) \]

and

\[ P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n(1 + \beta)_n}{n!} {}_2F_1 \left[ -n, 1 + \alpha + \beta + n; 1 + \beta; \frac{1+x}{2} \right]. \quad (1.4.10) \]

Each of (1.4.8)-(1.4.10) yields a finite series form for \( P_n^{(\alpha, \beta)}(x) \):

\[ P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^{n} \frac{(1 + \alpha)_n(1 + \alpha + \beta)_{n+k}}{k!(n-k)!(1 + \alpha)_k(1 + \alpha + \beta)_n} \left( \frac{x-1}{2} \right)^k , \quad (1.4.11) \]

\[ P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^{n} \frac{(1 + \alpha)_n(1 + \beta)_n}{k!(n-k)!(1 + \alpha)_k(1 + \beta)_{n-k}} \left( \frac{x-1}{2} \right)^k \left( \frac{x+1}{2} \right)^{n-k} \quad (1.4.12) \]

and

\[ P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^{n} \frac{(1)^{n-k}(1 + \beta)_n(1 + \alpha + \beta)_{n+k}}{k!(n-k)!(1 + \beta)_k(1 + \alpha + \beta)_n} \left( \frac{x+1}{2} \right)^k . \quad (1.4.13) \]
Equations (1.4.11), (1.4.12) and (1.4.13) are expanded forms of (1.4.8), (1.4.9) and (1.4.10) respectively.

By reversing the order of summation in (1.4.11), (1.4.12) and (1.4.13), respectively, one gets

\[
P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha + \beta)^{2n}}{n!(1 + \alpha + \beta)^n} \left( \frac{x - 1}{2} \right)^n \frac{2}{1 - x} \right),
\]

(1.4.14)

\[
P_n^{(\alpha, \beta)}(x) = \frac{(1 + \beta)^n}{n!} \left( \frac{x - 1}{2} \right)^n 2F_1 \left[ -n, -\alpha - n; \frac{1 + \beta}{1 + x} \right],
\]

(1.4.15)

and

\[
P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha + \beta)^{2n}}{n!(1 + \alpha + \beta)^n} \left( \frac{x + 1}{2} \right)^n \frac{2}{1 + x} \right),
\]

(1.4.16)

**Rice Polynomials**

Rice, S. O. [197] investigated a polynomial set,

\[
H_n(\xi, p, \nu) = \binom{3}{F_2} \left[ -n, n + 1, \xi; 1, p; \nu \right].
\]

(1.4.17)

In 1964, Khandekar [179] generalized Rice polynomial as

\[
H_n^{(\alpha, \beta)}(\xi, p, x) = \frac{(1 + \alpha)^n}{n!} \binom{3}{F_2} \left[ -n, 1 + \alpha + \beta + n, \xi; 1 + \alpha, p; x \right],
\]

(1.4.18)

where \((\alpha) > -1, \ Re(\beta) > -1.\)

**Bessel polynomials**

The simple Bessel polynomial \(y_n(x)\) is defined as (Krall and Frink [180])

\[
y_n(x) = \binom{2}{F_0} \left[ -n, 1 + n; \frac{1}{2} x \right].
\]

(1.4.19)

The generalized Bessel polynomials \(y_n(x, \alpha, \beta)\) is defined as (Krall and Frink work [180]; see also Grosswald [23])

\[
y_n(x, \alpha, \beta) = \binom{2}{F_0} \left[ -n, \alpha - 1 + n; \frac{x}{\beta} \right].
\]

(1.4.20)
1.5 Other Polynomial Systems

Bedient’s polynomials

Bedient [9] polynomials, denoted by \( R_n(\beta, \gamma; x) \) and \( G_n(\alpha, \beta; x) \), are respectively defined as:

\[
R_n(\beta, \gamma; x) = \frac{(\beta)_n(2x)^n}{n!} \, _3F_2 \left[ \begin{array}{c} -\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2} \gamma - \beta; \\ & \gamma, 1 - \beta - n; \\ \end{array} x^2 \right]
\]

and

\[
G_n(\alpha, \beta; x) = \frac{(\alpha)_n(\beta)_n(2x)^n}{n!(\alpha + \beta)_n} \, _3F_2 \left[ \begin{array}{c} -\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2} 1 - \alpha - \beta - n; \\ & 1 - \alpha - n, 1 - \beta - n; \\ \end{array} x^2 \right].
\]

Bateman’s polynomials

Bateman (see, for example, [196]) defined the polynomials \( Z_n(x) \) as follows:

\[
Z_n(x) = \, _2F_2 \left[ \begin{array}{c} -n, n + 1; \\ & 1, 1; \\ \end{array} x \right].
\]

In 1999, M. A. Khan and A. K. Shukla [77] defined the Jacobi type generalization of the Bateman’s polynomials \( Z_n(\alpha, \beta)(b, x) \) as follows:

\[
Z_n(\alpha, \beta)(b, x) = \, _2F_2 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n; \\ & 1 + \alpha, b; \\ \end{array} x \right].
\]

Also, Bateman (see, for example, [196]) defined the polynomials \( F_n(z) \) as follows:

\[
F_n(z) = \, _3F_2 \left[ \begin{array}{c} -n, n + 1, \frac{1}{2}(1 + z); \\ & 1, 1; \\ \end{array} 1 \right].
\]

Other Jacobi type generalization of the Bateman’s polynomials \( F_n(\alpha, \beta)(p, z) \) are defined by M. A. Khan and A. K. Shukla [77] as follows:

\[
F_n(\alpha, \beta)(p, z) = \, _3F_2 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n, \frac{1}{2}(1 + z); \\ & 1 + \alpha, p; \\ \end{array} 1 \right].
\]

Pasternak’s polynomials

The generalization of the Bateman’s Polynomial due to Pasternak (see [196]) is given below:

\[
F_n^m(z) = \, _3F_2 \left[ \begin{array}{c} -n, n + 1, \frac{1}{2}(1 + z + m); \\ & 1, m + 1; \\ \end{array} 1 \right],
\]

which is a generalization of Bateman’s polynomials \( F_n(z) \).
In 1999, M. A. Khan and A. K. Shukla [77] defined the Jacobi type generalization of the Pasternak’s polynomials $F^{(\alpha, \beta)}_{n, m}(z)$ as follows:

$$F^{(\alpha, \beta)}_{n, m}(z) = 3 F_2 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n, \frac{1}{2}(1 + z + m); \\ 1 + \alpha, 1 + m; 1 \end{array} \right].$$  \hspace{0.5cm} (1.5.8)

**Krawtchouk Polynomials**

The Krawtchouk polynomials are defined as (see [223], p. 75)

$$K_n(x; p, N) = 2 F_1 \left[ \begin{array}{c} -n, -x; \\ -N; p^{-1} \end{array} \right],$$  \hspace{0.5cm} (1.5.9)

$0 < p < 1$ and $x = 0, 1, 2, ..., N$.

**Meixner Polynomials**

The Meixner polynomials are defined as (see [223], p. 75)

$$M_n(x; \beta, c) = 2 F_1 \left[ \begin{array}{c} -n, -x; \\ \beta; 1 - \frac{1}{c} \end{array} \right],$$  \hspace{0.5cm} (1.5.10)

$\beta > 0$, $0 < c < 1$, $x = 0, 1, 2, ...$.

