CHAPTER II

AGRAVITATIONAL FIELD WITH CENTRAL SYMMETRY
2.1 Introduction:

A gravitational field with central symmetry may be produced by any centrally symmetric distribution of matter, for this, indeed, not only the distribution but also the motion of the matter must be centrally symmetric, i.e. the velocity at each point must be directed along the radius. Such solution of the Einstein equations was obtained by Schwarzschild (1916) and it completely gives the gravitational field in vacuum produced by any centrally symmetric distribution of masses. This solution is valid not only for masses at rest, but also when they are moving, so long as the motion has required symmetry i.e. a centrally symmetric pulsation. It is to be noted that the metric depends only on the total mass of the gravitating body, just as in the same problem in Newtonian theory. The central
symmetry of the gravitational field means that the expression for the interval $\text{ds}$, must be the same for all points situated at the same distance from the centre. In Euclidean spacetime this distance is equal to the radius vector. But in a non-Euclidean spacetime such as in the presence of gravitational field, there is no quantity which has all the features of the Euclidean radius vector. Hence, the choice of a radius vector is now arbitrary.

If $x^i = x^i(s)$ be the parametric equation of a certain curve with $s$ as the arc length measured from some point, then the vector $u^i = \frac{dx^i}{ds}$ be a unit vector tangent to the curve. If the curve is a geodesic, then along it $Du^i = 0$, which means that if the vector $u^i$ is subjected to a parallel displacement from a point $x^i$ on a geodesic curve to the other point $x^i + dx^i$ on the same curve, then it coincides with the vector $u^i + du^i$ tangent to the curve at the point $x^i + dx^i$. Hence, when the tangent to a geodesic moves along the curve, it is displaced parallel to itself. On the other hand, during parallel displacement of two vectors, the angle between them remains unaltered. Hence, one may conclude that during the parallel displacement of any vector along a geodesic curve, the angle between the vector and the tangent to the geodesic remains unaltered.
But in a curved spacetime the parallel displacement of a vector from one given point to another gives different result if the displacement is carried out over different paths. In particular, if we displace a vector parallel to itself along some closed curve then upon returing to the starting point, it does not coincide with its original value. The tensor $R^i_{k\ell m}$ is known as the curvature tensor or the Riemann tensor.

It is obvious that in a flat space the curvature tensor is zero i.e. $R^i_{k\ell m} = 0$. Because of tensor character of $R^i_{k\ell m}$ it is then equal to zero also in any other coordinate system. Hence, in a flat space, parallel displacement is a single valued operation, so that in making a circuit of closed contour a vector does not change. The converse is also valid, if $R^i_{k\ell m} = 0$, then the space is flat i.e. $R^i_{k\ell m} = 0$ the parallel displacement is a unique operation. Thus the vanishing or nonvanishing of curvature tensor is a criterion which provides us to determine whether a space is flat or curved. To obtain the equations determining the gravitational, it is necessary first to determine the action of this field. The required equations may then be obtained by varying the sum of the actions of field plus material particles.
One may obtain the energy momentum tensor of any physical system whose action reads

\begin{equation}
A = \int \wedge \left( q, \frac{\partial q}{\partial x^1} \right) \, dv \, dt
\end{equation}

\[ = \frac{1}{c} \int \wedge \, d\, \Omega \]

where \( \wedge \) is some function of the quantities \( q \), representing the state of the system, and of their first derivatives with respect to coordinates and time. It is to be noted that the space integral \( \int \wedge \, dv \) be the Lagrangian of the system, so that \( \wedge \) may be considered as the Lagrangian density. But in curvilinear coordinates this reads as

\begin{equation}
A = \frac{1}{c} \int \wedge \sqrt{-g} \, d\, \Omega .
\end{equation}

In Galilean coordinates \( g = -1 \) and \( A \) goes over into \( \int \wedge \, dv \, dt \). The energy momentum tensor of the electromagnetic field may be written as

\begin{equation}
T_{ik} = \frac{1}{4} \left( -F_{il} F_{k}^{l} + \frac{1}{4} F_{lm} F^{lm} g_{ik} \right) .
\end{equation}
For a macroscopic body the energy momentum tensor assumes the form

\[ T_{ik} = (p + \rho) u_i u_k - p g_{ik} \]

