CHAPTER 3
SEPARATION AXIOMS

The concept of fuzzy normal spaces was introduced by Bruce Hutton [18]. Tomasz Kubiak [66] established many interesting properties of fuzzy normal spaces. The concept of fuzzy regular spaces was introduced by Hutton and Reilly [32]. Separation axioms such as fuzzy pairwise $T_j$ ($j = 0, 1, 2, 3, 4$) spaces were studied by Ramadan, Abbas and Abd El-Latif [52]. In this chapter, fuzzy $\tilde{g} \cdot T_i$ ($i = 0, 1, 2, \frac{1}{2}$) spaces, fuzzy $\tilde{g}$-normality and fuzzy $\tilde{g}$-regularity are introduced in the sense of Sostak [60-62] and Ramadan [50-52]. Also, many interesting characterizations are established.
3.1 FUZZY $\tilde{g}$-$T_i$ SPACES

In this section, the concept of fuzzy $\tilde{g}$-$T_i$ ($i = 0, 1, 2, \frac{1}{2}$) spaces is introduced. Interesting properties and characterizations of such spaces are discussed.

**Definition 3.1.1** Let $(X, T)$ be a smooth fuzzy topological space. For any $\lambda, \mu \in I^X$ and $r \in I_0$, $\lambda$ is called

(1) $r$-fuzzy $\hat{g}$-closed if $C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\mu$ is $r$-fuzzy semi-open. The complement of an $r$-fuzzy $\hat{g}$-closed set is said to be an $r$-fuzzy $\hat{g}$-open set.

(2) $r$-fuzzy $*g$-closed if $C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\mu$ is $r$-fuzzy $\hat{g}$-open. The complement of an $r$-fuzzy $*g$-closed set is said to be an $r$-fuzzy $*g$-open set.

(3) $r$-fuzzy $^g$-semi-closed (briefly, $r$-$^fgs$-closed) if $SC_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\mu$ is $r$-fuzzy $^g$-open. The complement of an $r$-fuzzy $^g$-semi-closed set is said to be an $r$-fuzzy $^g$-semi-open set (briefly, $r$-$^fgs$-open set).

(4) $r$-fuzzy $\check{g}$-closed if $C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\mu$ is $r$-$^fgs$-open. The complement of an $r$-fuzzy $\check{g}$-closed set is said to be an $r$-fuzzy $\check{g}$-open set.

**Definition 3.1.2** Let $(X, T)$ be a smooth fuzzy topological space. For any $\lambda \in I^X$ and $r \in I_0$, 

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(1) \( \tilde{g}^{-1}_T ( \lambda, r ) = \lor \{ \mu \in I^X : \mu \leq \lambda, \mu \) is an r-fuzzy \( \tilde{g} \)-open set } is called the r-fuzzy \( \tilde{g} \)-interior of \( \lambda \).

(2) \( \tilde{g}^{-1}_C ( \lambda, r ) = \land \{ \mu \in I^X : \mu \geq \lambda, \mu \) is an r-fuzzy \( \tilde{g} \)-closed set } is called the r-fuzzy \( \tilde{g} \)-closure of \( \lambda \).

**Definition 3.1.3** Let \(( X, T )\) be a smooth fuzzy topological space. For any \( \lambda \in I^X \) and \( r \in I_0 \), \( \lambda \) is called r-generalized fuzzy \( \tilde{g} \)-closed ( briefly, \( r\)-gf\( \tilde{g} \)-closed ) iff \( \tilde{g}^{-1}_C ( \lambda, r ) \leq \mu \) whenever \( \lambda \leq \mu \), \( \mu \in I^X \) is r-fuzzy \( \tilde{g} \)-open. The complement of an r-generalized fuzzy \( \tilde{g} \)-closed set is r-generalized fuzzy \( \tilde{g} \)-open ( briefly, r-gf\( \tilde{g} \)-open ).

**Definition 3.1.4** Let \(( X, T )\) and \(( Y, S )\) be any two smooth fuzzy topological spaces. For \( r \in I_0 \), let \( f : ( X, T ) \to ( Y, S ) \) be a function.

(1) \( f \) is called fuzzy \( \tilde{g} \)-open ( resp. fuzzy \( \tilde{g} \)-closed ) if for each r-fuzzy \( \tilde{g} \)-open set \( \lambda \in I^X \), \( f ( \lambda ) \in I^Y \) is r-fuzzy \( \tilde{g} \)-open ( resp. r-fuzzy \( \tilde{g} \)-closed ).

(2) \( f \) is called fuzzy \( \tilde{g} \)-continuous if for each \( \lambda \in I^Y \) with \( S ( \lambda ) \geq r \), \( f^\diamond ( \lambda ) \in I^X \) is r-fuzzy \( \tilde{g} \)-open.

(3) \( f \) is called fuzzy \( \tilde{g} \)-irresolute if for each r-fuzzy \( \tilde{g} \)-open set \( \lambda \in I^Y \), \( f^\diamond ( \lambda ) \in I^X \) is r-fuzzy \( \tilde{g} \)-open.

(4) \( f \) is called fuzzy \( \tilde{g} \)-homeomorphism if \( f \) is one to one, onto, fuzzy \( \tilde{g} \)-irresolute and fuzzy \( \tilde{g} \)-open.
(5) \( f \) is called \( gfg \)-irresolute if for each \( r-gfg \) closed set \( \lambda \in \mathbf{I}^\gamma \), \( f^{-1}(\lambda) \in \mathbf{I}^X \) is \( r-gfg \)-closed.