**Hahn Polynomials**

The Hahn polynomial is defined as

$$Q_n(x; \alpha, \beta, N) = 3 F_2 \left[ \begin{array}{c} -n, -x, 1 + \alpha + \beta + n; \\ -N, 1 + \alpha; 1 \end{array} \right],$$  \hspace{0.5cm} (1.5.11)

$\alpha, \beta > -1$, $n, x = 0, 1, \cdots, N$.

**Gottlieb Polynomials**

The Gottlieb polynomials are given below (see [223], p. 185):

$$l_n(x; \lambda) = e^{-n\lambda} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{x}{k} \right) (1 - e^\lambda)^k.$$  \hspace{0.5cm} (1.5.12)

or

$$l_n(x; \lambda) = e^{-n\lambda} 2 F_1 \left[ \begin{array}{c} -n, -x; \\ 1; 1 - e^\lambda \end{array} \right].$$  \hspace{0.5cm} (1.5.13)
Poisson-Charlier Polynomials

The Poisson-Charlier Polynomial is defined by (cf. Erdelyi et al. [20], Vol. II, p. 226, Eq. (4); see also Szegő [228], p. 35, Eq. (2.81.2))

\[ C_n(x; \alpha) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{x}{k}\right)^k k! \alpha^{-k}, \quad (1.5.14) \]

\[ \alpha > 0, \; x = 0, 1, 2, \ldots . \]

1.6 Certain polynomials of Two Variables

Some polynomials of two variables used in our work are as follows:

Generalized Bessel polynomials of Two Variables

M. A. Khan and K. Ahmad [83] defined Bessel polynomials of two variables as given below:

\[ Y_n^{(\alpha, \beta)}(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(\alpha + n + 1)r(\beta + n + 1)s}{r!s!} \left(-\frac{x}{2}\right)^r \left(-\frac{y}{2}\right)^s. \quad (1.6.1) \]

The definition (1.6.1) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[ Y_n^{(\alpha, \beta)}(x, y) = F_{0;0;0}^{1;1;1} \left[ \begin{array}{ccc} -n & : & \alpha + n + 1; \beta + n + 1; \\ & -; & -; -; \end{array} \left(\frac{-x}{2}\right)^r \left(\frac{-y}{2}\right)^s \right]. \quad (1.6.2) \]

However, in the literature generalized Bessel polynomials of two variables have been denoted and defined in a different manner as given below:

\[ Y_n(x, a_1, b_1; y, a_2, b_2) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(a_1 - 1 + n)r(a_2 - 1 + n)s}{r!s!} \left(-\frac{x}{b_1}\right)^r \left(-\frac{y}{b_2}\right)^s. \quad (1.6.3) \]

For our results we shall stick to definition (1.6.3) for generalized Bessel polynomials of two variables.

The definition (1.6.3) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[ Y_n(x, a_1, b_1; y, a_2, b_2) = F_{0;0;0}^{1;1;1} \left[ \begin{array}{ccc} -n & : & a_1 - 1 + n; a_2 - 1 + n; \\ & -; & -; -; \end{array} \left(-\frac{x}{b_1}\right)^r \left(-\frac{y}{b_2}\right)^s \right]. \quad (1.6.4) \]
The definition (1.6.3) can also be represented as follows:

\[ Y_n(x, a_1, b_1; y, a_2, b_2) = \sum_{r=0}^{\infty} \frac{(-n)_r(a_1 - 1 + n)_r}{r!} \left( -\frac{x}{b_1} \right)^r Y_{n-r}(y, a_2 + r, b_2), \quad (1.6.5) \]

where \( Y_n(y, a_2, b_2) \) is the well-known generalized Bessel polynomials of one variable defined by (1.4.20).

The relationships between generalized Bessel polynomials of two variables defined by (1.6.3) and generalized Bessel polynomials of one variable are as follows:

\[ Y_n(x, a_1, b_1; 0, a_2, b_2) = Y_n(x, a_1, b_1), \quad (1.6.6) \]

and

\[ Y_n(0, a_1, b_1; y, a_2, b_2) = Y_n(y, a_2, b_2). \quad (1.6.7) \]

**Bedient Polynomials of Two Variables**

M. A. Khan, A. H. Khan and N. Ahmad [1] defined Bedient polynomials of two variables as given below:

\[ R_n(\beta, \gamma, \lambda, \mu; x, y) = \frac{(\beta + \lambda)_n (2xy)^n}{n!} \]

\[ \times \sum_{r=0}^{\left[ \frac{n}{2} \right]} \sum_{s=0}^{\left[ \frac{n-r}{2} \right]} \frac{(-n)_{2r+2s}(\gamma - \beta)_r(\mu - \lambda)_s}{2^{2r+2s}(1 - \beta - \lambda - n)_{r+s} (\gamma)_r(\mu)_s r! s!} x^{2r} y^{2s}. \quad (1.6.8) \]

\[ G_n(\alpha, \beta, \lambda, \mu; x, y) = \frac{(\alpha + \lambda)_n(\beta + \mu)_n (2xy)^n}{(\alpha + \beta + \lambda + \mu)_n n!} \]

\[ \times \sum_{r=0}^{\left[ \frac{n}{2} \right]} \sum_{s=0}^{\left[ \frac{n-r}{2} \right]} \frac{(-n)_{2r+2s}(1 - \alpha - \beta - \lambda - n)_{r+s}(1 - \beta - \mu - n)_{r+s} r! s! x^{2r} y^{2s}}{2^{2r+2s}(1 - \alpha - \lambda - n)_{r+s}(1 - \beta - \mu - n)_{r+s} r! s!} \quad (1.6.9) \]

In terms of double hypergeometric function, Bedient polynomials of two variables can also be written as

\[ R_n(\beta, \gamma, \lambda, \mu; x, y) = \frac{(\beta + \lambda)_n (2xy)^n}{n!} \]

\[ \times \, _2F^2_{1; 0; 0} \left[ \begin{array}{c} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2} : \gamma - \beta, \mu - \lambda; -; \frac{1}{x^2}, \frac{1}{y^2} \\ 1 - \beta - \lambda - n : \gamma, \mu; -; \end{array} \right]. \quad (1.6.10) \]
Krawtchouk polynomials of Two Variables

M. A. Khan and M. Akhlaq [2] defined Krawtchouk polynomials of two variables as given below:

\[ K_n(x, y; N, \frac{p_1}{q_1}, \frac{p_2}{q_2}) = \frac{(p_1 p_2)^n}{n!} (-N)_n \sum_{r=0}^{n-r} \frac{(p_1 p_2)^n}{n!} \left( \frac{(-N - x; -y)_r}{(1 + q_1)_r} \right) \left( \frac{(-N - x; -y)_r}{(1 + q_2)_r} \right)^s \].