It is always noted that \( T_{oo} \) is positive i.e.

\[ T_{oo} > 0 \]

The equations of the gravitational field - the basic equation of the general theory of relativity are Einstein equations

\[ R_{ik} - \frac{1}{2} R g_{ik} = \frac{8\pi k}{c^4} T_{ik} \]

or, in mixed components

\[ R^k_i - \frac{1}{2} R g^k_i = \frac{8\pi k}{c^4} T^k_i \]

Contracting on the indices i and k, one obtains
(2.8) \[ R = - \frac{8 \pi k}{c^4} T \]

where \( T = T^i_i \). Hence, the equations of the field may also assume the form

(2.9) \[ R_{ik} = \frac{8 \pi k}{c^4} \left( T_{ik} - \frac{1}{2} g_{ik} T \right) . \]

The Einstein equations are nonlinear, so for gravitational fields the principle of superposition is not valid. In empty space \( T_{ik} = 0 \), and the equations of the gravitational field reduce to the equation

(2.10) \[ R_{ik} = 0 . \]

It is to be noted that this does not at all mean that in vacuum, spacetime is flat; for this we need the stronger conditions \( R^i_{kilm} = 0 \).

2.2 The Metric and Field Equations:

The most general centrally symmetric expression for \( ds^2 \) reads
(2.11) \[ ds^2 = A(r,t) \, dr^2 + B(r,t) \, (\sin^2 \theta \, d\theta^2 + d\theta^2) + C(r,t) \, dt^2 + D(r,t) \, dr \, dt , \]

where \( A, B, C, D \) are some functions of \( r \) and \( t \).

In the general theory of relativity, there is the arbitrariness in the choice of a reference system, one may still subject the coordinates to any transformation without destroying the central symmetry of \( ds^2 \), which means that one may transform the coordinates \( r \) and \( t \) according as

(2.12) \[ r = f_1 (r', t') , \]

(2.13) \[ t = f_2 (r', t') , \]

where \( f_1 \) and \( f_2 \) are any functions of the new coordinates \( r' \) and \( t' \). Now one may select the coordinate \( r \) and \( t \) in such a way that: first of all the coefficient \( D(r,t) \) of \( dr \, dt \) vanishes i.e.

(2.14) \[ D = D(r,t) = 0 . \]
Secondly, the coefficient $B(r,t)$ becomes equal to $-r^2$ i.e.

$$B(r,t) = -r^2.$$  \hfill (2.15)

These conditions do not give the choice of the time coordinate uniquely. Still it may be subjected to any arbitrary transformation

$$t = f(t'),$$  \hfill (2.16)

not containing $r$.

Above, the latter condition gives that the radius vector $r$ is defined in such a way that the circumference of a circle with centre at the origin or coordinates is equal to $2\pi r$. i.e. the element of arc of a circle in the plane $\theta = \pi/2$ is equal to

$$d\Omega = r \, d\varphi.$$  \hfill (2.17)

It will be proper to write

$$A(r,t) = -e^\lambda$$  \hfill (2.18)
and

\[(2.19) \quad C(r,t) = c^2 e^\lambda ,\]

where \( \lambda \) and \( \nu \) are some functions of \( r \) and \( t \).

Hence, the eq. (2.11) assumes the form

\[(2.20) \quad ds^2 = e^\nu c^2 dt^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - e^\lambda dr^2 ,\]

where the coordinates are:

\[(2.21) \quad x^0 = ct ,\]
\[(2.22) \quad x^1 = r ,\]
\[(2.23) \quad x^2 = \theta , \quad \phi ,\]
\[(2.24) \quad x^3 = \phi .\]

The nonzero components of the metric tensor are given as

\[(2.25) \quad g_{00} = e^\nu ,\]
(2.26) \[ g_{11} = - e^\lambda \],

(2.27) \[ g_{22} = - r^2 \],

(2.28) \[ g_{33} = - r^2 \sin^2 \theta \].

and

(2.29) \[ g^{00} = e^{-\nu} \],

(2.30) \[ g^{11} = - e^{-\lambda} \],

(2.31) \[ g^{22} = - r^{-2} \],

(2.32) \[ g^{33} = - r^{-2} \sin^{-2} \theta \].