(6) \( f \) is called \( gfg \)-closed iff for any \( r-gfg \)-closed set \( \lambda \in \mathbf{I}^X \), \( f(\lambda) \in \mathbf{I}^\gamma \) is \( r-gfg \)-closed.

**Definition 3.1.5** For \( r \in \mathbf{I}_0 \), a smooth fuzzy topological space \( (\mathbf{X}, T) \) is called

1. fuzzy \( g \)-\( T_1 \) iff for \( \lambda, \mu \in \mathbf{I}^X \) with \( \lambda \not\subset \mu \), there exist r-fuzzy \( g \)-open sets \( \delta, \eta \in \mathbf{I}^X \) such that either \( \lambda \leq \delta \), \( \mu \not\subset \delta \) or \( \mu \leq \eta \), \( \lambda \not\subset \eta \).

2. fuzzy \( g \)-\( T_2 \) iff for \( \lambda, \mu \in \mathbf{I}^X \) with \( \lambda \not\subset \mu \), there exist r-fuzzy \( g \)-open sets \( \delta, \eta \in \mathbf{I}^X \) with \( \lambda \leq \delta \), \( \mu \leq \eta \) and \( \delta \not\subset \eta \).

**Definition 3.1.6** A smooth fuzzy topological space \( (\mathbf{X}, T) \) is called fuzzy \( g \)-\( T_{1/2} \) if every \( r-gfg \)-closed set is \( r \)-fuzzy \( g \)-closed, \( r \in \mathbf{I}_0 \).

**Proposition 3.1.1** Let \( (\mathbf{X}, T) \) be a smooth fuzzy topological space. For \( r \in \mathbf{I}_0 \), the following properties hold:

(a) For all \( r \)-fuzzy \( g \)-open set \( \lambda \in \mathbf{I}^X \), \( \lambda \not\subset \mu \) iff \( \lambda \not\subset \mathbf{C}_T(\mu, r) \), \( \mu \in \mathbf{I}^X \).

(b) \( \delta \not\subset \mathbf{C}_T(\lambda, r) \) iff \( \lambda \not\subset \mu \) for all \( r \)-fuzzy \( g \)-open sets \( \mu \in \mathbf{I}^X \) with \( \delta \leq \mu \), where \( \lambda, \delta \in \mathbf{I}^X \).

**Proof:** (a). Let \( \lambda \) be an \( r \)-fuzzy \( g \)-open set such that \( \lambda \not\subset \mu \). Since \( \mu > \mathbf{1} - \lambda \), \( \lambda \not\subset \mathbf{C}_T(\mu, r) \). Conversely, let \( \lambda \) be an \( r \)-fuzzy \( g \)-open set such that \( \lambda \not\subset \mu \). Then \( \mu \leq \mathbf{1} - \lambda \), this implies that
\[ \tilde{g} - C_T (\mu, r) \leq \tilde{g} - C_T (\bar{1} - \lambda, r) = \bar{1} - \lambda. \]

Now, \( \tilde{g} - C_T (\mu, r) \leq \bar{1} - \lambda. \) Thus, \( \lambda \not\Subset \tilde{g} - C_T (\mu, r) \) which is a contradiction. Hence the result.

(b). Let \( \delta \ q (\tilde{g} - C_T (\lambda, r)). \) Since \( \delta \leq \mu, \mu \ q (\tilde{g} - C_T (\lambda, r)). \) By (a), \( \mu \ q \lambda \) for all \( r \)-fuzzy \( \tilde{g} \)-open sets \( \mu \) with \( \delta \leq \mu. \) Conversely, suppose that \( \delta \not\Subset \tilde{g} - C_T (\lambda, r). \) Then, \( \delta \leq \bar{1} - (\tilde{g} - C_T (\lambda, r)). \)

Now, \( \mu = \bar{1} - (\tilde{g} - C_T (\mu, r)). \) Then \( \mu \) is an \( r \)-fuzzy \( \tilde{g} \)-open set.

Since \( \lambda \leq \tilde{g} - C_T (\lambda, r), \mu = \bar{1} - (\tilde{g} - C_T (\lambda, r)) \leq \bar{1} - \lambda, \) this implies that \( \lambda \not\Subset \mu, \) a contradiction. This proves the result.

**Proposition 3.1.2** Let \( (X, T) \) and \( (Y, S) \) be any two smooth fuzzy topological spaces. For \( r \in I_0, \) let \( f : (X, T) \rightarrow (Y, S) \) be a fuzzy \( \tilde{g} \)-irresolute, \( gf\tilde{g} \)-irresolute, and \( \tilde{g} \)-closed function. Then the following conditions hold:

(a) If \( f \) is injective and \( (Y, S) \) is a fuzzy \( \tilde{g} - T_{1/2} \) space, then \( (X, T) \) is a fuzzy \( \tilde{g} - T_{1/2} \) space.

(b) If \( f \) is surjective and \( (X, T) \) is a fuzzy \( \tilde{g} - T_{1/2} \) space, then \( (Y, S) \) is a fuzzy \( \tilde{g} - T_{1/2} \) space.

