Chapter 3

Bimerons in double layer quantum Hall systems

3.1 Recapitulation

We have already mentioned that a single meron cannot be a candidate for the lowest lying physical excitation in double layer quantum Hall system as its energy is logarithmically divergent. The possible candidates for such excitations are meron-antimeron bound pairs which have finite energies. We also explained in the last chapter that depending on the sense of winding of the transverse component of the pseudospin \( \sqrt{(1-m_z^2)} \) and the direction of \( m_z \) at the centre merons can be classified into four possible types \[25\], \[32\]. Now if one constructs a meron-antimeron pair with any two of these four types of merons which have opposite vorticities, its charge can be either \( \pm 1 \) or zero depending on the charges of the constituent (anti)merons. Since we are discussing only charged excitations we consider only the first of the two cases mentioned above in which each of the two merons have same charge but opposite vorticities. These type of excitations are known in the literature as
bimerons with topological charges \( \pm 1 \). Meron pairs with opposite vorticities as well as opposite charges will be neutral as a whole in zero temperature and may lead to a finite temperature Kosterliz - Thouless phase transition [31] in this system. In this thesis we shall not discuss about them any further.

In chapter 2 we have estimated the energy of such bimerons from the single meron solutions by approximating the bimeron as a pair of merons each carrying electric charge of \( \frac{1}{2} \) and logarithmically growing energy [34', [23]. The Coulomb energy between two merons is evaluated there by modelling the merons as point like charged objects separated by a distance \( R \). This approximation is only justifiable when the separation between two merons is large compared to the size of the each individual meron. This size is approximately given by the meron core radius that we have defined. In the following discussion we will examine the justification of such a picture of bimerons by directly evaluating bimeron solutions of the same field equations (2.98, 2.99), and their energy. We shall comment on the possible inaccuracy in estimating the Coulomb interaction energy between the merons as if they are rigid charges. Actually there will be distortion of the charge density of each meron by the presence of the other. All these features will be displayed when we directly calculate the bimeron solutions.

### 3.2 Equations for pseudospin texture

The energy functional from which we shall derive the profile of the bimeron texture is the following

\[
E(\vec{m}) = \int d^2 r \left[ \frac{1}{2} \rho^2 (\nabla m_z)^2 + \frac{1}{2} \rho^4 ( (\nabla m_x)^2 + (\nabla m_y)^2 ) + \frac{1}{2} m_2^2 \right] + C_1[m] + C_2[m]
\]  

(3.1)
where
\[ C_1[m] = \frac{1}{2} \int d\vec{r} d\vec{r}' V(\vec{r} - \vec{r}') \delta \rho(\vec{r}) \delta \rho(\vec{r}') \]  (3.2)
and \( C_2[m] \) is the \( C[m] \) defined in the eq. (2.91) in chapter II. Here \( \delta \rho(\vec{r}) \) is the topological charge density given by eq. (2.43).

One can see that the only difference between the expressions given in eq.(2.92) and the above one (3.1) is in the term \( C_1 \), where we have replaced \( V^+ \) by \( V \). We have already explained at the end of the section (2.4.2) the justification of such a replacement.

The non-local terms \( C_1 \) and \( C_2 \) in the energy functional (3.1) render field equations into coupled integro-differential equations. However unlike meron solution bimeron solution is not circularly symmetric. This makes the numerical exercise more difficult. While in the single meron case we did solve such an integro-differential equation (eq.2.98) ([34]), for the more complicated case of bimerons we will be content to solve the equations in the absence of the integral terms \( C_1 \) ansd \( C_2 \). The contributions of these terms can however be included in the total energy, but by using solutions of the local equations. In mild justification of this strategy, we will find later that the Coulomb energy \( C_2 \) for instance is less than half the energy from the local terms in eq. (2.95).

The coupled field equations for \( m_z \) and \( \alpha \equiv \tan^{-1} \left( \frac{m_y}{m_z} \right) \) resulting from eq. (3.1) in the absence of \( C_1 \) and \( C_2 \) are are also same as (2.98,2.99). For convenience of reading let us mention them once gain. They are
\[ \rho \nabla^2 m_z + \rho^4 m_z \left( \frac{(\nabla m_z)^2}{(1 - m_z^2)^2} + \frac{m_z \nabla^2 m_z}{1 - m_z^2} + \nabla^2 \alpha \right) - 2\beta m_z = 0 \]  (3.3)
and
\[ \nabla \cdot [(1 - m_z^2) \nabla \alpha] = 0 \]  (3.4)
The difficulties start hereafter as bimeron solutions are not symmetric in radial variable and boundary conditions they satisfy are also unusual. In the following section we are going to discuss these features of the bimeron solution in detail as well as the method we shall use to solve this problem.

