CHAPTER 5

ROBUST STABILIZATION OF SWITCHED
UNCERTAIN SINGULAR SYSTEMS

5.1 INTRODUCTION

In the past few years, singular systems have been extensively studied by many researchers because they can better describe and analyze physical systems than the state-space systems, and have extensive applications in electrical circuits, power systems, economics and other areas (Xu and Lam 2006). In particular, time delay systems have received much attention due to the fact that time delays may lead to instability and poor performance (see Balasubramaniam et al 2012, Kwon and Park 2006, Kwon et al 2006). A number of important and interesting results have been developed for singular time delay systems (Balasubramaniam et al 2012, Dai 1989, Wu et al 2011, Wu 2012, Yue et al 2005). The stability problem for singular system with the discrete and distributed delays were studied in (Feng and Lam 2012, Wang et al 2009). Wu and Zhou (2008) investigated the problem of delay-dependent robust stabilization for uncertain singular systems with discrete and distributed delays by using LMI approach. Xu et al (2003) addressed the problem of robust $H_\infty$ control for uncertain continuous singular systems with state delay in which the singular system under consideration involves state time delay and time-invariant norm bounded uncertainty.

Switched system is an important special class of hybrid dynamical systems. Switched systems have many applications in control of mechanical systems,
the automotive industry, air traffic control, and many other fields. Switched systems consist of a family of continuous-time or discrete time subsystems and a rule specifying which subsystem will be activated at each instant time. The study on switched time-delay systems is important in both theory and practice and thus it has attracted the interest of many researchers (Sun et al 2009, Ma et al 2008). Xing and Min (2010) investigated the problem of robust exponential admissibility for a class of continuous-time uncertain switched singular systems with interval time varying delay by defining a properly constructed decay-rate-dependent Lyapunov function and the average dwell time approach. The robust stability and $H_{\infty}$ control problems for discrete time uncertain singular time delay systems under arbitrary switching are discussed by using the switched Lyapunov function method in Ma et al (2008).

On the other hand, the problem of stabilization and $H_{\infty}$ control becomes theoretical and practical importance because the $H_{\infty}$ control design expresses the control problem as a mathematical optimization problem for finding the controller solution. Recently, the stability analysis and robust $H_{\infty}$ control problem for uncertain singular time-delay systems has been studied in (Boukas and Muthairi 2006, Wu et al 2010, Phat 2009, Phat and Ha 2009). The delay-dependent $H_{\infty}$ control problem for singular time-delay systems has been investigated in (Boukas and Muthairi 2006, Yang and Zhang 2005) and several sufficient conditions for the solvability of this problem were formulated by LMI approach.

Further, an important index for checking the conservatism of stability criteria is to find the maximum delay bounds. In this regard, the choice of a Lyapunov-Krasovskii functional and the derivation of a stability condition from the considered Lyapunov functional play an important role in the reduction of the conservatism of stability criteria (Kwon 2011). Recently, the delay partitioning or fractioning technique becomes an increasing interest of many researchers because it provides less conservative results while studying the dynamical behaviors of systems via LMI approach when the fractioning number goes thinner. Some results using the delay fractioning technique are reported in (Feng and Lam 2012, Sakhivel et al 2012). Feng et al (2011) studied the problem of delay-dependent $\alpha$- dissipativity analysis for continuous time
singular systems with time-delay using the delay fractioning technique.

Motivated by this consideration, in this chapter we consider the robust $H_\infty$ control problem for a class of switched singular system with mixed time delays. The main objective of this chapter is to obtain a robust $H_\infty$ controller design such that the resulting closed loop system is admissible for a given disturbance attenuation level $\gamma > 0$. The parameter uncertainties are assumed to be norm bounded. A new set of sufficient conditions are derived for achieving the required result of the considered switched singular system based on the delay fractioning approach together with the piecewise Lyapunov-Krasovskii functional.