The expressions for the Christoffel symbols in terms of the metric tensor may be evaluated

(2.33) \[ \Gamma^i_{kl} = g^{im} \frac{\partial g_{mk}}{\partial x^l} + g^{km} \frac{\partial g_{ml}}{\partial x^i} - g^{kl} \frac{\partial g_{im}}{\partial x^m} \].
The calculation leads to the following expressions:

\[(2.34) \quad \Gamma^{1}_{11} = \frac{\lambda'}{2}, \]

\[(2.35) \quad \Gamma^{0}_{10} = \frac{\nu'}{2}, \]

\[(2.36) \quad \Gamma^{2}_{33} = - \sin \theta \cos \theta, \]

\[(2.37) \quad \Gamma^{0}_{11} = \frac{\lambda}{2} e^{\lambda - \nu}, \]

\[(2.38) \quad \Gamma^{1}_{22} = - r e^{-\lambda}, \]

\[(2.39) \quad \Gamma^{1}_{00} = \frac{\nu'}{2} e^{\nu - \lambda}, \]

\[(2.40) \quad \Gamma^{2}_{12} = \Gamma^{3}_{13} = \frac{1}{r}, \]

\[(2.41) \quad \Gamma^{3}_{23} = C\theta, \]

\[(2.42) \quad \Gamma^{0}_{00} = \frac{\nu}{2}, \]

\[(2.43) \quad \Gamma^{1}_{10} = \frac{\lambda'}{2}. \]
(2.44) \[ \gamma_{33}^{1} = -r \sin^2 \theta \ e^{-\lambda} \]

where prime denotes differentiation with respect to \( r \) and dot for differentiation with respect to \( ct \). All other components are zero.

Now we define the Ricci tensor as

(2.45) \[ R^{i}_{\ l} = g^{lm} R^{\ l}_{\ \ lmk} = R^{i}_{\ ijk} \]

Hence, we obtain

(2.46) \[ R^{i}_{\ l} = \frac{\partial \gamma_{ik}^{l}}{\partial x^l} - \frac{\partial \gamma_{ik}^{l}}{\partial x^l} + \gamma_{ik}^{m} \gamma_{lm}^{\ m} - \gamma_{il}^{m} \gamma_{km}^{\ \ m} \]

This tensor is symmetric i.e.

(2.47) \[ R^{i}_{\ l} = R^{l}_{\ i} \]

By contracting \( R^{i}_{\ l} \), we obtain

(2.48) \[ R = g^{ik} R^{k}_{\ l} = g^{ik} g^{lm} R^{k}_{\ \ l}^{\ \ m} \]
which is known as the scalar curvature of the space. In the literature one may also find another definition of the tensor $R_{ik}$, using contraction of $R_{iklm}$ on the first and last indices. This definition differs in sign.

A simple calculation gives the following expressions:

\[(2.49) \quad e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = - \frac{8\pi k}{c^4} T_1^1, \]

\[(2.50) \quad \frac{1}{2} e^{-\lambda} \left( \nu'' + \frac{\nu'}{r} + \frac{\nu' - \lambda'}{r} - \frac{\lambda'}{2} \right) \]

\[- \frac{1}{2} e^{-\nu} \left( \nu'' + \frac{\nu'}{2} - \frac{\nu'}{2} \right) \]

\[= - \frac{8\pi k}{c^4} T_2^2 = - \frac{8\pi k}{c^4} T_{\theta 3}^3, \]

\[(2.51) \quad -e^{-\lambda} \left( \frac{\nu'}{r^2} \right) + \frac{1}{r^2} = \frac{8\pi k}{c^4} T_0^0, \]

\[(2.52) \quad -e^{-\lambda} \frac{\lambda'}{r} = \frac{8\pi k}{c^4} T_0^1, \]

the other components vanish identically.
2.3 The Solution:

Let us consider the very important case of a centrally symmetric field in vacuum, i.e., outside of the masses producing the field. In this case, we assume

\( T^i_k = 0 \) .