(1.6.12)

The definition (1.6.12) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[ K_n(x, y; N, \frac{p_1}{q_1}, \frac{p_2}{q_2}) = \frac{(p_1 p_2)^n}{n!} (-N)_n F_{1;1:1}^{1;1:1} \left[ \begin{array}{c} -n \quad -x \quad -y \\ N \quad -1 \\ 1 + \frac{q_1}{p_1} \quad 1 + \frac{q_2}{p_2} \end{array} \right]. \]

(1.6.13)

For this polynomial, he obtained the following generating functions:

\[ \sum_{n=0}^{\infty} K_n(x, y; N, \frac{p_1}{q_1}, \frac{p_2}{q_2}) t^n = (1 - p_1 p_2 t)^{N - x - y} (1 + p_2 q_1 t)^x (1 + p_1 q_2 t)^y \]

(1.6.14)

and

\[ \sum_{n=0}^{\infty} \binom{n + k}{k} K_{n+r}(x, y; N, \frac{p_1}{q_1}, \frac{p_2}{q_2}) t^n = (1 - p_1 p_2 t)^{N - x - y - r} (1 + p_2 q_1 t)^x (1 + p_1 q_2 t)^y \]

\[ \times (1 + p_1 q_2 t)^{y-r} K_r(x, y; N, \frac{p_1(1 + p_2 q_1 t)}{q_1(1 - p_1 p_2 t)}, \frac{p_2(1 + p_1 q_2 t)}{q_2(1 - p_1 p_2 t)}). \]

(1.6.15)

However, in the literature Krawtchouk polynomials of two variables have been denoted and defined in a different manner as given below:

\[ K_n(x; \lambda_1, N_1 \cdot y; \lambda_2, N_2) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (-x)_r (-y)_s}{r! s! (N_1)_r (N_2)_s} \left( \frac{1}{\lambda_1} \right)^r \left( \frac{1}{\lambda_2} \right)^s. \]

(1.6.16)
For our results we shall stick to definition (1.6.16) for Krawtchouk polynomials of two variables.

The definition (1.6.16) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[ K_n(x; \lambda_1, N_1 : y; \lambda_2, N_2) = F_{0:1:1}^{1:1:1} \left[ \begin{array}{c}
-n : -x; -y; 1 \cr - : -N_1; -N_2; \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \end{array} \right]. \] (1.6.17)

The definition (1.6.16) can also be represented as follows:

\[ K_n(x; \lambda_1, N_1 : y; \lambda_2, N_2) = \sum_{r=0}^{n} \binom{\beta}{n} (-n)_r (-x)_r \left( \frac{1}{\lambda_1} \right)^r \frac{1}{r!(N_1)_r} K_{n-r}(y; \lambda_2, N_2), \] (1.6.18)

where \( K_n(y; \lambda_2, N_2) \) is the well-known Krawtchouk polynomials of one variable defined by (1.5.9).

The relationships between Krawtchouk polynomials of two variables defined by (1.6.16) and Krawtchouk polynomials of one variable are as follows:

\[ K_n(x; \lambda_1, N_1 : 0; \lambda_2, N_2) = K_n(x; \lambda_1, N_1) \] (1.6.19)

and

\[ K_n(0; \lambda_1, N_1 : y; \lambda_2, N_2) = K_n(y; \lambda_2, N_2). \] (1.6.20)

Meixner polynomials of Two Variables

M. A. Khan and M. Akhlaq [150] defined Meixner polynomials of two variables as given below:

\[ m_n(x, y; \beta, c, d) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(\beta)_n (-n)_r (-x)_r (-y)_s (1-c^{-1})^r (1-d^{-1})^s}{r! s! (\beta)_{r+s}}. \] (1.6.21)

The definition (1.6.21) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[ m_n(x, y; \beta, c, d) = (\beta)_n F_{1:0:0}^{1:1:1} \left[ \begin{array}{c}
-n : -x; -y; 1 \cr \beta : -; -; 1 - c^{-1}, 1 - d^{-1} \end{array} \right]. \] (1.6.22)
For this polynomial, he obtained the following generating functions:

\[
\sum_{n=0}^{\infty} m_n(x, y; \beta, c, d) \frac{t^n}{n!} = (1 - t)^{-\beta - x - y} \left(1 - \frac{t}{c}\right)^x \left(1 - \frac{t}{d}\right)^y, \quad |t| < \min(1, |c|, |d|) \tag{1.6.23}
\]

and

\[
\sum_{n=0}^{\infty} m_{n+k}(x, y; \beta, c, d) \frac{t^n}{n!} = (1 - t)^{-\beta - k - x - y} \left(1 - \frac{t}{c}\right)^x \left(1 - \frac{t}{d}\right)^y m_k \left(x, y; \beta, \frac{c - t}{1 - t}, \frac{d - t}{1 - t}\right). \tag{1.6.24}
\]

However, in the literature Meixner polynomials of two variables have been denoted and defined in a different manner as given below:

\[
M_n(x; \beta_1, \lambda_1 : y; \beta_2, \lambda_2) = (\beta_1)_n \left(\beta_2\right)_n \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_r (-x)_r (-y)_s}{r! s! (\beta_1)_r (\beta_2)_s} \left(1 - \frac{1}{\lambda_1}\right)^r \left(1 - \frac{1}{\lambda_2}\right)^s. \tag{1.6.25}
\]

For our results we shall stick to definition (1.6.25) for Meixner polynomials of two variables.

The definition (1.6.25) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[
M_n(x; \beta_1, \lambda_1 : y; \beta_2, \lambda_2) = (\beta_1)_n (\beta_2)_n F_{1:1;1}^{1:1;1} \left[ -n : -x; -y; \begin{array}{c} -n : -x : -y : \beta_1; \beta_2; 1 - \frac{1}{\lambda_1}, 1 - \frac{1}{\lambda_2} \end{array} \right]. \tag{1.6.26}
\]

The definition (1.6.25) can also be represented as follows:

\[
M_n(x; \beta_1, \lambda_1 : y; \beta_2, \lambda_2) = (\beta_1)_n \sum_{r=0}^{n} \frac{(-n)_r (-x)_r}{r! (\beta_1)_r} M_{n-r}(y; \beta_2, \lambda_2), \tag{1.6.27}
\]

where \(M_n(y; \beta_2, \lambda_2)\) is the well-known Meixner polynomials of one variable defined by (1.5.10).

The relationships between Meixner polynomials of two variables defined by (1.6.25) and Meixner polynomials of one variable are as follows:

\[
M_n(x; \beta_1, \lambda_1 : 0; \beta_2, \lambda_2) = (\beta_2)_n M_n(x; \beta_1, \lambda_1) \tag{1.6.28}
\]

and

\[
M_n(0; \beta_1, \lambda_1 : y; \beta_2, \lambda_2) = (\beta_1)_n M_n(y; \beta_2, \lambda_2). \tag{1.6.29}
\]
Hahn polynomials of Two Variables

The Hahn polynomials of two variables $Q_n(x; \alpha_1, \beta_1, N_1 : y; \alpha_2, \beta_2, N_2)$ are defined as:

$$Q_n(x; \alpha_1, \beta_1, N_1 : y; \alpha_2, \beta_2, N_2) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(1 + \alpha_1 + \beta_1 + n)_r(1 + \alpha_2 + \beta_2 + n)_s(-x)_r(-y)_s}{r!s!(1 + \alpha_1)_r(1 + \alpha_2)_s(-N_1)_r(-N_2)_s}. \tag{1.6.30}$$

The definition (1.6.30) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

$$Q_n(x; \alpha_1, \beta_1, N_1 : y; \alpha_2, \beta_2, N_2) = {}_2F_1(1 + \alpha_1 + \beta_1 + n, -x; 1 + \alpha_1, -N_1; 1 + \alpha_2, -N_2; y). \tag{1.6.31}$$

The definition (1.6.30) can also be represented as follows:

$$Q_n(x; \alpha_1, \beta_1, N_1 : y; \alpha_2, \beta_2, N_2) = \sum_{r=0}^{n} \frac{(-n)_{r+s}(1 + \alpha_1 + \beta_1 + n)_r(-x)_r(-y)_s}{r!(1 + \alpha_1)_r(-N_1)_r} Q_{n-r}(y; \alpha_2, \beta_2 + r, N_2), \tag{1.6.32}$$

where $Q_n(y; \alpha_2, \beta_2, N_2)$ is the well-known Hahn polynomials of one variable defined by (1.5.11).