3.2.1 Bipolar co-ordinate system

To find bimeron solutions we have to numerically solve the coupled non-linear partial differential equations (NLPDE) (3.3) and (3.4). The defining boundary condition of a bimeron is \( m_z = \pm 1 \) at the points \((0, \pm a)\). The only advantage here compared to the earlier case is that unlike the single meron solution the bimeron solution is analytically known in the exact non-linear sigma model (NLSM) limit. Our strategy will be to use the known exact solution of these equations in the NLSM limit, and solve the full equations iteratively starting with the NLSM solution. As already discussed in the earlier chapter in the NLSM limit \((d \to 0) \rho^d = \rho^s\), i.e. the stiffness is isotropic and further that the capacitance coefficient \(\beta\) vanishes. Then, with \(C_1\) and \(C_2\) also neglected, the action in (3.1) is again just that of the NLSM, all of whose solutions are exactly known [30]. In that chapter it was also mentioned that they are conveniently described by the complex field \(w(z)\) which represents the stereographic projection of the unit sphere of textures \(m\), given by the relation

\[
w(z) = \frac{m_x + im_y}{(1 - m_z)}
\]

where \(z = x + iy\). Our texture variables \(m_z\) and \(\alpha\) are related to \(w(z)\) by

\[
m_z = \frac{|w|^2 - 1}{|w|^2 + 1}
\]

\[and\quad \alpha = \arg(w)\]

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Any analytic function \( w(z) \) will be a solution of the NLSM. In particular the function

\[
 w(z) = \frac{z - a}{z + a}
\]

represents the bimeron, with the points \((0, -a)\) and \((0, a)\) representing the centers of the two merons, where the solution gives \( m_z = \pm 1 \) respectively. It may be checked that (3.7) satisfies the coupled equations (3.3) and (3.4) in the isotropic limit.

When the interlayer separation \( d \) is not zero, we have to cope with the coupled field equations (3.3) and (3.4) with both the anisotropic stiffness and capacitance terms present. Some analysis of this system was done long ago by Ogilvie and Guralnik [38] who studied the NLSM with the mass (capacitance) term included but no anisotropy. (An ansatz suggested in ref ([38]) does not work, as we will show below.) Meanwhile Watanabe and Otsu [39] studied the anisotropic NLSM but without the mass term. Both made considerable progress analytically, but neither offered exact or numerical solutions. Here we shall try to solve (3.3) and (3.4) numerically after including both the capacitance and anisotropic terms.

To do so, it will be convenient to use a bipolar coordinate system to describe the x-y plane, as might be expected when we have to impose boundary conditions at two finite points \((0, -a)\) and \((0, a)\). These coordinates, \( \eta \) and \( \phi \), are defined by

\[
\eta \equiv \ln \left| \frac{z - a}{z + a} \right|
\]

and

\[
\phi \equiv \arg \left( \frac{z - a}{z + a} \right)
\]

This coordinate set has many advantages [36]. The points \((0, -a)\) and \((0, a)\) at which we have to impose boundary conditions are now mapped into \( \eta \to \pm \infty \).
The full x-y plane is mapped in $(\eta, \phi)$ coordinates to an infinite strip with $\eta = [-\infty, +\infty]$ and $\phi = [-\pi, \pi]$. Finally, it is clear upon comparing eq(3.8) to eq (3.7) that this set of coordinates is closely related to the exact NLSM bimeron solution. Clearly the the exact NLSM solution (2.101) corresponds to the simple expressions

$$m_z = \tanh(\eta)$$

$$\text{and} \quad \alpha = \phi$$

(3.9)

One of the major features of such a solution is that inspite of it is having two centres (at $x = \pm a$) its topological charge density given by the formula (2.43) looks like that of a single object as it has its maxima at the centre of the co-ordinate plane ($x = 0, y = 0$ or $\eta = 0, \phi = \pi$). This does not resemble the charge distributions of two single merons. This can be easily verified in bipolar co-ordinate system where the topological charge density given by the eq. (2.43) is

$$\dot{\rho}(\eta, \phi) = \frac{\text{sech}^2(\eta)}{4\pi Q^2}$$

(3.10)

where

$$Q^2(\eta, \phi) = \frac{a^2}{(\cosh \eta - \cos \phi)^2}$$

(3.11)

is the Jacobian of this coordinate transformation.

Away from the NLSM limit, since this is an orthogonal coordinate system with simple expressions for the gradient, divergence and Laplacian, the equations (3.3) and (3.4) become
It should be pointed out that gradient and laplacian operators in this co-ordinate systems are simply given as

\[
\left[ \left( \frac{\rho^x - \rho^d}{\rho^d} \right) + \frac{1}{1 - m_z^2} \right] (\partial^2_\eta m_z + \partial^2_\phi m_z) + \frac{m_z(\partial_\eta m_z + \partial_\phi m_z)^2}{(1 - m_z^2)^2} + m_z((\partial_\eta \alpha + \partial_\phi \alpha)^2 - \frac{2\beta}{\rho^d} Q^2(\eta, \phi)) = 0
\]

(3.12)

\[(1 - m_z^2)(\partial^2_\eta \alpha + \partial^2_\phi \alpha) - 2m_z(\partial_\eta m_z \partial_\eta \alpha + \partial_\phi m_z \partial_\phi \alpha). = 0 \quad (3.13)\]

It should be pointed out that gradient and laplacian operators in this co-ordinate systems are simply given as

\[
\vec{\nabla} = \frac{1}{Q^2}(\frac{\partial}{\partial \eta} \hat{\eta} + \frac{\partial}{\partial \phi} \hat{\phi})
\]

(3.14)

\[
\nabla^2 = \frac{1}{Q^2}(\frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \phi^2})
\]

(3.15)

Now let us analyse these equations as different terms are included in stages.