5.2 PRELIMINARIES AND SYSTEM DESCRIPTION

Consider the following uncertain switched singular system with mixed time delays described by,

$$
\begin{align*}
E \dot{x}(t) &= (A_{\sigma(t)} + \Delta A_{\sigma(t)}(t))x(t) + (B_{\sigma(t)} + \Delta B_{\sigma(t)}(t))x(t - \tau(t)) + (D_{\sigma(t)} + \Delta D_{\sigma(t)}(t))u(t) + (C_{\sigma(t)} + \Delta C_{\sigma(t)}(t)) \int_{t-h}^{t} x(s) ds + B_{\sigma(t)}w(t), \\
\dot{z}(t) &= G_{\sigma(t)}x(t) + H_{\sigma(t)}u(t) + J_{\sigma(t)}w(t), \\
x(t) &= \phi(t), \text{ for every } t \in [-\tau, 0], \tau = [\tau_2, h],
\end{align*}
$$

(5.2.1)

where $\sigma(t) = i$ is a piecewise constant function of time called a switching signal, which takes its values in the finite set $\mathcal{S} = \{1, 2, \ldots, k\}$; where $k > 1$ is the number of subsystems; $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^r$ is the control input vector; $w(t) \in \mathbb{R}^p$ is the disturbance input vector; $z(t) \in \mathbb{R}^q$ is the controlled output vector; $A_i$, $B_i$, $C_i$, $D_i$, $B_{wi}$, $G_i$, $H_i$ and $J_i$ are known real constant matrices with appropriate dimensions; $E \in \mathbb{R}^{n \times n}$ may be singular with rank$(E) = r \leq n$; $\phi(t)$ is the initial function; $\tau(t)$ denotes the time varying delay satisfying,

$$
0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \quad \dot{\tau}(t) \leq \eta < 1,
$$

(5.2.2)

where $\tau_1$, $\tau_2$ and $\eta$ are positive constants representing minimum, maximum time delay and time derivative delay limit respectively; $h > 0$ represents the constant distributed
delay. Here the time varying delay is represented into two parts: constant part and
time-varying part as in Hu et al (2008), that is, \( \tau(t) = \tau_1 + \tau^*(t) \), and \( \tau^*(t) \) satisfies
\( 0 \leq \tau^*(t) \leq \tau_2 - \tau_1 \).

The parameter uncertainties \( \Delta A_i(t), \Delta B_i(t), \Delta C_i(t), \Delta D_i(t) \) are time
varying matrices with appropriate dimensions and defined as,

\[
\begin{bmatrix}
\Delta A_i(t) & \Delta B_i(t) & \Delta C_i(t) & \Delta D_i(t)
\end{bmatrix}
= M_i F_i(t)
\begin{bmatrix}
N_{1i} & N_{2i} & N_{3i} & N_{4i}
\end{bmatrix},
\]

(5.2.3)

where \( N_{1i}, N_{2i}, N_{3i}, N_{4i} \) and \( M_i \) are known constant matrices of appropriate dimen-
sions and \( F_i(t) \) is an unknown time varying matrix with Lebesgue measurable elements
bounded by, \( F_i^T(t)F_i(t) \leq I \).

The control law employed in this section to deal the problem of robust \( H_{\infty} \) control for the singular system (5.2.1) is

\[
u(t) = K_i x(t),
\]

(5.2.4)

where \( K_i \) is the gain matrix of the controller and defined as \( K_i = W_i V_i^{-1} \).

Then the robust switched singular system (5.2.1) can be written as,

\[
\begin{align*}
E \dot{x}(t) &= (A_i + \Delta A_i(t) + D_i K_i + \Delta D_i(t) K_i) x(t) + (B_i + \Delta B_i(t)) \\
&\quad x(t - \tau(t)) + (C_i + \Delta C_i(t)) \int_{t-\tau(t)}^{t} x(s) ds + B_i \omega(t), \\
\end{align*}
\]

(5.2.5)

\[
z(t) = (G_i + H_i K_i) x(t) + J_i \omega(t).
\]

Before stating the main result, we introduce the following useful definitions and lemmas.