The relationships between Hahn polynomials of two variables and Hahn polynomials of one variable are as follows:

$$Q_n(x; \alpha_1, \beta_1, N_1 : 0; \alpha_2, \beta_2, N_2) = Q_n(x; \alpha_1, \beta_1, N_1) \tag{1.6.33}$$

and

$$Q_n(0; \alpha_1, \beta_1, N_1 : y; \alpha_2, \beta_2, N_2) = Q_n(y; \alpha_2, \beta_2, N_2). \tag{1.6.34}$$

Gottlieb polynomials of Two Variables

M. A. Khan and M. Akhlaq [108] defined Gottlieb polynomials of two variables $I_n(x; \lambda : y; \mu)$ as given below:

$$I_n(x; \lambda : y; \mu) = e^{-n(\lambda+\mu)} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-x)_r(-y)_s}{r!s!(1)_r(1)_s} (1 - e^{\lambda})^r (1 - e^{\mu})^s. \tag{1.6.35}$$
The definition (1.6.35) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[ I_n(x; \lambda_1 : y; \lambda_2) = e^{-n(\lambda_1 + \mu)} F_{1;1}^{1:1;1}_{0;0;0;0;0;0} \left[ \begin{array}{c} -n : -x; -y; \\
1: -; -; 1 - e^\lambda, 1 - e^\mu 
\end{array} \right]. \] (1.6.36)

The relationships between Gottlieb polynomials of two variables and Gottlieb polynomials of one variable are as follows:

\[ I_n(x; \lambda : 0; \mu) = I_n(x; \lambda) \] (1.6.37)

and

\[ I_n(0; \lambda : y; \mu) = I_n(y; \lambda). \] (1.6.38)

**Poisson-Charlier polynomials of Two Variables**

The Poisson-Charlier polynomials of two variables \( C_n(x; \alpha_1 : y; \alpha_2) \) are defined by M. A. Khan and S. Ahmed [146] as follows:

\[ C_n(x; \alpha_1 : y; \alpha_2) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-x)^r(-y)^s}{r!s!} \left( -\frac{1}{\alpha_1} \right)^r \left( -\frac{1}{\alpha_2} \right)^s. \] (1.6.39)

The definition (1.6.39) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[ C_n(x; \alpha_1 : y; \alpha_2) = F_{1;1}^{1:1;1}_{0;0;0;0;0;0} \left[ \begin{array}{c} -n : -x; -y; \\
1: -; -; 1 - \frac{1}{\alpha_1}, 1 - \frac{1}{\alpha_2} 
\end{array} \right]. \] (1.6.40)

The definition (1.6.39) can also be represented as follows:

\[ C_n(x; \alpha_1 : y; \alpha_2) = \sum_{r=0}^{n} \frac{(-n)_r(-x)^r}{r!} \left( -\frac{1}{\alpha_1} \right)^r C_{n-r}(y; \alpha_2), \] (1.6.41)

where \( C_n(y; \alpha_2) \) is the well-known Poisson-Charlier polynomials of one variable defined by (1.5.14).

The relationships between Poisson-Charlier polynomials of two variables and Poisson-Charlier polynomials of one variable are as follows:

\[ C_n(x; \alpha_1 : 0; \alpha_2) = C_n(x; \alpha_1) \] (1.6.42)

and

\[ C_n(0; \alpha_1 : y; \alpha_2) = C_n(y; \alpha_2). \] (1.6.43)
1.7 Generating Functions

The name 'generating function' was introduced by Laplace in 1812. Since then the theory of generating functions has been developed into various directions and found wide applications in various branches of science and technology.

A generating functions may be used to define a set of functions, to determine a differential recurrence relation or a pure recurrence relation, to evaluate certain integrals, etcetera.

**Linear Generating Functions**

Consider a two-variable function $F(x, t)$ which possesses a formal (not necessarily convergent for $t \neq 0$) power series expansion in $t$ such that

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x) t^n$$

(1.7.1)

where each member of the coefficient set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of $t$. Then the expansion (1.7.1) of $F(x, t)$ is said to have generated the set $\{f_n(x)\}$ and $F(x, t)$ is called a linear generating function (or, simply, a generating function) for the set $\{f_n(x)\}$.

The definition (1.7.1) may be extended slightly to include a generating function of the type:

$$G(x, t) = \sum_{n=0}^{\infty} c_n g_n(x) t^n,$$

(1.7.2)

where the sequence $\{c_n\}_{n=0}^{\infty}$ may contain the parameters of the set $g_n(x)$, but is independent of $x$ and $t$.

A set of functions may have more than one generating function. However, if $G(x, t) = \sum_{n=0}^{\infty} h_n(x) t^n$ then $G(x, t)$ is the unique generator for the set $h_n(x)$ as the coefficient set.

We now, extend our definition of a generating relation to include functions which possess Laurent series expansion. Thus, if the set $\{f_n(x)\}$ is defined for $n = 0, \pm 1, \pm 2, \ldots$
the definition (1.7.2) may be extended in terms of Laurent series expansion

\[ F^*(x, t) = \sum_{n=-\infty}^{\infty} \gamma_n f_n(x)t^n, \]  

(1.7.3)

where the sequence \( \{\gamma_n(x)\}_{n=-\infty}^{\infty} \) is independent of \( x \) and \( t \).

**Bilinear Generating Functions**

If a three-variable function \( F(x, y, t) \) possesses a formal power series expansion in \( t \) such that

\[ F(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_n(x)f_n(y)t^n, \]  

(1.7.4)

where the sequence \( \{\gamma_n\} \) is independent of \( x, y \) and \( t \), then \( F(x, y, t) \) is called a bilinear generating function for the set \( \{f_n(x)\} \).

More generally, if \( F(x, y, t) \) can be expanded in powers of \( t \) in the form

\[ F(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x)f_{\beta(n)}(y)t^n, \]  

(1.7.5)

where \( \alpha(n) \) and \( \beta(n) \) are the functions of \( n \), which are not necessarily equal, we shall still call \( F(x, y, t) \) a bilinear generating function for the set \( \{f_n(x)\} \).

**Bilateral Generating Functions**

Suppose that a three-variable function \( H(x, y, t) \) has a formal power series expansion in \( t \) such that

\[ H(x, y, t) = \sum_{n=0}^{\infty} h_n f_n(x)g_n(y)t^n, \]  

(1.7.6)

where the sequence \( \{h_n\} \) is independent of \( x, y \) and \( t \), and the sets of functions \( \{f_n(x)\}_{n=0}^{\infty} \) and \( \{g_n(x)\}_{n=0}^{\infty} \) are different. Then \( H(x, y, t) \) is called a bilateral generating function for the set \( \{f_n(x)\} \) or \( \{g_n(x)\} \).