**Proof:** (a). Let \( \lambda \in I^X \) be an \( r \)-gf\(\tilde{g} \)-closed set. Since \( f \) is \( gf\tilde{g} \)-closed, \( f (\lambda) \in I^Y \) is \( r \)-gf\(\tilde{g} \)-closed. Since \( (Y, S) \) is fuzzy \( \tilde{g} - T_{1/2}, \) \( f (\lambda) \) is \( r \)-fuzzy \( \tilde{g} \)-closed. Now, \( \lambda = f^* (f (\lambda)) \) is \( r \)-fuzzy \( \tilde{g} \)-closed. Hence \( (X, T) \) is a fuzzy \( \tilde{g} - T_{1/2} \) space.

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(b). Let \( \mu \in I^Y \) be an \( r\)-\( gf\tilde{g}\)-closed set. Since \( f \) is \( gf\tilde{g}\)-irresolute, \( f^- (\mu) \in I^X \) is an \( r\)-\( gf\tilde{g}\)-closed set. Since \((X, T)\) is a fuzzy \( \tilde{g}\)-\( T_{1/2}\) space, \( f^- (\mu) \) is an \( r\)-fuzzy \( \tilde{g}\)-closed set. Therefore, \( \mu = f ( f^- (\mu)) \) is \( r\)-fuzzy \( \tilde{g}\)-closed. Hence \((Y, S)\) is a fuzzy \( \tilde{g}\)-\( T_{1/2}\) space.

**Proposition 3.1.3** Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. Let \( f : (X, T) \to (Y, S) \) be a fuzzy \( \tilde{g}\)-irresolute, and injective function. If \((Y, S)\) is fuzzy \( \tilde{g}\)-\( T_2\) (resp. fuzzy \( \tilde{g}\)-\( T_1\)), then \((X, T)\) is fuzzy \( \tilde{g}\)-\( T_2\) (resp. fuzzy \( \tilde{g}\)-\( T_1\)), \( r \in I_0\).

**Proof:** Let \( \lambda_1, \lambda_2 \in I^Y \) be such that \( \lambda_1 \notin \lambda_2 \). Since \( f \) is injective, \( f (\lambda_1) \notin f (\lambda_2) \). Since \((Y, S)\) is a fuzzy \( \tilde{g}\)-\( T_2\) space, there exist \( r\)-fuzzy \( \tilde{g}\)-open sets \( \lambda, \mu \in I^Y \) such that \( f (\lambda_1) \leq \lambda \) and \( f (\lambda_2) \leq \mu \) such that \( \lambda \notin \mu \). That is, \( \lambda \leq \bar{1} - \mu \) which implies that \( f^- (\lambda) \notin f^- (\mu) \).

Now, \( \lambda_1 \leq f^- (\lambda) \) and \( \lambda_2 \leq f^- (\mu) \). Since \( f \) is fuzzy \( \tilde{g}\)-irresolute, \( f^- (\lambda), f^- (\mu) \in I^X \) are \( r\)-fuzzy \( \tilde{g}\)-open sets. Hence \((X, T)\) is a fuzzy \( \tilde{g}\)-\( T_2\) space. Similarly we prove the case of fuzzy \( \tilde{g}\)-\( T_1\) space.

### 3.2 Fuzzy \( \tilde{g}\)-Normal Spaces and Its Characterizations

In this section, the concept of fuzzy \( \tilde{g}\)-normal spaces is introduced. Interesting properties and characterizations of such spaces are discussed.

**Definition 3.2.1** A smooth fuzzy topological space \((X, T)\) is said to be fuzzy \( \tilde{g}\)-normal if for every \( r\)-fuzzy \( \tilde{g}\)-closed set \( \lambda \in I^X \) and \( r\)-fuzzy
\( \tilde{g} \)-open set \( \mu \in \tilde{I}^X \) with \( \lambda \leq \mu \), there exists a \( \gamma \in \tilde{I}^X \) such that
\[ \lambda \leq \tilde{g} \cdot I_T (\gamma, r) \leq \tilde{g} \cdot C_T (\gamma, r) \leq \mu, \ r \in I_0. \]

**Proposition 3.2.1** For any smooth fuzzy topological space \( (X, T) \) and \( \lambda, \mu, \delta \in \tilde{I}^X, r \in I_0 \), the following statements are equivalent:

(a) \( (X, T) \) is fuzzy \( \tilde{g} \)-normal.

(b) For each \( r \)-fuzzy \( \tilde{g} \)-closed set \( \lambda \) and each \( r \)-fuzzy \( \tilde{g} \)-open set \( \mu \) with \( \lambda \leq \mu \), there exists an \( r \)-fuzzy \( \tilde{g} \)-open set \( \delta \) such that
\[ \tilde{g} \cdot C_T (\lambda, r) \leq \delta \leq \tilde{g} \cdot C_T (\delta, r) \leq \mu. \]

(c) For each \( r \)-gf\( \tilde{g} \)-closed set \( \lambda \) and \( r \)-fuzzy \( \tilde{g} \)-open set \( \mu \) with \( \lambda \leq \mu \), there exists an \( r \)-fuzzy \( \tilde{g} \)-open set \( \delta \) such that
\[ \tilde{g} \cdot C_T (\lambda, r) \leq \delta \leq \tilde{g} \cdot C_T (\delta, r) \leq \mu. \]