**Definition 5.2.1.** The pair \( (E, A_i) \) is said to be

- regular if for each mode \( i \in \mathcal{S} \), \( \text{dct}(sE - A_i) \) is not identically zero.
- impulse free if for each mode \( i \in \mathcal{S} \), \( \text{deg}(\text{det}(sE - A_i)) = \text{rank } E \).
Definition 5.2.2. The singular system (5.2.5) is said to be

- regular and impulse free, if for each mode \( i \in \mathbb{S} \), the pairs \( (E, A_i), (E, (A_i + B_i)) \) are regular and impulse free.

- admissible, if it is regular, impulse free and stable.

Definition 5.2.3. Singular system (5.2.5) is said to be robustly stable with disturbance attenuation \( \gamma \) if for all \( w(t) \in L_2[0, \infty) \), the response \( z(t) \) under the zero initial condition, i.e., \( \phi(t) = 0 \) satisfies

\[
\int_0^\infty z^T(t)z(t)dt \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt.
\]

Lemma 5.2.4. (Mei 2009) Given matrices \( \phi = \phi^T \), \( M \), \( N \) and \( F(t) \) with appropriate dimensions \( \phi + MF(t)N + N^T F(t)^T M^T < 0 \) for all \( F(t) \) satisfying \( F(t)^T F(t) \leq I \), if and only if there exists a scalar \( \epsilon > 0 \) such that \( \phi + \epsilon MM^T + \epsilon^{-1} N^T N < 0 \).

5.2.1 \( H_\infty \) Controller Design

In this section, we prove the \( H_\infty \) performance problem for the singular system (5.2.5) when \( u(t) = 0 \). We consider the nominal form of singular system (5.2.5) as follows;

\[
E\dot{x}(t) = A_i x(t) + B_i x(t - \tau(t)) + C_i \int_{t-h}^t x(s)ds + B_{u_i}w(t),
\]

\[
z(t) = G_i x(t) + J_i w(t). \tag{5.2.6}
\]

More precisely, we apply the delay fractioning approach and the piecewise Lyapunov function technique to investigate the admissibility for the considered singular system.

Theorem 5.2.5. Given an integers \( l, m \geq 1 \), scalars \( \mu > 1 \) and \( 0 \leq \eta < 1 \), the singular system (5.2.6) is said to be admissible with the \( H_\infty \) performance level \( \gamma > 0 \), if there exists symmetric positive definite matrices \( P_{11}, R_{u_i}, u = 1, 2, 3, 4 \), \( Q_{vi}, S_{vi}, v = \ldots \)
1, 2, 3, any appropriately dimensioned matrices \( V_{vi}, T_{vi}, U_{vi}, \ v = 1, 2, 3 \) and constant matrix \( R \in \mathbb{R}^{n \times (n-r)} \) satisfying \( E^T R = 0 \) with \( \text{rank}(R) = (n - r) \), such that the following LMIs hold for \( i, j \in S, \ i \neq j \):

\[
\begin{bmatrix}
\theta + \theta_1 \quad \Theta_1^T \\
* & -I \\
* & * \quad S_{1i} \\
* & * & * \quad S_{2i}
\end{bmatrix}
\begin{bmatrix}
\sqrt{T_1} W_{ai} T_1^T \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\sqrt{T_2} - \tau_1 W_{1i} T_1^T \\
0 \\
0 \\
0 \\
0
\end{bmatrix} < 0, \quad (5.2.7)
\]

\[
\begin{bmatrix}
\theta + \theta_1 \quad \Theta_1^T \\
* & -I \\
* & * \quad S_{1i} \\
* & * & * \quad S_{2i}
\end{bmatrix}
\begin{bmatrix}
\sqrt{T_1} W_{ai} T_1^T \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\sqrt{T_2} - \tau_1 W_{1i} T_1^T \\
0 \\
0 \\
0 \\
0
\end{bmatrix} < 0, \quad (5.2.8)
\]

\[
P_{1i} \leq \mu P_{1i}, \quad R_{vi} \leq \mu R_{vi}, \quad Q_{vi} \leq \mu Q_{vi}, \quad S_{vi} \leq \mu S_{vi}, \quad (5.2.9)
\]

where

\[
\theta = W_{P_i}^T \bar{P}_i W_{P_i} + W_{Q}^T \bar{Q} W_{Q} + W_{R_1}^T \bar{R}_1 W_{R_1} + W_{R_2}^T \bar{R}_2 W_{R_2}
\]