The above definition of a bilateral generating function, used earlier by Rainville ([196], p. 170) and McBride ([192], p. 19), may be extended to include bilateral generating functions of the type:

\[ F(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x)g_{\beta(n)}(y)t^n, \]  

(1.7.7)
Where the sequence \( \{\gamma_n\} \) is independent of \( x, y \) and \( t \), the sets of functions \( \{f_n(x)\}_{n=0}^{\infty} \) and \( \{g_n(x)\}_{n=0}^{\infty} \) are different, and \( \alpha(n) \) and \( \beta(n) \) are functions of \( n \) which are not necessarily equal.

### 1.8 Jacobi Polynomials of Two and Three Variables

In 1991, S. F. Ragab [195] defined Laguerre polynomials of two variables \( L_n^{(\alpha, \beta)}(x, y) \) as follows:

\[
L_n^{(\alpha, \beta)}(x, y) = \frac{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{n!} \sum_{r=0}^{n} \frac{(-y)^r L_n^{(\alpha)}(x)}{\Gamma(\alpha + n - r + 1) \Gamma(\beta + r + 1)}, \tag{1.8.1}
\]

where \( L_n^{(\alpha)}(x) \) is the well-known Laguerre polynomials of one variable defined by (1.4.6).

The definition (1.8.1) is equivalent to the following explicit representation of \( L_n^{(\alpha, \beta)}(x, y) \), given by S. F. Ragab [195]

\[
L_n^{(\alpha, \beta)}(x, y) = \frac{(\alpha + 1)_n (\beta + 1)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)^{r+s} x^s y^r}{(\alpha + 1)_s (\beta + 1)_r r! s!}. \tag{1.8.2}
\]

In 1997, M. A. Khan and A. K. Shukla [68] extended Laguerre polynomials of two variables to Laguerre polynomials of three variables and later to Laguerre polynomials of \( m \)-variables [71].

In 1998, M. A. Khan and G. S. Abukhammash [73] defined Hermite polynomials of two variables \( H_n(x, y) \) as follows:

\[
H_n(x, y) = \sum_{r=0}^{[n/2]} \sum_{s=0}^{n-2r} \frac{n! (-y)^r H_{n-2r}(x)}{r!(n-2r)!}, \tag{1.8.3}
\]

where \( H_n(x) \) is the well-known Hermite polynomial of one variable defined by (1.4.2).

The definition (1.8.3) is equivalent to the following explicit representation of \( H_n(x, y) \), given by M. A. Khan and G. S. Abukhammash [73]

\[
H_n(x, y) = \sum_{r=0}^{[n/2]} \sum_{s=0}^{[n-2r]} \frac{(-n)_{2r+2s} (2x)^{n-2r-2s} (-y)^r (-1)^s}{r! s!}. \tag{1.8.4}
\]
In 2000, H. S. P. Shrivastava [202] defined the Jacobi polynomials of two and three variables, denoted by \( P_{n}^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(x, y) \) and \( P_{n}^{(\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \beta_3)}(x, y, z) \) respectively, as follows:

\[
P_{n}^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(x, y) = \frac{(1 + \alpha_1)_n (1 + \alpha_2)_n}{(n!)^2} \times \sum_{r=0}^{n} \sum_{s=0}^{n-r} (-n)_{r+s} (1 + \alpha_1 + \beta_1 + n)_r (1 + \alpha_2 + \beta_2 + n)_s \frac{1}{r! s!} \left( \frac{1-x}{2} \right)^r \left( \frac{1-y}{2} \right)^s \text{ (1.8.5)}
\]

and

\[
P_{n}^{(\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \beta_3)}(x, y, z) = \frac{(1 + \alpha_1)_n (1 + \alpha_2)_n (1 + \alpha_3)_n}{(n!)^3} \times \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{k=0}^{n-r-s} (-n)_{r+s+k} (1 + \alpha_1 + \beta_1 + n)_r (1 + \alpha_2 + \beta_2 + n)_s (1 + \alpha_3 + \beta_3 + n)_k \frac{1}{r! s! k!} \left( \frac{1-x}{2} \right)^r \left( \frac{1-y}{2} \right)^s \left( \frac{1-z}{2} \right)^k. \text{ (1.8.6)}
\]

The relations (1.8.5) and (1.8.6) can also be expressed in the following respective forms:

\[
P_{n}^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(x, y) = \frac{(1 + \alpha_1)_n (1 + \alpha_2)_n}{(n!)^2} \times F_2 \left[ -n, 1 + \alpha_1 + \beta_1 + n, 1 + \alpha_2 + \beta_2 + n; 1 + \alpha_1, 1 + \alpha_2; \frac{1-x}{2}, \frac{1-y}{2} \right], \text{ (1.8.7)}
\]

where \( F_2 \) denotes one of the Appell’s function of two variables defined by (1.3.2), and

\[
P_{n}^{(\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \beta_3)}(x, y, z) = \frac{(1 + \alpha_1)_n (1 + \alpha_2)_n (1 + \alpha_3)_n}{(n!)^3} \times F_3 \left[ -n, 1 + \alpha_1 + \beta_1 + n, 1 + \alpha_2 + \beta_2 + n, 1 + \alpha_3 + \beta_3 + n; 1 + \alpha_1, 1 + \alpha_2, 1 + \alpha_3; \frac{1-x}{2}, \frac{1-y}{2}, \frac{1-z}{2} \right]. \text{ (1.8.8)}
\]

where \( F_3 \) denotes Lauricella’s triple hypergeometric function defined by (1.3.34) with \( n = 3 \).
The definition (1.8.5) can also be represented as follows:

\[ P_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(x, y) = \frac{\Gamma(1 + \alpha_1 + n) \Gamma(1 + \alpha_2 + n)}{n!} \times \sum_{r=0}^{n} \frac{(-1)^r (1 + \alpha_1 + \beta_1 + n)_r (1-x)^r}{r! \Gamma(1 + \alpha_1 + r) \Gamma(1 + \alpha_2 + n - r)} P_{n-r}^{(\alpha_2, \beta_2 + r)}(y), \]

where \( P_n^{(\alpha, \beta)}(y) \) is the well-known Jacobi polynomial of one variable defined by (1.4.8).

In chapter II, some generating functions of Jacobi polynomials of two and three variables have been obtained. Also double and triple generating functions of Jacobi polynomials of one, two and three variables have been studied in chapter III and chapter IV respectively.

1.9 Generalized Rice Polynomials of Two Variables

In chapter V, we gave generating functions of generalized Rice polynomials of two variables. Further, double and triple generating functions of generalized Rice polynomials of one and two variables have been obtained in chapter VI.