**Proof**: (a) \( \Rightarrow \) (b). The proof is trivial.

(b) \( \Rightarrow \) (c). Let \( \lambda \) be any \( r \)-gf\( \tilde{g} \)-closed set and \( \mu \) be any \( r \)-fuzzy \( \tilde{g} \)-open set such that \( \lambda \leq \mu \). Since \( \lambda \) is \( r \)-gf\( \tilde{g} \)-closed, \( \tilde{g} \cdot C_T (\lambda, r) \leq \mu \). Now, \( \tilde{g} \cdot C_T (\lambda, r) \) is \( r \)-fuzzy \( \tilde{g} \)-closed and \( \mu \) is \( r \)-fuzzy \( \tilde{g} \)-open. By (b), there exists an \( r \)-fuzzy \( \tilde{g} \)-open set \( \delta \) such that
\[ \tilde{g} \cdot C_T (\lambda, r) \leq \delta \leq \tilde{g} \cdot C_T (\delta, r) \leq \mu. \]

(c) \( \Rightarrow \) (a). The proof is trivial.

**Proposition 3.2.2** Let \( (X, T) \) and \( (Y, S) \) be any two smooth fuzzy topological spaces. If \( f : (X, T) \rightarrow (Y, S) \) is fuzzy \( \tilde{g} \)-homeomorphism and \( (Y, S) \) is fuzzy \( \tilde{g} \)-normal, then \( (X, T) \) is fuzzy \( \tilde{g} \)-normal.
Proof: Let $\lambda \in \mathcal{I}^X$ be any r-fuzzy $\tilde{g}$-closed set and $\mu \in \mathcal{I}^X$ be any r-fuzzy $\tilde{g}$-open set such that $\lambda \leq \mu$ where $r \in I_0$. Since $f$ is fuzzy $\tilde{g}$-homeomorphism, it is also fuzzy $\tilde{g}$-closed. Hence $f(\lambda) \in \mathcal{I}^Y$ is r-fuzzy $\tilde{g}$-closed. Since $f$ is fuzzy $\tilde{g}$-open, $f(\mu) \in \mathcal{I}^Y$ is r-fuzzy $\tilde{g}$-open.

Since $(Y, S)$ is fuzzy $\tilde{g}$-normal, there exists a $\gamma \in \mathcal{I}^Y$ such that

$$f(\gamma) \leq \tilde{g} - \mathcal{I}_T(\gamma, r) \leq \tilde{g} - \mathcal{C}_T(\gamma, r) \leq f(\mu).$$

Now,

$$f^{-1}(f(\lambda)) = \lambda \leq f^{-1}(\tilde{g} - \mathcal{I}_T(\gamma, r)) \leq f^{-1}(\tilde{g} - \mathcal{C}_T(\gamma, r)) \leq f^{-1}(f(\mu)) = \mu.$$  

That is, $\lambda \leq \tilde{g} - \mathcal{I}_T(f^{-1}(\gamma), r) \leq \tilde{g} - \mathcal{C}_T(f^{-1}(\gamma), r) \leq \mu$. Therefore $(X, T)$ is fuzzy $\tilde{g}$-normal.

Proposition 3.2.3 Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces. If $f: (X, T) \rightarrow (Y, S)$ is fuzzy $\tilde{g}$-homeomorphism and $(X, T)$ is a fuzzy $\tilde{g}$-normal space, then $(Y, S)$ is fuzzy $\tilde{g}$-normal.

Proof: Let $\lambda \in \mathcal{I}^Y$ be any r-fuzzy $\tilde{g}$-closed set and $\mu \in \mathcal{I}^Y$ be any r-fuzzy $\tilde{g}$-open set such that $\lambda \leq \mu$ where $r \in I_0$. Since $f$ is fuzzy $\tilde{g}$-irresolute, $f^{-1}(\lambda) \in \mathcal{I}^X$ is r-fuzzy $\tilde{g}$-closed and $f^{-1}(\mu) \in \mathcal{I}^X$ is r-fuzzy $\tilde{g}$-open.

Since $(X, T)$ is fuzzy $\tilde{g}$-normal, there exists a $\gamma \in \mathcal{I}^X$ such that

$$f^{-1}(\lambda) \leq \tilde{g} - \mathcal{I}_T(\gamma, r) \leq \tilde{g} - \mathcal{C}_T(\gamma, r) \leq f^{-1}(\mu).$$

Now,

$$f\left(f^{-1}(\lambda)\right) = \lambda \leq f\left(\tilde{g} - \mathcal{I}_T(\gamma, r)\right) \leq f\left(\tilde{g} - \mathcal{C}_T(\gamma, r)\right) \leq f\left(f^{-1}(\mu)\right) = \mu.$$  

That is, $\lambda \leq \tilde{g} - \mathcal{I}_T(f(\gamma), r) \leq \tilde{g} - \mathcal{C}_T(f(\gamma), r) \leq \mu$. Therefore $(Y, S)$ is fuzzy $\tilde{g}$-normal.
Proposition 3.2.4 Let \((X, T)\) be a fuzzy \(\tilde{g}\)-normal space. Let
\[
\{\lambda_i\}_{i \in J} \subseteq I_X^\mathbb{R} \quad \text{and} \quad \{\mu_j\}_{j \in J} \subseteq I_X^\mathbb{R}, \quad r \in I_0.
\]
If there exist \(\lambda, \mu \in I_X^\mathbb{R}\) such that
\[
\tilde{g} - C_T (\lambda_i, r) \leq \tilde{g} - C_T (\lambda, r) \leq \tilde{g} - I_T (\mu_j, r)
\]
and
\[
\tilde{g} - C_T (\lambda_i, r) \leq \tilde{g} - I_T (\mu, r) \leq \tilde{g} - I_T (\mu_j, r)
\]
for all \(i, j = 1, 2, \ldots\)
and \(r \in I_0\), then there exists a \(\gamma \in I_X^\mathbb{R}\) such that
\[
\tilde{g} - C_T (\lambda_i, r) \leq \tilde{g} - I_T (\gamma, r) \leq \tilde{g} - C_T (\gamma, r) \leq \tilde{g} - I_T (\mu_j, r)
\]
for all \(i, j = 1, 2, \ldots\)