(a) In the NLSM limit, our exact solution has \(\alpha = \phi\). Then (3.13) forces \(m_z\) to be a function of \(\eta\) alone, \(m_z = m_z(\eta)\). Upon inserting this into the other equation (3.12) it becomes an ordinary non-linear differential equation. This is the advantage of this choice of coordinates. The solution can be verified to be \(m_z = \tanh(\eta)\).

(b) Next let us include anisotropy \((\rho^x \neq \rho^d)\), while still keeping the capacitance term zero \((\beta = 0)\). Once again we can set \(\alpha = \phi\), and consequently \(m_z = m_z(\eta)\), which will obey again an ordinary differential equation given by

\[
\left[ \left( \frac{\rho^x - \rho^d}{\rho^d} \right) + \frac{1}{1 - m_z^2} \right] (\partial^2_\eta m_z + \frac{m_z(\partial_\eta m_z)^2}{(1 - m_z^2)^2} + m_z = 0
\]

(3.16)
Figure 3.1: The solution of equation (3.16). The three continuous curves correspond, as you go outwards, to three different values of layer separation $d$ equal to 0.5, 0.6 and 0.7 respectively in the unit of magnetic length $l$. The dotted curve corresponds to the exact solution of NLSM i.e. $m_z = \tanh(\eta)$. 
This has no analytic solution, but can be solved relatively easily numerically, being just an ordinary differential equation in the variable $\eta$. As boundary conditions we impose $m_z = 0$ at $\eta = 0$ and $m_z = 1$ at $\eta = \infty$.

(Note that the equation above is symmetric under $\eta \rightarrow -\eta$, so that we can choose the solution to be antisymmetric, i.e. $m_z(-\eta) = -m_z(\eta)$). The resulting numerical solutions for different values of layer separation $d$ (on which the anisotropy depends), are shown in Fig. (3.1). One can see that with increasing the layer separation, and hence increasing anisotropy in the stiffness, the pseudospin component $m_z$ reaches its asymptotic value more slowly.

(c) Finally let us also include the capacitance term and consider the equations (3.12 and 3.13) in full. Now the ansatz $\alpha = \phi$ is no longer sustainable, in contrast to what has been suggested in ref ([38]). The substitution of the ansatz $\alpha = \phi$ in equation (3.13) would again force $\partial_{\phi} m_z = 0$, i.e. $m_z = m_z(\eta)$. But now this is in contradiction with equation (3.12) which has an explicit $\phi$ dependence through the last (capacitance) term $\frac{2\alpha}{\rho^2} Q^2 (\eta, \phi)$. Therefore, once one includes the capacitance term in equation (3.12) both $\alpha$ and $m_z$ become functions of both $\eta$ and $\phi$. One has unavoidably to solve the coupled non-linear PDE for $m_z = m_z(\eta, \phi)$ and $\alpha = \alpha(\eta, \phi)$.

We do this by employing what we believe is a good ansatz for $\alpha$ which approximately satisfies (3.13). We then solve the other equation (3.12) numerically after inserting that ansatz for $\alpha$. Our ansatz is been motivated by the following arguments. One can see from equation (3.12) that the troublesome $\phi$ dependent term $Q^2$ is negligibly small in the large $\eta$ region ($Q \sim sech(\eta)$) and is most dominant in the small $\eta$ region. Hence $\alpha$ will still approach $\phi$ as $\eta \rightarrow \infty$ but needs to be modified substantially in the small $\eta$ region where
however $m_z \ll 1$. When $m_z \ll 1$ equation (3.4) can be approximated by

$$\nabla^2 \alpha = 0 \quad (3.17)$$

This is just Laplace’s equation in two dimension whose solutions are all harmonic functions. With this in mind we choose our ansatz for $\alpha$ as follows

$$\alpha = \sigma - B \exp(-\eta) \sin(\phi) \quad (3.18)$$

where

$$\kappa \equiv \left( \frac{2\beta}{\rho^2} \right)^{1/2} \sigma \quad (3.19)$$

This solves Laplace’s equation and satisfies all the required boundary conditions and asymptotic behaviour, namely

$$\begin{align*}
\alpha &\to \phi \quad \text{as} \quad \eta \to \pm \infty \\
\alpha &= 0 \quad \text{when} \quad \phi = 0 \\
\alpha &= \pi \quad \text{when} \quad \phi = \pi \\
\alpha &= \phi \quad \text{when} \quad \kappa = 0 \quad (3.20)
\end{align*}$$

Note that the ansatz has a cusp at $\eta = 0$. This need not cause concern. Some such cusps can be expected on physical grounds and are familiar in soliton physics. The point is that each meron feels some force due to the other (Coulomb plus a logarithmic force) at arbitrary separation. We would expect them to move because of this force, and cannot strictly speaking expect a static bimeron solution to exist at arbitrary separation. But a cusp, like the one in the above ansatz, amounts to a delta function in the second derivative and can be interpreted as an external force just at $\eta = 0$ which can "hold the two merons together" at arbitrary separation. For more discussion of this point see Rajaraman and Perring and Skyrme [40] where this technique was used to get intersoliton forces between one dimensional solitons.
The constant $B$ is chosen by minimising the energy. Substituting this ansatz in equation (3.12) we then solved it numerically subject to the boundary condition

$$m_z = 0 \quad \text{at} \quad \eta = 0$$
$$m_z = \pm 1 \quad \text{when} \quad \eta = \pm \infty \quad (3.21)$$

It is sufficient to solve the equation in the first quadrant i.e. $(\eta[0, \infty]$ and $\phi[0, \pi])$. For the rest of the quadrants solutions can be obtained by writing

$$m_z(\eta, \phi) = -m_z(\eta, \phi) = -m_z(\eta, -\phi)$$

$$\alpha(\eta, \phi) = \alpha(\eta, \phi) = -\alpha(\eta, -\phi) \quad (3.22)$$

which is consistent with the invariance of equations (3.12) and (3.13) under the transformation $\eta \rightarrow -\eta$ and $\phi \rightarrow -\phi$.