In view of eqs. (2.49), (2.51) and (2.52), we get

\[
(2.54) \quad e^{-\lambda \left( \frac{\nu'}{r} + \frac{1}{r^2} \right)} - \frac{1}{r^2} = 0 ,
\]

\[
(2.55) \quad e^{\lambda \left( \frac{\nu'}{r} - \frac{1}{r^2} \right)} + \frac{1}{r^2} = 0 ,
\]

\[
(2.56) \quad \lambda = 0 .
\]

Hence, we have not taken eq. (2.50), since it follows from the eqs. (2.54) - (2.56). In view of eq. (2.56), it is obvious that does not depend on the time. By adding eqs. (2.54) and (2.55), one obtains

\[
(2.57) \quad e^{\lambda \left( \frac{\nu'}{r} + \frac{1}{r^2} + \frac{\nu'}{r} - \frac{1}{r^2} \right)} = 0 ,
\]
or,

\[
(2.58) \quad \frac{\eta}{r} (v' + \lambda') = 0 ,
\]

which gives

\[
(2.59) \quad v' + \lambda' = 0
\]

or,

\[
(2.60) \quad \lambda + v = f(t)
\]

where \( f(t) \) as the function of time only. Still there is possibility of an arbitrary transformation of the time of the form

\[
(2.61) \quad t = f(t')
\]

It is equivalent to adding to \( v \) an arbitrary function of the time, which means

\[
(2.62) \quad f(t) = 0 ,
\]

in eq. (2.60). So, without loss in generality, one may put
\[(2.63) \quad \lambda + \nu = 0.\]

It is to be noted that the centrally symmetric gravitational field in vacuum is static. Now let us integrate eq. (2.58) to obtain

\[(2.64) \quad \frac{-\lambda}{e} = e^\nu = 1 + \frac{\text{constant}}{r}.\]

Now as \( r \to \infty \), we obtain

\[(2.65) \quad \frac{-\lambda}{e} = e^\nu = 1 ,\]

showing that far from the gravitating bodies the metric automatically reduces to Galilean. The constant in eq. (2.64) may be expressed in terms of mass of the body by demanding that at large distances, where the field is very weak, Newton's law should hold. In other words

\[(2.66) \quad g_{00} = 1 + \frac{2\varphi}{c^2} ,\]

where \( \varphi \) be the potential and its Newtonian value reads

\[(2.67) \quad \varphi = - \frac{\text{km}}{r} .\]
Hence, the constant assumes the value

\[(2.68) \quad \text{Constant} = - \frac{2Km}{c^2} \quad .\]

It has the dimensions of length and known as the gravitational radius \(r_g\) of the body i.e.

\[(2.69) \quad r_g = \frac{2Km}{c^2} \quad .\]

Hence, the metric assumes the form

\[(2.70) \quad ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(\sin^2 \theta \, d\phi^2 + d\theta^2\right) \quad .\]

This solution of the Einstein equations was obtained by Schwarzschild (1916) and it describes the gravitational field in vacuum generated by any centrally symmetric distribution of masses.
2.4 Interpretation of Result:

The spatial metric is obtained as

\[ d\mathbf{l}^2 = \frac{dr^2}{1-r_g/r} + r^2 \left( \sin^2 \theta \, d\phi^2 + d\theta^2 \right). \]  

(2.71)

The geometrical meaning of the coordinate \( r \) is obtained by the fact that the circumference of circle is \( 2\pi r \) with its centre at the centre of the field. The distance between two points \( r_1 \) and \( r_2 \) along the same radius as

\[ \int_{r_1}^{r_2} \frac{dr}{\sqrt{1-r_g/r}} > r_2 - r_1 \]  

(2.72)

It is observed that \( g_{\theta\theta} < 1 \) and

\[ d\bar{e} = \sqrt{g_{\theta\theta}} \, dt \]  

(2.73)

where \( \bar{e} \) be the proper time, such that

\[ d\bar{e} \ll dt \]  

(2.74)
At infinity $t = \infty$. Hence, at finite distances there is a slowing down of the time compared with the time at infinity.