The generalized Rice polynomials of two variables \( H_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(\xi_1, \xi_2, p_1, p_2, x, y) \) are defined by

\[ H_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(\xi_1, \xi_2, p_1, p_2, x, y) = \frac{(1 + \alpha_1)_r (1 + \alpha_2)_r}{(n!)^2} \times \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (1 + \alpha_1 + \beta_1 + n)_r (1 + \alpha_2 + \beta_2 + n)_s (\xi_1)_r (\xi_2)_s}{r! s! (1 + \alpha_1)_r (1 + \alpha_2)_s (p_1)_r (p_2)_s} x^r y^s. \]

The definition (1.9.1) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[ H_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(\xi_1, \xi_2, p_1, p_2, x, y) = \frac{(1 + \alpha_1)_n (1 + \alpha_2)_n}{(n!)^2} \times \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (1 + \alpha_1 + \beta_1 + n)_r (1 + \alpha_2 + \beta_2 + n)_s (\xi_1)_r (\xi_2)_s}{r! s! (1 + \alpha_1)_r (1 + \alpha_2)_s (p_1)_r (p_2)_s} x^r y^s. \]

\[ \text{1.9.2} \]
The definition (1.9.1) can also be represented as follows:

\[ H_n^{(\alpha_1; \alpha_2)}(\xi_1, \xi_2, p_1, p_2, x, y) = \frac{\Gamma(1 + \alpha_1 + n)\Gamma(1 + \alpha_2 + n)}{n!} \times \sum_{r=0}^{n} \frac{(-1)^r(1 + \alpha_1 + \beta_1 + n)_r(\xi_1)_r(\xi_2)_r}{r!(1 + \alpha_1 + r)\Gamma(1 + \alpha_2 + n - r)(\Gamma(p_1)_r)} H_{n-r}^{(\alpha_2, \beta_2+r)}(\xi_2, p_2, y), \] (1.9.3)

where \( H_n^{(\alpha_1, \beta)}(\xi_2, p_2, y) \) is the well-known generalized Rice polynomial of one variable defined by (1.4.18).

The relationships between generalized Rice polynomials of two variables and generalized Rice polynomials of one variable are as follows:

\[ H_n^{(\alpha_1, \alpha_2, \beta_2)}(\xi_1, \xi_2, p_1, p_2, x, 0) = H_n^{(\alpha_1, \beta_1)}(\xi_1, p_1, x) \] (1.9.4)

and

\[ H_n^{(\alpha_1, \alpha_2, \beta_2)}(\xi_1, \xi_2, p_1, p_2, 0, y) = H_n^{(\alpha_2, \beta_2)}(\xi_2, p_2, y). \] (1.9.5)

1.10 Generalized Bateman’s and Pasternak’s Polynomials of Two Variables

Generating functions of generalized Bateman’s and Pasternak’s polynomials of two variables have been studied in chapter VII.

The generalized Bateman’s polynomials of two variables \( Z_n^{(\alpha_1, \alpha_2; \beta_1, \beta_2)}(b_1, x; b_2, y) \) are defined by

\[ Z_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(b_1, x; b_2, y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_r(1 + \alpha_1 + \beta_1 + n)_r(1 + \alpha_2 + \beta_2 + n)_s x^r y^s}{r!(1 + \alpha_1)_r(1 + \beta_1)_r(1 + \alpha_2)_s(1 + \beta_2)_s}. \] (1.10.1)

The definition (1.10.1) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[ Z_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(b_1, x; b_2, y) = F_1^{1:1:1:1}_{0:2:2} \left[ \begin{array}{c} -n \colon 1 + \alpha_1 + \beta_1 + n; 1 + \alpha_2 + \beta_2 + n; \\ - \colon 1 + \alpha_1, 1 + \beta_1; 1 + \alpha_2, 1 + \beta_2; x, y \end{array} \right]. \] (1.10.2)

The definition (1.10.1) can also be represented as follows:

\[ Z_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(b_1, x; b_2, y) = \sum_{r=0}^{n} \frac{(-n)_r(1 + \alpha_1 + \beta_1 + n)_r x^r}{r!(1 + \alpha_1)_r(1 + \beta_1)_r} Z_{n-r}^{(\alpha_2, \beta_2+r)}(b_2, y), \] (1.10.3)
where \( Z_n^{(\alpha, \beta)}(b_2, y) \) is the well-known generalized Bateman’s polynomial of one variable defined by (1.5.4).

The relationships between generalized Bateman’s polynomials of two variables and generalized Bateman’s polynomials of one variable are as follows:

\[
Z_n^{(\alpha_1; \beta_1; \alpha_2, \beta_2)}(b_1, x; b_2, 0) = Z_n^{(\alpha_1, \beta_1)}(b_1, x) \tag{1.10.4}
\]

and

\[
Z_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(b_1, 0; b_2, y) = Z_n^{(\alpha_2, \beta_2)}(b_2, y). \tag{1.10.5}
\]

Another generalized Bateman’s polynomials of two variables \( F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(p_1, z_1; p_2, z_2) \) are defined by

\[
F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(p_1, z_1; p_2, z_2) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-n)^{r+s}(1 + \alpha_1 + \beta_1 + n)_r}{r!s!(1 + \alpha_1)_r(1 + \alpha_2)_s(p_1)_r(1 + \alpha_2)_s(p_2)_s} \frac{(1 + z_1)_r(1 + \beta_1 + n)_s}{(1 + z_2)_s}. \tag{1.10.6}
\]

The relation (1.10.6) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[
F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(p_1, z_1; p_2, z_2) = F_{1:2;2}^{0:2;2}\left[ -n : 1 + \alpha_1 + \beta_1 + n, \frac{1}{2}(1 + z_1); 1 + \alpha_2 + \beta_2 + n, \frac{1}{2}(1 + z_2); 1 + \alpha_1, p_1 ; 1 + \alpha_2, p_2 ; 1, 1 \right]. \tag{1.10.7}
\]

The definition (1.10.6) can also be represented as follows:

\[
F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(p_1, z_1; p_2, z_2) = \sum_{r=0}^{\infty} \frac{(-n)_r(1 + \alpha_1 + \beta_1 + n)_r(1 + z_1)_r}{r!(1 + \alpha_1)_r(p_1)_r} F_n^{(\alpha_2, \beta_2)_r}(p_2, z_2), \tag{1.10.8}
\]

where \( F_n^{(\alpha, \beta)}(p_2, z_2) \) is the well-known generalized Bateman’s polynomial of one variable defined by (1.5.6).

The relationships between generalized Bateman’s polynomials of two variables and generalized Bateman’s polynomials of one variable are as follows:

\[
F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(p_1, z_1; p_2, -1) = F_n^{(\alpha_1, \beta_1)}(p_1, z_1) \tag{1.10.9}
\]
and

\[
F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(p_1, -1; p_2, z_2) = F_n^{(\alpha_2, \beta_2)}(p_2, z_2).
\]  
(1.10.10)

The generalized Pasternak’s polynomials of two variables \(F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(z_1, z_2)\) are defined by

\[
F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(z_1, z_2) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} (-n)_{r+s} \frac{(1 + \alpha_1 + \beta_1 + n)_r (1 + \alpha_2 + \beta_2 + n)_s (1 + z_1 + m_1)_r (1 + z_2 + m_2)_s}{r! s! (1 + \alpha_1)_r (1 + m_1)_r (1 + \alpha_2)_s (1 + m_2)_s}.
\]  
(1.10.11)

The relation (1.10.11) can also be expressed in terms of double hypergeometric function defined by (1.3.32) as follows:

\[
F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(z_1, z_2) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} (-n)_{r+s} \frac{(1 + \alpha_1 + \beta_1 + n)_r (1 + \alpha_2 + \beta_2 + n)_s (1 + z_1 + m_1)_r (1 + z_2 + m_2)_s}{r! s! (1 + \alpha_1)_r (1 + m_1)_r (1 + \alpha_2)_s (1 + m_2)_s}.
\]  
(1.10.12)

The definition (1.10.11) can also be represented as follows:

\[
F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(z_1, z_2) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} (-n)_{r+s} \frac{(1 + \alpha_1 + \beta_1 + n)_r (1 + \alpha_2 + \beta_2 + n)_s (1 + z_1 + m_1)_r (1 + z_2 + m_2)_s}{r! s! (1 + \alpha_1)_r (1 + m_1)_r (1 + \alpha_2)_s (1 + m_2)_s}.
\]  
(1.10.13)