### 3.3 Numerical procedure

The cost one has to pay for resorting to an unfamiliar co-ordinate system like the bipolar co-ordinate system is that an unwanted singularity is incorporated in the equation (3.12) through the last term at the origin of this co-ordinate system. This happens because all points at spatial infinity in the cartesian or plane-polar co-ordinate system are mapped onto a single point in the bipolar co-ordinate system namely at $\eta = 0$ & $\phi = 0$. As a result the scale factor $Q(\eta, \phi)$ diverges at this point. To get rid of this divergence $m_z$ need to behave near this point in a way such that other terms in the equation can generate proper counter-terms to get this divergence cancelled. This requires us to do a careful asymptotic analysis of the equation (3.12) which
we shall present in the following subsection. Apart from its analytical value, a study of these singularities of the non-linear partial differential equation is going to be essential for the numerical calculation also for the following reasons. Ideally the code for numerical computation on its own should settle the issue of asymptotic behaviour if we can provide the proper boundary values of either the functions or its derivatives completely. However one of the major handicaps of this problem is that we do not know ab initio all the necessary boundary conditions. In fact in course of discussing the numerical procedure in detail we will explain how we have generated the missing boundary conditions at different stages of the numerical computations by using what is known as relaxation technique. The second point to note in this context is that as the equation is highly non-linear the truncation error is not negligible unless one makes the lattice size very small. Particularly near the points where the equation has diverging terms this truncation error can be large enough if associated with such divergent terms. This may offset the possibility of reaching the proper asymptotic behaviour by itself. Hence it is necessary to provide the proper asymptotic form of the function near the point of divergence so that all large terms in the equation are mutually cancelled against each other over a neighbourhood of the troublesome point yielding a smooth solution. Our numerical calculation confirms this necessity.

3.3.1 Asymptotic analysis

Following the discussion in the preceding section we shall now analyse the behaviour of these equations near the point $\eta = 0$ and $\phi = 0$. We have already mentioned that the last term on the left hand side of the equation
(3.12) is divergent.

\[
\lim_{\eta, \phi \to 0} Q = \lim_{\eta, \phi \to 0} \frac{a}{\cosh(\eta) - \cos(\phi)} = \infty \quad (3.23)
\]

Now after expanding \(\cosh(\eta)\) and \(\cos(\phi)\) in their respective power series and only retaining the leading order terms one obtains

\[
Q \approx \frac{a}{\frac{\eta^2 + \phi^2}{2} + \frac{\eta^4 - \phi^4}{4!}} \quad (3.24)
\]

\[
= \frac{a}{\frac{\eta^2 + \phi^2}{2} (1 + \frac{\eta^4 - \phi^4}{12})} \quad (3.25)
\]

The purpose of all these is to find as how \(Q\) diverges at the origin of the bipolar co-ordinate system as a function of \(\eta\) and \(\phi\). To simplify our investigation we will now introduce the following variables namely \(\rho\) and \(\theta\) such that \(\eta = \rho \cos(\theta)\) and \(\phi = \rho \sin(\theta)\). One can easily sense that this new set of variables are having normal polar-cartesian relation with the bipolar co-ordinate system. Using these new set of variables we can easily trace that in this limit the diverging capacitance term will go like

\[
\lim_{\eta, \phi \to 0} \frac{2\beta}{\rho_0} Q^2 \rightarrow \frac{2\beta}{\rho^d \left(1 + \frac{\rho^2 \cos(2\theta)}{12}\right)}^2 = \frac{8\beta a^2}{\rho^d \rho^4 \left(1 + \frac{\rho^2 \cos(2\theta)}{12}\right)} - \frac{8\beta a^2 \cos(2\theta)}{\rho^d \left(\frac{1}{\rho^4} - \frac{\cos(2\theta)}{6\rho^2}\right)} \quad (3.26)
\]

As a result leading two singularities will go like \(\frac{1}{\rho^1}\) and \(\frac{1}{\rho^5}\). To treat the equation 3.12 asymptotically in order to make it singularity free it is good enough to concentrate on the following modified form of the equation. The
approximations made to reach this form are the followings. We note

\[ \lim_{\eta, \phi \to 0} m_z = 0 \]  
(3.27)

So,

\[ \lim_{\eta, \phi \to 0} (1 - m_z^2) \approx 1 \]  
(3.28)

and,

\[ \lim_{\eta, \phi \to 0} m_z (\nabla^2 m_z)^2 \approx m_z^3 \]  
(3.29)