Let us now consider an approximate expression for $ds^2$ at large distances i.e.

\begin{equation}
(2.75) \quad ds^2 = ds_0^2 - \frac{2Km}{c^2 r} \left( c^2 dt^2 + dr^2 \right),
\end{equation}

where $ds_0^2$ be the Galilean metric and second term represents a small correction. At large distances from the masses, every fields appear centrally symmetric. Now, we consider the behaviour of a centrally symmetric gravitational field in the interior of the gravitating masses. In view of eq. (2.51) for $r \rightarrow 0$, $\lambda$ must also vanish at least like $r^2$. Integrating eq. (2.51) with $\lambda|_{r=0} = 0$, one obtains

\begin{equation}
(2.76) \quad \lambda = - \ln \left[ 1 - \frac{8\pi k}{c^4 r} \int_0^h r^2 dr \right]
\end{equation}

It is obvious that $\lambda > 0$ i.e.

\begin{equation}
(2.77) \quad \frac{\lambda}{\varepsilon} > 1.
\end{equation}
Now,

\[(2.78) \quad (T^0_0 - T^1_1) = \frac{\kappa}{r} \left( v' + \lambda' \right) \frac{c^4}{8\pi k} \]

\[(2.79) \quad \frac{8\pi k}{c^4} (T^0_0 - T^1_1) = \left( \rho + p \right) \frac{(1 + v^2/c^2)}{1 - v^2/c^2} \geq 0 \]

showing that

\[(2.80) \quad v' + \lambda' \geq 0 \]

Hence, overall space

\[(2.81) \quad v + \lambda \leq 0 \]

Since \( \lambda > 0 \), we get \( v \leq 0 \) i.e.

\[(2.82) \quad e^v < 1 \]

The inequalities show that the properties of the spatial metric and the feature of clocks in a centrally symmetric field in vacuum apply equally well to the field in the interior of the gravitating masses. If a spherical body of radius a produces the gravitational field then for \( r > a \), we get
(2.83) \[ T^0_0 = 0 \]

Hence,

(2.84) \[ \lambda = - \ln \left[ 1 - \frac{8\pi k}{c^4 r} \int \frac{d}{T^0_0} r^2 dr \right] \]

On the other for vacuum, one obtains

(2.85) \[ \lambda = - \ln \left( 1 - \frac{2Km}{c^2 r} \right) \]

In view of eqs. (2.84) and (2.85), we get

(2.86) \[ m = \frac{4\pi}{c^2} \int \frac{d}{T^0_0} r^2 dr \]

which gives total mass of the body in terms of its energy momentum tensor.

2.5 Concluding Remarks:

We have presented gravitational field with central symmetry and obtained the solution of the Einstein field equations with matter and without matter
i.e. in vacuum and hence, recovered the Schwarzschild solution. Thus, we have interpreted our results. We have obtained the mass formula

\[
m = \frac{4\pi}{c^2} \int_0^\alpha T_0^0 r^2 \, dr.
\]

In particular, for a static distribution of matter in a body, one obtains

\[
T_0^0 = \mathcal{F},
\]

so that

\[
m = \frac{4\pi}{c^2} \int_0^\alpha \rho r^2 \, dr.
\]

It is to be noted the fact that the integration is taken with respect to \(4\pi r^2 \, dr\), where as the element of spatial volume for the metric is \(dV = 4\pi r^2 e^{\frac{\lambda}{2}} \, dr\), where according \(e^{\frac{\lambda}{2}} > 1\). This difference gives the gravitational mass defect of the body.

*****