Proof: First, we shall show by induction that for all \(n \geq 2\) there exists a collection \(\{\gamma_i, \delta_i / 1 \leq i \leq n\}\) contained in \(I_X^\mathbb{R}\) such that the conditions
\[
\begin{align*}
\tilde{g} - C_T (\lambda_i, r) &\leq \tilde{g} - I_T (\gamma_i, r); \\
\tilde{g} - C_T (\delta_i, r) &\leq \tilde{g} - I_T (\mu_j, r); \\
\tilde{g} - C_T (\lambda, r) &\leq \tilde{g} - I_T (\delta_j, r); \\
\tilde{g} - C_T (\gamma_i, r) &\leq \tilde{g} - I_T (\mu, r); \\
\tilde{g} - C_T (\gamma_i, r) &\leq \tilde{g} - I_T (\delta_j, r),
\end{align*}
\]
hold for all \(i, j = 1, 2, \ldots, n-1\). Clearly \((S_2)\) follows at once from the fuzzy \(\tilde{g}\)-normality of \((X, T)\). Now, suppose that for \(n \geq 2\), \(\gamma_i, \delta_i \in I_X^\mathbb{R}\) \((i < n)\) such that \((S_n)\) holds.

Since, \(\tilde{g} - C_T (\lambda_n, r) \leq \tilde{g} - C_T (\lambda, r) \leq \tilde{g} - I_T (\delta_j, r)\) \((j < n)\) and \(\tilde{g} - C_T (\lambda_n, r) \leq \tilde{g} - I_T (\mu, r)\) by fuzzy \(\tilde{g}\)-normality of \((X, T)\), there exists a \(\gamma_n \in I_X^\mathbb{R}\) such that
\[
\tilde{g} - C_T (\lambda_n, r) \leq \tilde{g} - I_T (\gamma_n, r) \leq \tilde{g} - C_T (\gamma_n, r) \leq \tilde{g} - I_T (\{ j < \delta \land \mu \}).
\]
Similarly, Since \(\tilde{g} - C_T (\lambda, r) \leq \tilde{g} - I_T (\mu_n, r)\) and

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\[ \dot{g} - C_T (\gamma_i, r) \leq \dot{g} - I_T (\mu_n, r) \quad (i \leq n), \]

there exists a \( \delta_n \in I^X \) such that

\[
\bigvee_{i \leq n} \gamma_i \bigvee \lambda \leq \dot{g} - I_T (\delta_n, r) \leq \dot{g} - C_T (\delta_n, r) \leq \dot{g} - I_T (\mu_n, r). \]

Thus \( S_{n+1} \) holds.

Proposition 3.2.5 Let \((X, T)\) be a smooth fuzzy topological space which is also a fuzzy \( \dot{g} \)-normal space. If \( \{q_j\}_{j \in Q} \) and \( \{q'_j\}_{j \in Q} \) are monotone increasing collections of respectively, fuzzy \( \dot{g} \)-closed and fuzzy \( \dot{g} \)-open subsets of \((X, T)\) \((Q \text{ is the set of all rational numbers})\) such that \( q_j \leq \mu_s \) whenever \( q < s \), then there exists a collection \( \{q'_j\}_{j \in Q} \in I^X \) such that \( \lambda_{q_j} \leq \dot{g} - I_T (\gamma_s, r) \), \( \dot{g} - C_T (\gamma_{q'_j}, r) \leq \dot{g} - I_T (\gamma_s, r) \) and \( \dot{g} - C_T (\gamma_{q'_j}, r) \leq \mu_s \) whenever \( q < s \), \( r \in I_0 \).

Proof: Let us arrange into a sequence \( \{q_n\} \) of all rational numbers \((\text{without repetitions})\). For every \( n \geq 2 \) we shall define inductively a collection \( \{\gamma_i / 1 \leq i \leq n\} \in I^X \) such that

\[
\lambda_q \leq \dot{g} - I_T (\gamma_i, r) \quad \text{if } q < q_i; \]

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\[ \tilde{g} \cdot C_T(\gamma_i, r) \leq \mu_q \quad \text{if } q_i < q; \quad (S_n) \]

\[ \tilde{g} \cdot C_T(\gamma_i, r) \leq \tilde{g} \cdot I_T(\gamma_j, r) \quad \text{if } q_i < q_j, \]

for all \( 1 \leq i, j < n \). It is clear that the countable collections \( \{ \lambda_q / q < q_1 \} \) and \( \{ \mu_q / q > q_1 \} \) together with \( \lambda_\sim \) and \( \mu_\sim \) satisfy all hypotheses of Proposition 3.2.4, so that there exists \( \delta_1 \in I^X \) such that

\[ \lambda_q \leq \tilde{g} \cdot I_T(\delta_1, r) \quad \text{for all } q < q_1 \quad \text{and} \]

\[ \tilde{g} \cdot C_T(\delta_1, r) \leq \mu_q \quad \text{for all } q > q_1. \]

Letting \( \gamma_\sim = \delta_1 \), we get \((S_2)\). Assume that the fuzzy subsets \( \gamma_\sim \) are already defined for \( i < n \) and satisfy \((S_n)\).