After incorporating this approximation the equation (3.12) reduces to the following asymptotic form

\[ \nabla^2 m_z + m_z (\nabla \alpha)^2 - \frac{2 \beta}{\rho^2} Q^2 = 0 \]  
(3.30)

where all vector operators are in the bipolar co-ordinate. As \( \rho \) and \( \theta \) co-ordinate system is the polar version of \( \eta \) and \( \phi \), the vector operators in this \( \rho, \theta \) co-ordinate system will have the standard polar form. After writing these operators explicitly equation (3.30) will look like

\[ \frac{1}{\rho} \partial^2_{\rho} m_z + \frac{1}{\rho} \partial_{\rho} m_z + \frac{1}{\rho^2} \partial^2_{\phi} m_z + m_z ((\nabla \alpha)^2 - \frac{2 \beta}{\rho^2} Q^2) = 0 \]  
(3.31)

At this stage we will transform the variable \( m_z \) to \( u \) such that \( u = \sqrt{\rho} m_z \) (as is done in the case of hydrogen atom). After this substitution the above equation will be simplified to

\[ \frac{1}{\sqrt{\rho}} \partial^2_{\rho} u + \frac{u}{4 \rho^{5/2}} + \frac{1}{\rho^{5/2}} \partial^2_{\phi} u + \frac{u}{\sqrt{\rho}} ((\nabla \alpha)^2 - k'^2 (\frac{1}{\rho^4} + \frac{\cos(2\theta)}{6 \rho^2})) = 0 \]  
(3.32)

where

\[ k' = \sqrt{\frac{8 \beta a^2}{\rho^2}} \equiv 2 \kappa \]  
(3.33)
Hence for the most dominant singularity i.e. $\frac{1}{\rho^2}$ to be cancelled $u$ has to behave like

$$\lim_{\rho \to 0} u = e^{-\frac{k'}{\rho}} G(\rho, \theta)$$

for all $\theta$. This corresponds, in more familiar polar coordinates $(r, \theta)$ to writing

$m_z$ in the form $[\exp(-\frac{k' r}{2\rho^2})] \tilde{G}(r, \theta)$. That $m_z$ will suffer such an exponential fall-off as $r \to \infty$ can also be inferred directly from the "mass term" $2\beta m_z$ in the original field equation (2.98).

Finally to get rid of the subdominant singularities we set

$$u = \exp(-\frac{k'}{\rho}) \sin\left(\frac{\theta}{2}\right) G(\rho, \theta)$$

After Substituting this expression one gets

$$\nabla^2 m_z = \frac{2k'}{\rho^{5/2}} G' u - \frac{2k'}{\rho^{5/2}} \frac{k'^2}{\rho^{5/2}} u + \frac{1}{\sqrt{\rho}} G'' u - \frac{1}{\rho^{5/2}} \cot\left(\frac{\theta}{2}\right) \frac{u \tilde{G}}{G} - \frac{u \tilde{G}}{G \rho^{5/2}}$$

Now from the section (3.2.1)

$$\lim_{\rho, \theta \to 0} (\nabla \alpha)^2 = (1 + B)^2$$

Hence after the removal of this most dominant singularity the asymptotic form of the equation will be

$$\frac{2k'G'}{\rho^2} - \frac{2k'G}{\rho^3} + G'' - \frac{1}{\rho^2} \cot\left(\frac{\theta}{2}\right) \tilde{G} - \frac{2}{\rho^2} \cos(2\theta) = 0$$

To remove the subdominant singularities it is helpful to expand $G$ in the following manner i.e

$$G = \rho + f_2(\theta) \rho^2 + f_3(\theta) \rho^3 + \rho^4 g(\rho, \theta)$$

Inserting this we will find that the next dominant singularity get automatically cancelled and it comes out very clearly that one has to set $f_2(\theta) = \frac{k \cos(2\theta)}{12}$ to get the lowest order singularity i.e. $\frac{1}{\rho}$ out.
3.3.2 Algorithm: Newton-Raphson relaxation technique

Given this functional form of $m_{z}$ near the origin of the $\eta, \phi$ plane, the boundary conditions (3.21), and the ansatz (3.18) for $\alpha$ we solved equation (3.12) through an iterative procedure. We start with the solution for $\kappa=0$ but with full anisotropy, which can be obtained relatively easily from the ordinary differential equation (3.12). The numerical procedure in brief is the following. First one discretise the equations on a lattice in 2-d which will convert the differential equations into a set of finite difference equations. Then from the initial guess solution (3.12) we will assign the initial value of the field $m_{z}$ at each lattice point. Let us denote each lattice site by an index $i$ and the l.h.s of the difference equation at that lattice site as $F_{i}$. Now the value of the $m_{z}$ at each lattice point is an independent co-ordinate and $F_{i}$ is a function of these co-ordinates. Hence the finite difference equations are actually non-linear algebraic equations whose root we have to find out. The technique is known as Newton-Raphson method ([37]). Let us denote this independent co-ordinate as a column vector $m_{z}$ which has $N$ elements denoted as $m_{zj}$. Here $N$ is the no. of lattice points.

$$F_{k}(m_{z} + \Delta m_{z}) = F_{k}(m_{z}) + \sum_{j=1}^{N} \frac{\partial F_{k}}{\partial m_{zj}} \delta m_{zj} + O(\delta m_{z})^{2} \quad (3.40)$$

The subscript $k$ of $F_{k}$ also runs from 1 to $N$ as we have one non-linear finite difference equation at each lattice site. The Jacobian $J = \sum_{j=1}^{N} \frac{\partial F_{k}}{\partial m_{zj}}$, which is a $N \times N$ dimensional matrix guides the transformation between the co-ordinates $m_{zj}$ and $F_{k}$.