The relationships between generalized Pasternak’s polynomials of two variables and generalized Pasternak’s polynomials of one variable are as follows:

\[
F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(z_1, -1 - m_2) = F_n^{(\alpha_1, \beta_1)}(z_1)
\]  
(1.10.14)

and

\[
F_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(-1 - m_1, z_2) = F_n^{(\alpha_2, \beta_2)}(z_2).
\]  
(1.10.15)
1.11 Pseudo Jacobi Polynomials

Shivley (see, for example, [32]; see also [[196], p. 298, Eq. (1)]; [[34], p. 127, Eq. 47] and [[35], p. 1758, Eq. (3)]) has defined the polynomial $R_n(a, x)$ by the equation

$$R_n(a, x) = \frac{(a + n)_n}{n!} \, _{1}F_{1} \left( \begin{array}{c} -n; \\ a + n, \end{array} \right),$$

(1.11.1)

in which $n$ is any non-negative integer, and $a$ is independent of $n$.

The pseudo-Laguerre polynomial $R_n(a, x)$ may also be written as

$$R_n(a, x) = \frac{(a)_2^n}{n!(a)_n} \, _{1}F_{1} \left( \begin{array}{c} -n; \\ a + n, \end{array} \right),$$

(1.11.2)

which are related to the proper simple Laguerre polynomial

$$L_n(x) = \, _{1}F_{1} \left( \begin{array}{c} -n; \\ 1, \end{array} \right)$$

by

$$R_n(a, x) = \frac{1}{(a - 1)^n} \sum_{k=0}^{n} \frac{(a - 1)_n L_{n-k}(x)}{k!}.$$  

(1.11.3)

Toscano (See, for example, [196], p. 298, Eq. (3)) had already shown that

$$\sum_{n=0}^{\infty} R_n(a, x)t^n = (1 - 4t)^{-\frac{1}{2}} \left( \frac{2}{1 + \sqrt{1 - 4t}} \right)^{a-1} \exp \left( \frac{-4xt}{(1 + \sqrt{1 - 4t})^2} \right).$$  

(1.11.4)

Shively obtained Toscano’s other generating relation

$$\sum_{n=0}^{\infty} \frac{R_n(a, x)t^n}{\left( \frac{1}{2} + \frac{1}{2}a \right)_n} = e^{2t} \, _{0}F_{1} \left[ \begin{array}{c} \frac{1}{2}; \\ \frac{1}{2} + \frac{1}{2}a, \end{array} \right] t^2 - xt$$

(1.11.5)

and extended Toscano’s (1.11.4) to

$$\sum_{n=0}^{\infty} S_n(x)t^n = (1 - 4t)^{-\frac{1}{2}} \left( \frac{2}{1 + \sqrt{1 - 4t}} \right)^{a-1} pFq \left[ \begin{array}{c} \alpha_1, ..., \alpha_p; \\ \beta_1, ..., \beta_q, \end{array} \right] \left( \frac{-4xt}{(1 + \sqrt{1 - 4t})^2} \right),$$

(1.11.6)

in which

$$S_n(x) = \frac{(a)_2^n}{n!(a)_n} \, _{p+1}F_{q+1} \left( \begin{array}{c} -n, \alpha_1, ..., \alpha_p; \\ a + n, \beta_1, ..., \beta_q, \end{array} \right).$$

(1.11.7)
For the particular choice \( p = 0, q = 1, \beta_1 = 1, a = 1 \), the \( S_n(x) \) becomes
\[
\sigma_n(x) = \frac{(2n)!}{(n!)^2} \ _1F_2 \left[ \begin{array}{c} -n; \\ 1 + n, 1; \end{array} x \right],
\]
for which Shively has the additional generating relation
\[
\sum_{n=0}^{\infty} \frac{\sigma_n(x)t^n}{(2n)!} = \ _0F_1 \left[ \begin{array}{c} -; \\ t - \sqrt{4xt + t^2} \end{array} \right] \ _0F_1 \left[ \begin{array}{c} -; \\ t + \sqrt{4xt + t^2} \end{array} \right].
\]

The \( R_n(a, x) \) of (1.11.1) is of Sheffer A-type zero, as pointed out by Shively. He obtains many other properties of \( R_n(a, x) \).

Motivated by the work of Shively [196], pseudo-Jacobi polynomials have been defined and studied in chapter VIII.

The pseudo-Jacobi polynomials denoted by \( A_n(a, b, x) \) are defined as:
\[
A_n(a, b, x) = \frac{(a)_{2n}}{n!(a)_n} \ _2F_1 \left[ \begin{array}{c} -n, 1 + a + b + n; \\ a + n; \end{array} \frac{1 - x}{2} \right].
\]

From (1.11.10) it follows that \( A_n(a, b, x) \) is a polynomial of degree \( n \) and that
\[
A_n(a, b, 1) = \frac{(a)_{2n}}{n!(a)_n}.
\]

Applying Euler’s transformation (1.2.26) to (1.11.10) yields
\[
A_n(a, b, x) = \frac{(a)_{2n}}{n!(a)_n} \left( \frac{x + 1}{2} \right)^n \ _2F_1 \left[ \begin{array}{c} -n, -1 - b; \\ a + n; \end{array} \frac{x - 1}{x + 1} \right].
\]

Both (1.11.10) and (1.11.12) yield a finite series form for \( A_n(a, b, x) \):
\[
A_n(a, b, x) = \sum_{k=0}^{n} \frac{(a)_{2n}(1 + a + b)_{n+k}}{k!(n - k)!(a)_{n+k}(1 + a + b)_n} \left( \frac{x - 1}{2} \right)^k
\]
and
\[
A_n(a, b, x) = \sum_{k=0}^{n} \frac{(a)_{2n}(-1 - b)_k}{k!(n - k)!(a)_{n+k}} \left( \frac{1 - x}{2} \right)^k \left( \frac{x + 1}{2} \right)^{n-k}.
\]

Equations (1.11.13) and (1.11.14) are expanded forms of (1.11.10) and (1.11.12), respectively.
By reversing the order of summation in (1.11.13) and (1.11.14), one can get respectively

\[ A_n(a, b, x) = \frac{(1 + a + b)^{2n}}{n!(1 + a + b)^n} \left( \frac{x - 1}{2} \right)^n 2F_1 \left[ \begin{array}{c} -n, 1 - a - 2n; 2 \\ -a - b - 2n; 1 - x \end{array} \right] \]  

(1.11.15)

and

\[ A_n(a, b, x) = \frac{(-1 - b)^n}{n!} \left( \frac{1 - x}{2} \right)^n 2F_1 \left[ \begin{array}{c} -n, 1 - a - 2n; x + 1 \\ 2 + b - n; x - 1 \end{array} \right]. \]  

(1.11.16)

1.12 Pseudo Two variables Jacobi Polynomials

In chapter IX, we studied pseudo two variables Jacobi polynomials which have been defined on the pattern of Hermite polynomials of two variables due to M. A. Khan and G. S. Abukhammash [73].