Define \( \lambda = \vee \{ \gamma_i / i < n, q_i < q_n \} \vee \lambda_\sim \) and \( \mu = \wedge \{ \gamma_j / j < n, q_j < q_n \} \wedge \mu_\sim \).

Then, \( \tilde{g} \cdot C_T(\gamma_i, r) \leq \tilde{g} \cdot C_T(\lambda, r) \leq \tilde{g} \cdot I_T(\gamma_j, r) \) and

\[ \tilde{g} \cdot C_T(\gamma_i, r) \leq \tilde{g} \cdot I_T(\mu, r) \leq \tilde{g} \cdot I_T(\gamma_j, r) \quad \text{whenever } q_i < q_n < q_j \]

(\( i, j < n \)) as well as

\[ \lambda_q \leq \tilde{g} \cdot C_T(\lambda, r) \leq \mu_s \quad \text{and} \]

\[ \lambda_q \leq \tilde{g} \cdot I_T(\mu, r) \leq \mu_s \quad \text{whenever } q < q_n < s. \]

This shows that the countable collections

\[ \{ \gamma_i / i < n, q_i < q_n \} \vee \{ \lambda_q / q < q_n \} \quad \text{and} \]

\[ \{ \gamma_j / j < n, q_j > q_n \} \vee \{ \mu_q / q > q_n \} \]

together with \( \lambda \) and \( \mu \) satisfy all hypotheses of Proposition 3.2.4.

Hence there exists a \( \delta_n \in I^X \) such that

\[ \lambda_q \leq \tilde{g} \cdot I_T(\delta_n, r) \quad \text{if } q < q_n, \]

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\[ \mathcal{g}\mathcal{-C}_\mathcal{T}(\gamma_{\gamma'}, r) \leq \mathcal{g}\mathcal{-I}_\mathcal{T}(\delta_n, r) \text{ if } q_i < q_n, \]
\[ \mathcal{g}\mathcal{-C}_\mathcal{T}(\delta_n, r) \leq \mu_q \text{ if } q_n < q, \]
\[ \mathcal{g}\mathcal{-C}_\mathcal{T}(\delta_n, r) \leq \mathcal{g}\mathcal{-I}_\mathcal{T}(\gamma_{\gamma'}, r) \text{ if } q_n < q_j \text{ where } 1 \leq i, j \leq n - 1. \]

Letting \( \gamma_{\gamma'} = \delta_n \), we obtain fuzzy subsets \( \gamma_{\gamma'}, \gamma_{\gamma'}, \ldots, \gamma_{\gamma'} \) that satisfy the result (\( S_{n+1} \)). Therefore the collection \{ \gamma_{\gamma'} \mid i = 1, 2, \ldots \} has the required properties. This completes the proof.

### 3.3 Fuzzy \( \mathcal{g}\mathcal{-}\text{Regular Space}\) and its Characterizations

In this section, the concept of fuzzy \( \mathcal{g}\mathcal{-}\)regular spaces is introduced. Some interesting characterizations are established.

**Definition 3.3.1** A smooth fuzzy topological space \((X, T)\) is called a fuzzy \( \mathcal{g}\mathcal{-}\)regular space if for every \( r\)-fuzzy \( \mathcal{g}\mathcal{-}\)closed set \( \lambda \) and each \( \alpha \in \mathcal{I}^X \) with \( \alpha \not\approx \lambda \), there exist \( \mu, \delta \in \mathcal{I}^X \) with \( T(\mu) \geq r, T(\delta) \geq r \) and \( \delta \not\approx \mu \) such that \( \alpha \leq \delta, \lambda \leq \mu, r \in \mathcal{I}_0 \).

**Proposition 3.3.1** Let \((X, T)\) be a smooth fuzzy topological space. Then the following statements are equivalent:

(a) \((X, T)\) is fuzzy \( \mathcal{g}\mathcal{-}\)regular.

(b) For each \( \alpha \in \mathcal{I}^X \) and \( r\)-fuzzy \( \mathcal{g}\mathcal{-}\)open set \( \lambda \in \mathcal{I}^X \) with \( \alpha \not\approx \lambda \), there exists a \( \delta \in \mathcal{I}^X \) with \( T(\delta) \geq r, \alpha \leq \delta \) such that \( C_T(\delta, r) \leq \lambda, r \in \mathcal{I}_0 \).

**Proof:** (a) \( \Rightarrow \) (b). Let \( \lambda \) be any \( r\)-fuzzy \( \mathcal{g}\mathcal{-}\)open set with \( \alpha \not\approx \lambda \). By hypothesis, there exist \( \mu, \delta \in \mathcal{I}^X \) with \( T(\mu) \geq r, T(\delta) \geq r \) and \( \delta \not\approx \mu \) such that \( \overline{1 - \lambda} \leq \mu \) and \( \alpha \leq \delta \).
Since $\delta \not\in \mu$ implies $\delta \leq 1 - \mu$, $C_T(\delta, r) \leq C_T(\bar{1} - \mu, r) = \bar{1} - \mu$.