Now if we start with our set of four field equations and a very small value of $\kappa = \epsilon$ we can expect that our starting ansatz which is exact solution for the
case \( \kappa = 0 \) is still very close to the actual solution for the full equations. This ensures that the truncation of the Taylor expansion in eq.(5.10) is justified and the right hand side will converge to the actual solution very fast. For the actual solution l.h.s of the equation is 0. Substituting this we found the correction to the starting ansatz at a given iteration is

\[
\Delta \delta m_{\tau\nu} = F_k J^{-1}
\]  

(3.41)

where \( J^{-1} \) is the inverse of the mentioned Jacobian. This correction is then added to the starting ansatz to improve it and the same process is iterated with this improved ansatz. The process converges to the desired solution if this correction starts becoming smaller in successive steps. Finally we stop where this correction saturates to a very small number which should be less than some pre-defined constant in the computer code. This is how improving upon the starting ansatz iteratively we finally reach the desired solution. The solution for \( \epsilon \) is then used as input to obtain the solution for \( 2\epsilon \) and so on. This procedure is repeated until one reaches the desired value of \( \kappa \). The advantage of this procedure is that one can make \( \epsilon \) arbitrarily small to make the Newton-Raphson method converge. In this way we obtained solutions for different values of the ansatz parameter \( B \) for each value of \( \kappa \).

### 3.3.3 Boundary conditions

Coming to the issue of fixing the boundary condition we should first note that we have the boundary condition at \( \eta = \pm \infty \) for any \( \phi \) because these points correspond to the centre of the two merons. However we have no such boundary conditions at the end points of the co-ordinate \( \phi \) namely \( \phi = 0 \) and \( \pi \) axis. To generate that we shall try the following trick. At the \( \kappa = \epsilon \) stage we shall retain the boundary condition at \( \kappa = 0 \). As \( \epsilon \) is very small
we assume these boundary conditions should be very close to those of the actual solutions at $\kappa = \epsilon$. Using these boundary conditions once we obtain the solutions at the interior points we replace the old boundary condition by a new one obtained from extrapolating from these new set internal points. We shall now use these as boundary conditions for the stage $\kappa = 2\epsilon$. This procedure is then repeated iteratively. This is how boundary conditions are generated at every stage.

### 3.4 Results

Our solutions of equations (3.12) and along with the ansatz (3.18) give us the value of the pseudospin vector $\vec{m}$ as a function of $\eta$ and $\phi$, or equivalently, the value of the vector-field $\vec{m}$ on a lattice of points on the parent x-y plane. We repeated this calculation for a set of values of the parameter $B$ in the ansatz (3.18). We found that as one varies $B$ starting from 0, the energy does not vary much as $B$ goes 0 to 0.1, but then it increases sharply after $B=0.1$. This behaviour is seen to be common to all $\kappa$s and all $\kappa$s. Hence we take $B$ to be equal to 0.1. and solve the PDE for a number of layer separation $d$, and bimeron separation $a$ ($a$ is actually half of the meron-antimeron separation). Together all these solutions represent a large body of calculated data. But it is neither feasible nor very interesting to try to display it all or them. Instead we will try to bring out salient features of our solutions through examples.

#### 3.4.1 Pseudospin profile of the bimeron solutions

Recall from (3.16) that in the absence of the capacitance term $m_z$ had no $\phi$-dependence. To give some feel for how the $m_z$ varies with $\phi$ in the presence of the capacitance term, we plot in Fig.(3.2) the solution $m_z(\eta)$ of equation

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(3.12) for a set of values for $\phi$. This solution corresponds to $d = 0.7$ and $a = 3.158$. The sequence of curves shown correspond to $\phi$ equal to 0, 0.2$\pi$, 0.47$\pi$, and 0.94$\pi$ respectively with the outermost one belonging to $\phi$ equal to 0. As we have discussed earlier, as $\eta$ and $\phi$ tend to zero, the solution should damp exponentially as $\exp(-\frac{2\kappa}{\sqrt{\eta^2 + d^2}})$. Correspondingly we see in Fig. (3.2) that the low $\phi$ curves rise very slowly as $\eta$ increases away from zero. To make this behaviour more explicit in Fig. 3.3 we have again provided the solution $m_\pm(\eta)$ however for the two different values of $\kappa$ where $\kappa$ is given by the equation (3.19). Though each of the figures correspond to bimerons of same size, the lower one belongs to higher value of layer separation and consequently higher $\kappa$. Hence near the region $\eta = 0$ and $\phi = 0$ it grows at a slower rate. We also give for comparison, in the form of the dotted curve, the function $\tanh(\eta)$ which is the solution in the NLSM limit. The comparison shows that the restructuring of the pseudospin texture due to the inclusion of the capacitance and term and anisotropy is considerable.