\[
+ W_{S_1}^T \bar{S}_1 W_{S_1} + \text{sym}(W_{\alpha_1} \bar{V}^T W_V + W_{\alpha_1} T E W_T + W_{\alpha_1} U^T R^T W_u),
\]

\[
\bar{P}_1 = \begin{bmatrix}
0 & E^T P_{1i} \\
E^T P_{1i} & 0
\end{bmatrix}, \quad \bar{Q} = \text{diag} \{ Q_{1i}, -Q_{1i}, Q_{2i}, -Q_{2i}, Q_{3i}, -Q_{3i} \},
\]

\[
\bar{R}_1 = \text{diag} \{ R_{1i}, -R_{1i}, R_{2i}, -R_{2i} \}, \quad \bar{S}_1 = \text{diag} \{ S_{1i}, S_{2i}, S_{3i}, -S_{3i} \},
\]

\[
\bar{R}_2 = \text{diag} \{ R_{3i}, -R_{3i}, -R_{3i}, R_{4i}, -R_{4i}, -R_{4i} \}, \quad \Theta_1 = [ G_i \ 0_{n,(m+1+9)n} \ J_i ],
\]

\[
W_{\gamma} = \begin{bmatrix} 0_{n,(m+l+9)n} \ I_n \end{bmatrix}, \quad \bar{V} = [ V_{1i} \ V_{2i} \ V_{3i} ], \quad \theta_1 = \gamma^2 W_{\gamma}^T W_{\gamma},
\]

\[
W_{P_i} = \begin{bmatrix}
I_n & 0_{n,(m+1+9)n} \\
0_{n,(m+3)n} & I_n & 0_{n,(l+6)n}
\end{bmatrix},
\]

\[
W_{Q} = \begin{bmatrix}
I_{mn} & 0_{mn,(l+10)n} \\
0_{mn} & I_{mn} & 0_{mn,(l+9)n} \\
I_n & 0_{n,(l+m+9)n} & \sqrt{1 - \eta} I_n & 0_{n,(l+8)n} \\
0_{n,(m+1)n} & I_n & 0_{n,(l+k-9)n} & 0_{n,(l+7)n} \\
0_{n,(m+2)n} & I_n & 0_{n,(l+6)n} & 0_{n,(l+7)n}
\end{bmatrix},
\]

\[
W_{\alpha_1} = \begin{bmatrix}
I_n & 0_{n,(m+1+9)n} \\
0_{n,(m+1)n} & I_n & 0_{n,(l+8)n} \\
0_{n,(m+3)n} & I_n & 0_{n,(l+6)n}
\end{bmatrix}, \quad W_u = \begin{bmatrix} 0_{n,(m+3)n} \ I_n \ 0_{n,(l+6)n} \end{bmatrix},
\]
\[
W_{R_{12}} = \begin{bmatrix}
\sqrt{\tau_2 - \tau_1} I_n & 0_{n,(l+m+9)n} \\
0_{n,(m+8)n} & 1 I_n & 0_{n,(l+1)n} \\
\sqrt{\tau_2 - \tau_1} I_n & 0_{n,(l+m+9)n} \\
0_{n,(m+8)n} & 1 I_n & 0_{n,(l+1)n} \\
\sqrt{\tau_2 - \tau_1} I_n & 0_{n,(l+m+9)n} \\
0_{n,(m+7)n} & 1 I_n & 0_{n,(l+2)n}
\end{bmatrix},
\]

\[
W_{R_{11}} = \begin{bmatrix}
0_{tn,(m+5)n} & I_n & 0_{tn,5n} \\
0_{tn,(m+6)n} & I_n & 0_{tn,4n} \\
\sqrt{\frac{t}{n}} I_n & 0_{n,(l+m+9)n} \\
0_{n,(m+5)n} & \sqrt{\frac{t}{n}} I_n & 0_{n,(l+4)n}
\end{bmatrix},
\]