But, $1 - \lambda \leq \mu$ gives $1 - \mu \leq \lambda$. That is, $C_T(\delta, r) \leq \lambda$. Hence the result.

**(b) ⇒ (a).** Let $\gamma$ be any r-fuzzy $\tilde{g}$-closed set with $\alpha \not\in \gamma$ for any $\alpha \in I^X$.

Now, $1 - \gamma$ is r-fuzzy $\tilde{g}$-open. By hypothesis, there exists a $\delta \in I^X$ with $T(\delta) \geq r$, $\alpha \leq \delta$ such that $C_T(\delta, r) \leq 1 - \gamma$. Then, $\gamma \leq 1 - C_T(\delta, r)$.

Now, $\delta \leq 1 - (1 - C_T(\delta, r))$ such that $\alpha \leq \delta$ and $\gamma \leq 1 - C_T(\delta, r)$. Therefore $(X, T)$ is fuzzy $\tilde{g}$-regular.

**Proposition 3.3.2** Let $(X, T)$ be a smooth fuzzy topological space.

Then $(X, T)$ is fuzzy $\tilde{g}$-regular iff for every r-fuzzy $\tilde{g}$-closed set $\lambda \in I^X$ and $\alpha \in I^X$ with $\alpha \not\in \lambda$, there exist $\mu$, $\delta \in I^X$ with $T(\mu) \geq r$, $T(\delta) \geq r$ such that $\alpha \leq \delta$, $\lambda \leq \mu$ then $\mu \not\in C_T(\delta, r)$, $r \in I_0$.

**Proof:** Let $(X, T)$ be a fuzzy $\tilde{g}$-regular space. Let $\lambda \in I^X$ be any r-fuzzy $\tilde{g}$-closed set and $\alpha$ be such that $\alpha \not\in \lambda$. Since $(X, T)$ is fuzzy $\tilde{g}$-regular, there exist $\mu$, $\delta$ with $T(\mu) \geq r$, $T(\delta) \geq r$, $\delta \not\in \mu$ such that $\alpha \leq \delta$, $\lambda \leq \mu$.

Now, $\delta \not\in \mu$ implies that $C_T(\delta, r) \leq C_T(\bar{1} - \mu, r) = \bar{1} - \mu$. That is, $\mu \not\in C_T(\delta, r)$. Hence the result. Converse part is trivial.

**Proposition 3.3.3** Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces. If $f : (X, T) \rightarrow (Y, S)$ is bijective, fuzzy $\tilde{g}$-irresolute, fuzzy open and if $(X, T)$ is a fuzzy $\tilde{g}$-regular space, then $(Y, S)$ is fuzzy $\tilde{g}$-regular.

**Proof:** Let $\lambda \in I^Y$ be any r-fuzzy $\tilde{g}$-closed set and $\beta \in I^Y$ be such that $\beta \not\in \lambda$, $r \in I_0$. Since $f$ is fuzzy $\tilde{g}$-irresolute, $f^-(\lambda) \in I^X$ is r-fuzzy
\( \tilde{g} \)-closed. Let \( f(\alpha) = \beta \) for any \( \alpha \in \mathcal{I}^X \). Since \( f \) is bijective, \( \alpha = f^{-1}(\beta) \).

Since \( (X, T) \) is fuzzy \( \tilde{g} \)-regular and \( \alpha \notin f^{-1}(\lambda) \) there exist \( \mu, \delta \in \mathcal{I}^Y \) with \( T(\mu) \geq r, T(\delta) \geq r \) and \( \delta \nsubseteq \mu \) such that \( \alpha \leq \delta \) and \( f^{-1}(\lambda) \leq \mu \).

Since \( f \) is fuzzy open and bijective, \( f(\alpha) \leq f(\delta) \) implies that \( \beta \leq f(\delta) \), \( \lambda \leq f(\mu) \) and \( S(f(\delta)) \geq r \), \( S(f(\mu)) \geq r \) with \( f(\delta) \nsubseteq f(\mu) \). Hence \( (Y, S) \) is fuzzy \( \tilde{g} \)-regular.

**Proposition 3.3.4** Let \( (X, T) \) and \( (Y, S) \) be any two smooth fuzzy topological spaces. If \( f : (X, T) \to (Y, S) \) is fuzzy \( \tilde{g} \)-closed, fuzzy continuous, injective and \( (Y, S) \) is fuzzy \( \tilde{g} \)-regular then \( (X, T) \) is fuzzy \( \tilde{g} \)-regular.

**Proof:** Let \( \lambda \in \mathcal{I}^X \) be any \( r \)-fuzzy \( \tilde{g} \)-closed set and \( \alpha \in \mathcal{I}^X \) be such that \( \alpha \notin \lambda, r \in I_0 \). Since \( f \) is fuzzy \( \tilde{g} \)-closed, \( f(\lambda) \in \mathcal{I}^Y \) is \( r \)-fuzzy \( \tilde{g} \)-closed and \( f(\alpha) \notin f(\lambda) \). Since \( (Y, S) \) is fuzzy \( \tilde{g} \)-regular, there exist \( \mu, \delta \in \mathcal{I}^Y \) with \( S(\mu) \geq r, S(\delta) \geq r \) and \( \delta \nsubseteq \mu \) such that \( f(\alpha) \leq \mu \) and \( f(\lambda) \leq \delta \).