As an alternate representation of our results, we show in Fig. 3.4 the projection of $\vec{m}$ on the x-y plane, for the example of $d$ equal to 0.7 and $\kappa$ equal to 4.4. (All lengths throughout this article are in units of the magnetic length $l$). The length of each arrow gives the magnitude of its easy-plane projection $\sqrt{m_x^2 + m_y^2}$ and its direction gives the azimuthal angle of the projected vector, namely, $\alpha = \tan^{-1}\left(\frac{m_y}{m_x}\right)$. The plot clearly shows that our "bimeron" solution is indeed a meron-antimeron pair.

Note that, as desired, $\vec{m}$ lies along the x-axis asymptotically. This picture closely resembles the general structure obtained by Brey et.al. [32]. The data corresponding to all other values of $d$ and $a$ we studied have a similar behavior.
Figure 3.2: The solution $m_z(\eta)$ of field equation (with both capacitance and anisotropy) for a set of values for $\phi$. The curves correspond, as you go inwards, to $\phi = 0, 0.2\pi, 0.47\pi, 0.94\pi$ respectively with the outermost one corresponds to $\phi$ equal to 0. The layer separation $d$ is equal to 0.7$l$ and bimeron separation $a$ is equal to 3.158$l$. The dotted curve at the top again corresponds to $m_z = \tanh(\eta)$. 
Figure 3.3: The solution of field equation (with both capacitance and anisotropy) at two different layer separation (but same bimeron separation, \( a = 3.5l \)). Fig (a) correspond to layer separation .6l and \( \kappa = 4.41 \) and figure (b) correspond to layer separation .8l and \( \kappa = 6.4 \). In each figure as we go from the innermost curve to the outermost one \( \phi \) takes values \( 0, 0.2\pi, 0.47\pi, 0.94\pi \) respectively.
Figure 3.4: This figure gives the magnitude and direction of x-y projection of m at different points on the plane. The layer separation and the bimeron separation are same as in Fig.3.2
Figure 3.5: A contour plot of the topological charge density of the bimeron when both the capacitance term and the anisotropy term is incorporated. This particular plot corresponds to a layer separation \( d \) equal to 0.7 and bimeron separation \( a \) equal to 3.158 both in the unit of magnetic length \( l \). The number against each contour (shown by broken curves) denotes the corresponding charge density.
In Fig. 3.5 we plot the topological charge density given in eq. (2.43) as a function of $\eta$ and $\phi$ in the presence of all the local terms in the field equations, including anisotropic ones. In viewing this figure it may be helpful to remember that large $|\eta|$ corresponds to the meron centers while $\eta = 0, \phi = 0$ corresponds to spatial infinity. $\phi = \pi$ corresponds to the line joining the two merons. As topological charge density is symmetric when either of the coordinate variable changes sign we show the contours only in the first quadrant where both $\eta$ and $\phi$ are positive.

3.4.2 Energy of the bimerons

Next let us turn to the energetics of these bimeron solutions.

In Fig 3.6 we plot the Coulomb energy $C_1$ evaluated using our solution of the equation (3.12), as a function of the bimeron separation. The continuous curve is the best fit to our calculated points.

As we have already mentioned that a phenomenological estimate of bimeron energetics is made assuming that it can be viewed as a bound pair of two merons, each symmetrical, undistorted by the other and carrying a charge of $\frac{\xi}{2}$. Such a pair would have a Coulomb energy of $\frac{1}{8a}$ (in units of $\xi^2$ that we are using). To see how good an approximation this simple picture is, we give in the same 3.6, in the form of a broken line the plot of this function $\frac{1}{8a}$. We see that the value of the Coulomb energy we get from the actual bimeron solution is much larger than what the simple two-charge picture would give. This is presumably because each meron is considerably squashed (polarised) by the close proximity of the other. In our earlier work on single merons [34], we had found that at the layer separation ($d = 0.7$) used in Fig. 3.6, the core-radius of individual merons is about 2, which is of the same order as the meron-separation in Fig. 3.6. In fact we can see that the gap between
Figure 3.6: This figure gives the plot of the coulomb energy as a function of bimeron separation $a$ in units of magnetic length $l$. The unit of energy is $e^2$. The upper curve is our computed value of the coulomb energy integral $C_1$, using the solution of equation (3.12)(points). The continuous line is the best curve to these points. The form of the best fitted curve is $E = \frac{A}{a^B}$ where $A$ and $B$ are found out to be 0.847 and 0.821 respectively. The dotted curve at the bottom corresponds to the Coulomb energy that the bimeron would have, if viewed as a bound pair of two point charges of $\frac{e}{2}$ each, separated by a distance $2a$. This data corresponds to a layer separation $d$ equal to 0.7.
the two curves in Fig 3.6 is higher for smaller \( a \) where the individual \( \pm \)ons are squeezed together more. Of course our results, while indicative, may not be quantitatively unambiguous. For instance, recall that our solution was obtained using only the local terms in the differential equation and the Coulomb energy was calculated by substituting this solution into the integral \( C_1 \). The non-local Coulomb term's influence on the solution has not been included.

In Fig 3.7 we plot the variation of three terms in the energy functional namely the contribution from the local terms (capacitance + gradient energy) \( C_1 \) and \( C_2 \) as a function of the bimeron separation. The data presented here corresponds to layer separation \( d \) equal to 0.7\( l \) but this behaviour is representative of almost all the layer separations (0.5, 0.6, 0.7 and 0.8 for which we have found solutions.