\[
W_{S_1} = \begin{bmatrix}
0_{n,(m+3)n} & \sqrt{\frac{t}{m}} I_n & 0_{n,(l+6)n} \\
0_{n,(m+3)n} & \sqrt{\tau_2 - \tau_1} I_n & 0_{n,(l+6)n} \\
0_{n,(m+3)n} & \sqrt{\frac{t}{n}} I_n & 0_{n,(l+6)n} \\
0_{n,(m+4)n} & \sqrt{\frac{t}{n}} I_n & 0_{n,(l+5)n}
\end{bmatrix},
\]

\[
W_T = \begin{bmatrix}
0_{n,(m+1)n} & I_n - I_n & 0_{n,(l+7)n} \\
0_{n,mn} & I_n - I_n & 0_{n,(l+8)n} \\
I_n - I_{mn} & 0_{n,(l+9)n}
\end{bmatrix},
\]

\[
W_V = \begin{bmatrix}
A_i & 0_{n,mn} & B_i & 0_n & -I_n & 0_n & C_i & 0_{n,(l+3)n} & B_{wi}
\end{bmatrix}.
\]

**Proof.** In order to prove that the switched singular system (5.2.6) is admissible, we first prove that the system (5.2.6) is regular and impulse free for any time varying delay \( \tau(t) \) satisfying \( \tau_1 \leq \tau(t) \leq \tau_2 \). From (5.2.7) and (5.2.8), it follows that

\[
\begin{bmatrix}
\Theta_{11} & \Theta_{12} & \Theta_{13} \\
* & \Theta_{22} & \Theta_{23} \\
* & * & \Theta_{33}
\end{bmatrix} < 0,
\]

where

\[
\begin{align*}
\Theta_{11} &= 2V_1^T A_i + 2T_{31} E, \quad \Theta_{12} = 2V_1^T B_i + 2A_i^T V_2 + 2T_{11} E - 2T_{21} E + 2E^T T_{32}^T, \\
\Theta_{13} &= E^T P_{11} - 2V_1^T + 2A_i^T V_3 + U_1^T R^T + 2E^T T_{32}^T, \quad \Theta_{22} = 2V_2^T B_i + 2T_{12} E \\
& \quad - 2T_{22} E, \quad \Theta_{23} = -2V_2^T + B_i^T V_3 + U_2^T R^T + 2E^T T_{13}^T - 2E^T T_{23}^T, \\
\Theta_{33} &= -2V_3^T + U_3^T R^T.
\end{align*}
\]
Let $V = \begin{bmatrix} I & 0 & A_i \\ 0 & I & B_i \end{bmatrix}$, pre and post multiplying (5.2.10) by $V$ and $V^T$ respectively yields

$$\Lambda_i = \begin{bmatrix} \Lambda_{i1} & \Lambda_{3i} \\ * & \Lambda_{3i} \end{bmatrix} < 0,$$

where

$$\Lambda_{i1} = \Theta_{11} + A_i^T \Theta_{13}^T + \Theta_{13} A_i + A_i^T \Theta_{33} A_i, \quad \Lambda_{2i} = \Theta_{12} + A_i^T \Theta_{23}^T + \Theta_{13} B_i + A_i^T \Theta_{33} B_i.$$

Since rank$(E) = r < n$, there must exists two non-singular matrix $\hat{G}$ and $\hat{H} \in \mathbb{R}^{n \times r}$ such that,

$$\hat{E} = \hat{G} E \hat{H} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $R$ can be parameterized as $R = \hat{G}^T \begin{bmatrix} 0 \\ \phi \end{bmatrix}$, where $\phi \in \mathbb{R}^{(n-r) \times (n-r)}$ is any non-singular matrix. Similarly, we define

$$\hat{\Lambda_i} = \hat{G} \Lambda_i \hat{H} = \begin{bmatrix} A_{11i} & A_{12i} \\ A_{21i} & A_{22i} \end{bmatrix}, \quad \hat{B}_i = \hat{G} B_i \hat{H} = \begin{bmatrix} B_{11i} & B_{12i} \\ B_{21i} & B_{22i} \end{bmatrix},$$