Since \( f \) is fuzzy continuous, \( f^{-1}(\mu) \), \( f^{-1}(\delta) \in \mathcal{I}^X \) with \( T(f^{-1}(\mu)) \geq r \) and \( T(f^{-1}(\delta)) \geq r \). Also, \( \alpha \leq f^{-1}(\mu) \), \( \lambda \leq f^{-1}(\delta) \) and \( f^{-1}(\delta) \nsubseteq f^{-1}(\mu) \).

Therefore \( (X, T) \) is fuzzy \( \tilde{g} \)-regular.

**Proposition 3.3.5** Let \( (X, T) \) be a smooth fuzzy topological space. Then the following statements are equivalent:

(a) \( (X, T) \) is fuzzy \( \tilde{g} \)-regular.

(b) For every \( r \)-fuzzy \( \tilde{g} \)-open set \( \lambda \in \mathcal{I}^X \) and \( \alpha \in \mathcal{I}^X \) such that \( \alpha \leq \lambda \) there exists a \( \gamma \in \mathcal{I}^X \) with \( T(\gamma) \geq r \) such that \( \alpha \leq \gamma \leq C_T(\gamma, r) \leq \lambda \).
(c) For every r-fuzzy \( \tilde{g} \)-open set \( \lambda \in I^X \) and \( \alpha \in I^X \) such that \( \alpha \leq \lambda \), there exists a \( \delta \in I^X \) with \( T(\delta) \geq r \) and \( \delta = I_T(\Delta, r) \), \( T(\tilde{\lambda} - \Delta) \geq r \) such that \( \alpha \leq \delta \leq C_T(\delta, r) \leq \lambda \).

(d) For every r-fuzzy \( \tilde{g} \)-closed set \( \mu \in I^X \) and \( \alpha \in I^X \) such that \( \alpha \notin \mu \) there exist \( \gamma, \lambda \in I^X \) with \( T(\gamma) \geq r \) and \( T(\lambda) \geq r \) such that \( \alpha \leq \gamma, \mu \leq \lambda \) with \( C_T(\gamma, r) \notin C_T(\lambda, r) \).

**Proof:** (a) \( \Rightarrow \) (b). Let \( \lambda \) be an r-fuzzy \( \tilde{g} \)-open set such that \( \alpha \leq \lambda \). Now, \( \tilde{1} - \lambda \) is an r-fuzzy \( \tilde{g} \)-closed set. By (a), \( \alpha \notin \tilde{1} - \lambda \). Since \((X, T)\) is fuzzy \( \tilde{g} \)-regular, there exist \( \gamma, \delta \in I^X \) with \( T(\gamma) \geq r \), \( T(\delta) \geq r \) and \( \gamma \notin \delta \) such that \( \alpha \leq \gamma, \tilde{1} - \lambda \leq \delta \). Since \( \gamma \notin \delta \), \( \gamma \leq \tilde{1} - \delta \).

Hence, \( C_T(\gamma, r) \leq C_T(\tilde{1} - \delta, r) = \tilde{1} - \delta \), since \( T(\delta) \geq r \). Now, \( \tilde{1} - \delta \leq \lambda \). Therefore, \( \alpha \leq \gamma \leq C_T(\gamma, r) \leq \lambda \).

(b) \( \Rightarrow \) (c). Let \( \lambda \) be an r-fuzzy \( \tilde{g} \)-open set such that \( \alpha \leq \lambda \). By (b), there exists a \( \gamma \in I^X \) with \( T(\gamma) \geq r \) such that \( \alpha \leq \gamma \leq C_T(\gamma, r) = \Delta \leq \lambda \). Let \( \delta = I_T(\Delta, r) \) where \( \Delta = C_T(\gamma, r) \). Also, \( \alpha \leq \delta \leq C_T(\delta, r) \) \( = C_T(I_T(\Delta, r), r) \leq C_T(\Delta, r) = C_T(C_T(\gamma, r), r) = C_T(\gamma, r) \leq \lambda \). Thus, \( \alpha \leq \delta \leq C_T(\delta, r) \leq \lambda \).

(c) \( \Rightarrow \) (d). Let \( \mu \) be an r-fuzzy \( \tilde{g} \)-closed set with \( \alpha \notin \mu \). Then \( \tilde{1} - \mu \) is an r-fuzzy \( \tilde{g} \)-open set with \( \alpha \leq \tilde{1} - \mu \). By (c), there exists a \( \delta \in I^X \) with \( T(\delta) \geq r \) such that \( \alpha \leq \delta \leq C_T(\delta, r) \leq \tilde{1} - \mu \) where \( \delta = I_T(\Delta, r) \) for
some \( \Delta \in I^x \) with \( T(\bar{I} - \Delta) \geq r \). Again by hypothesis there exists a \( \gamma \in I^x \) such that

\[
\alpha \leq \gamma \leq C_I(\gamma, r) \leq \delta. \text{ Let } \lambda = \bar{I} - C_I(\delta, r). \tag{3.1.1}
\]

Then, \( \alpha \leq \gamma, \mu \leq \lambda \). By (3.1.1), \( \lambda \leq \bar{I} - \delta \).

Now, \( C_I(\lambda, r) \leq \bar{I} - \delta \leq \bar{I} - C_I(\gamma, r) \). Thus, \( C_I(\gamma, r) \notin C_I(\lambda, r) \).

(d) \( \Rightarrow \) (a). The proof is trivial.