The trend of all three contributions is the same for the other layer separations also with only slight changes in the slope of the curves.

Our calculations were done for different bimeron separations \( a \), for each layer separation \( d \). In reality, the exact solution should exist only for some optimal bimeron separation \( a \) for each value of \( d \). To obtain such a solution one need to solve the field equations with capacitance term as well as \( C_1 \) and \( C_2 \). While the former will be minimised for a bimeron of zero size \( (a = 0) \) the latter two terms will prefer a bimeron with infinite \( a \). Hence the relative strength of these terms will fix the size of the bimeron. One can ask if our calculations would reveal this by minimising the total energy at some particular \( a \). To see this, we have shown in Fig. 3.8 the total energy at \( d=0.7 \) (i.e. the sum of all three contributions plotted in the Fig. (3.8) as a function of bimeron separation \( a \).

As we can see from Fig. (3.8), the total energy keeps decreasing with
Figure 3.7: this figure gives a relative estimate of the contribution of the three type of terms in the action, namely, the local terms, $C_1$ and $C_2$, as a function of bimeron separation $a$. The units are as specified in the earlier figures. This data also corresponds to a layer separation $0.7l$. 

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a, all the way to about \( a = 3.2 \), which is the highest value up to which we could calculate, given limitations of our computing facilities. As we have not solved the equations including \( C_1 \) and \( C_2 \) but evaluate these terms with already obtained solutions, these data points are not quantitatively precise. However, the decrease is clearly levelling off and is indicating that a minimum may exist at around \( a = 4 \) or 5. What we have done, in drawing Fig. 3.8, is to obtain a best-fit-curve of the data points up to \( a = 3.2 \) and extrapolate that curve up to \( a = 4.5 \). For what it is worth such extrapolation indicates a minimum at about \( a = 4 \). This corresponds to a meron-antimeron separation of of about 8, larger than what Yang and MacDonald found by entirely different methods (see their fig. 2) \cite{23}. Their value of the meron separation for \( d = 0.7 \) is about 4.5. We attribute this discrepancy to the fact, noted already in our discussion of Fig. 3.6, that the Coulomb energy in our explicit calculation of the bimeron solution is higher than the undistorted meron pair estimate used in ref\cite{23}. The actual larger Coulomb repulsion is, we believe responsible for the larger optimal meron separation that we get.

We saw that the Coulomb interaction energy between the two merons as given by the term \( C_1 \) in the present calculation differs quite a bit from the simple picture of the bimeron as a pair of undistorted merons of charge \( \frac{\xi}{2} \) each. One can ask if there is a similar discrepancy in the non-Coulombic energy as well. This is the subject of Table 2. In the picture of a bimeron as a pair of merons \cite{25} \cite{23} \cite{34}, it will have energy equal to

\[
E_{\text{prev}} \equiv 2E_{mc} + 2\pi \rho d \ln \left( \frac{2a}{R_{mc}} \right) \tag{3.42}
\]

where \( E_{mc} \) and \( R_{mc} \) are respectively the core energy and radius of a single merons, which have a logarithmic interaction with each other because of the logarithmic divergence of the self energy of single merons. (As stated already
we are leaving out their Coulomb interaction in the comparison being done in this table.) This $E_{prev}$ has been calculated in our previous work [34]. It can be compared with the local part of the energy in the present calculation. Such a comparison is given in Table 2 for different values of $d$, using the optimal value of the meron separation $a$ which minimises $E_{local}$. We see that the comparison is not bad considering the completely different ways of estimating this energy in this paper and in earlier literature.

![Figure 3.8: A plot of the total energy $E(total)$ as a function of bimeron separation $a$, for a layer separation of 0.7. This curve was obtained by extrapolating the curve fitted to the calculated values going upto $a = 3.2$.](image-url)
Table 2: The optimal bimeron separation (a), the bimeron local energy ($E_{\text{local}}$) and meron pair energy ($E_{\text{prev}}$) from our previous work [34] as a function of the layer separation $d$. The unit of energy is $\frac{\varepsilon^2}{\ell}$ and the unit of length is $l$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$a$</th>
<th>$E_{\text{local}}$</th>
<th>$E_{\text{prev}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>3.30</td>
<td>.270</td>
<td>.217</td>
</tr>
<tr>
<td>0.6</td>
<td>3.16</td>
<td>.248</td>
<td>.226</td>
</tr>
<tr>
<td>0.7</td>
<td>2.72</td>
<td>.223</td>
<td>.224</td>
</tr>
<tr>
<td>0.8</td>
<td>2.39</td>
<td>.201</td>
<td>.214</td>
</tr>
</tbody>
</table>

In conclusion, our solution for the bimeron obtained by directly solving the coupled partial differential equations that the bimeron texture obeys provides an alternate way of obtaining the profiles and energies of these objects. As far as the local part of the energy is concerned, the results are in broad agreement with microscopic derivations earlier. But the Coulomb energy we obtain is higher by a factor of about 2 from earlier simple estimates because in actuality, the two merons in close proximity will not behave like undistorted symmetrical merons. This may be attributable to the possibility that a true bimeron solution does not constitute of two isolated single meron profiles but is distorted by each others presence.