The pseudo two variables Jacobi polynomials \( P_n^{(\alpha, \beta)}(x, y) \) are defined as:

\[ P_n^{(\alpha, \beta)}(x, y) = \frac{(1 + \alpha)^n}{n!} 2F_1 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n; \sqrt{y + 1} - x \\ 1 + \alpha; \frac{\sqrt{y + 1}}{2} \end{array} \right]. \]  

(1.12.1)

It may be noted that for \( y = 0 \), pseudo two variables Jacobi polynomials \( P_n^{(\alpha, \beta)}(x, y) \) are reduced to Jacobi polynomials of one variable \( P_n^{(\alpha, \beta)}(x) \). Thus

\[ P_n^{(\alpha, \beta)}(x, 0) = P_n^{(\alpha, \beta)}(x). \]  

(1.12.2)

Further,

\[ P_n^{(\alpha, \beta)}(1, 0) = \frac{(1 + \alpha)^n}{n!} \]  

(1.12.3)

and

\[ P_n^{(\alpha, \beta)}(0, y) = \frac{(1 + \alpha)^n}{n!} 2F_1 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n; 1 \\ 1 + \alpha; \frac{1}{2} \end{array} \right]. \]  

(1.12.4)

Applying Euler’s transformation (1.2.26) to (1.12.1) yields

\[ P_n^{(\alpha, \beta)}(x, y) = \frac{(1 + \alpha)^n}{n!} \left( \frac{\sqrt{y + 1} + x}{2\sqrt{y + 1}} \right)^n 2F_1 \left[ \begin{array}{c} -n, -\beta - n; x - \frac{\sqrt{y + 1}}{x + \sqrt{y + 1}} \\ 1 + \alpha; \frac{\sqrt{y + 1}}{x + \sqrt{y + 1}} \end{array} \right]. \]  

(1.12.5)
Another $2F_1$ form for $P_n^{(\alpha, \beta)}(x, y)$ have been obtained as follows:

$$P_n^{(\alpha, \beta)}(x, y) = \frac{(-1)^n(1 + \beta)_n}{n!} 2F_1 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n; \\ 1 + \beta; \end{array} \frac{\sqrt{y + 1} + x}{2\sqrt{y + 1}} \right]. \quad (1.12.6)$$

Each of (1.12.1), (1.12.5) and (1.12.6) yield a finite series form for $P_n^{(\alpha, \beta)}(x, y)$ as given below:

$$P_n^{(\alpha, \beta)}(x, y) = \sum_{k=0}^{n} \frac{(1 + \alpha)_n(1 + \beta)_{n+k}}{k!(n-k)!((1 + \alpha)_k(1 + \alpha + \beta)_n} \left( \frac{x - \sqrt{y + 1}}{2\sqrt{y + 1}} \right)^k \left( \frac{x + \sqrt{y + 1}}{2\sqrt{y + 1}} \right)^{n-k} \quad (1.12.7)$$

and

$$P_n^{(\alpha, \beta)}(x, y) = \sum_{k=0}^{n} \frac{(-1)^{n-k}(1 + \beta)_n(1 + \alpha + \beta)_{n+k}}{k!(n-k)!((1 + \beta)_k(1 + \alpha + \beta)_n} \left( \frac{x + \sqrt{y + 1}}{2\sqrt{y + 1}} \right)^k \left( \frac{x - \sqrt{y + 1}}{2\sqrt{y + 1}} \right)^{n-k} \quad (1.12.8)$$

By reversing the order of summation in (1.12.7), (1.12.8) and (1.12.9) respectively, one can obtain

$$P_n^{(\alpha, \beta)}(x, y) = \frac{(1 + \alpha + \beta)_{2n}}{n!(1 + \alpha + \beta)_n} \left( \frac{x - \sqrt{y + 1}}{2\sqrt{y + 1}} \right)^n 2F_1 \left[ \begin{array}{c} -n, -\alpha - n; \\ -\alpha - 2n; \frac{2\sqrt{y + 1}}{\sqrt{y + 1} - x} \end{array} \right], \quad (1.12.10)$$

$$P_n^{(\alpha, \beta)}(x, y) = \frac{(1 + \beta)_n}{n!} \left( \frac{x - \sqrt{y + 1}}{2\sqrt{y + 1}} \right)^n 2F_1 \left[ \begin{array}{c} -n, -\alpha - n; \\ -\alpha + \beta - 2n; \frac{2\sqrt{y + 1}}{\sqrt{y + 1} + x} \end{array} \right], \quad (1.12.11)$$

and

$$P_n^{(\alpha, \beta)}(x, y) = \frac{(1 + \alpha + \beta)_{2n}}{n!(1 + \alpha + \beta)_n} \left( \frac{x + \sqrt{y + 1}}{2\sqrt{y + 1}} \right)^n 2F_1 \left[ \begin{array}{c} -n, -\beta - n; \\ -\alpha - \beta - 2n; \frac{2\sqrt{y + 1}}{x + \sqrt{y + 1}} \end{array} \right]. \quad (1.12.12)$$

### 1.13 Integral Representations of The Product of Certain Polynomials of Two Variables

In 1938, Watson [232] gave the integral representation of the product $L_m^{(\alpha)}(x)L_n^{(\beta)}(y)$, which was generalized by L. Carlitz [13] in the form
\[ L_m^{(\alpha)}(x)L_n^{(\beta)}(y) = \frac{2^{\alpha+\beta+m+n}}{\pi^2} \frac{\Gamma(\alpha + m + 1)\Gamma(\beta + n + 1)}{\Gamma(\alpha + \beta + m + n + 1)} \]
\[ \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\phi i + (\alpha-\beta)\theta i} \cos^{m+n} \phi \cos^{\alpha+\beta} \theta \]
\[ \times L_{m+n}^{(\alpha+\beta)} \left( \frac{x e^{(\theta-\phi)i} + ye^{-(\theta-\phi)i}}{\cos \phi} \cos \theta \right) d\phi d\theta, \quad (1.13.1) \]

where \( L_n^{(\alpha)}(x) \) denotes the generalized Laguerre polynomials defined by (1.4.6).

Following the method adopted by Carlitz [13], in 1963, Chatterjea [15] gave the integral representation of the product of two generalized Bessel polynomials of one variable defined by (1.4.20). Further, in 1964, Chatterjea [16] gave the integral formula of the product of two Jacobi polynomials of one variable defined by (1.4.8). Also, in 1969, H. L. Manocha, [187] gave the integral representation of the product of two generalized Rice polynomials of one variable defined by (1.4.18).

In 1976, H. M. Srivastava and R. Panda [224] derived an integral representation of the product of two Jacobi polynomials of one variable and also gave some generalization involving Kampé de Fériet’s double hypergeometric functions. Also in 2004, Shy-Der Lin, Yi Shan Chao and H. M. Srivastava [183] investigated several families of hypergeometric polynomials of one variable and their associated single-, double-, and triple- integral representations. They also gave several special cases of their defined polynomials involving classical orthogonal polynomials, such as Laguerre, Jacobi, Hermite and Bessel polynomials of one variable, and various other related polynomials of one variable.

Recently in 2011, M. A. Khan, A. H. Khan and Manoj Singh [138] gave the integral representations of similar product of several other polynomials of one variable, e.g. Meixner, Krawtchouk, Gottlieb and Poisson-Charlier polynomials of one variable.

Motivated by the above mentioned works the last chapter deals with similar works using two variables polynomials. Besides, some new results have also been investigated. For this the chapter is divided into three sections.