$$\hat{P}_{ii} = \hat{G} P_{ii} \hat{H} = \begin{bmatrix} P_{11i} & P_{12i} \\ P_{21i} & P_{22i} \end{bmatrix}, \quad \hat{S}_{1i} = \hat{G} S_{1i} \hat{H} = \begin{bmatrix} S_{11i} & S_{12i} \\ S_{21i} & S_{22i} \end{bmatrix},$$

$$\hat{S}_{2i} = \hat{G} S_{2i} \hat{H} = \begin{bmatrix} S_{21i} & S_{22i} \\ S_{21i} & S_{22i} \end{bmatrix}, \quad \hat{S}_{3i} = \hat{G} S_{3i} \hat{H} = \begin{bmatrix} S_{31i} & S_{32i} \\ S_{31i} & S_{32i} \end{bmatrix},$$

$$\hat{U} = \hat{H}^T U = \begin{bmatrix} \hat{U}_{1i} \\ \hat{U}_{2i} \end{bmatrix}.$$

Now, pre and post multiplying $\Lambda_{i1}$ by $\hat{H}^T$ and $\hat{H}$, we get

$$\Lambda_{i1} = \hat{H}^T \Lambda_{i1} \hat{H} = \begin{bmatrix} \hat{\Lambda}_{1i1} & \hat{\Lambda}_{12i} \\ * & \hat{\Lambda}_{22i} \end{bmatrix}.$$

(5.2.11)
From (5.2.11), it is easy to see that,

\[ \widehat{A}_{22i}^T \phi \widehat{U}_{21i}^T + \widehat{U}_{21i} \phi^T \widehat{A}_{22i} < 0 \]  

(5.2.12)

and thus \( \widehat{A}_{22i} \) is non-singular. Suppose that \( \widehat{A}_{22i} \) is singular, there must exist a non-zero vector \( \rho \in \mathbb{R}^{n-r} \), which ensures that \( \widehat{A}_{22i} \rho = 0 \). Thus we conclude that \( \rho^T (\widehat{A}_{22i}^T \phi \widehat{U}_{21i}^T + \widehat{U}_{21i} \phi^T \widehat{A}_{22i}) \rho = 0 \), this is contradicts (5.2.12). So \( \widehat{A}_{22i} \) is non-singular. Then, it can be shown that

\[
\det(zE - A_i) = \det(G^{-1}) \det(zE - \widehat{A}_i) \det(H^{-1}) \\
= \det(G^{-1}) \det \left[ (zI_r - \widehat{A}_{11i})(-\widehat{A}_{22i}) - \widehat{A}_{12i} \widehat{A}_{21i} \right] \det(H^{-1}) \\
= \det(G^{-1}) \det(-\widehat{A}_{22i}) \det \left[ (zI_r - \widehat{A}_{11i} + \widehat{A}_{12i} \widehat{A}_{21i} \widehat{A}_{22i}^{-1}) \right] \det(H^{-1})
\]

which implies that \( \det(zE - A_i) \) is not identically zero for each mode \( i \in \mathbb{S} \) and \( \det(zE - A_i) = r = \text{rank}(E) \). Thus, the pair \( (E, A_i) \) is regular and impulse free for each mode \( i \in \mathbb{S} \).

Next, we prove that the singular system (5.2.6) is asymptotically stable, for this we consider the following Lyapunov-Krasovskii functional based on the delay and integral fractioning idea:

\[ V(t, x(t)) = \sum_{i=1}^{5} V_i(t, x(t)), \]

(5.2.13)

where

\[
V_1(t, x(t)) = x^T(t) E^T P_3 E x(t), \\
V_2(t, x(t)) = \int_{t-\tau_m}^{t} \gamma_1^T(s) Q_1 \gamma_1(s) ds + \int_{t-\tau(t)}^{t} x^T(s) Q_2 x(s) ds \\
+ \int_{t-\tau_2}^{t} x^T(s) Q_3 x(s) ds, \\
V_3(t, x(t)) = \int_{t-\tau_m}^{t} \gamma_2^T(s) R_1 \gamma_2(s) ds + \int_{t-\tau_2}^{t} \int_{t+\theta}^{t} x^T(s) R_2 x(s) d\theta ds \\
+ \int_{t-\tau_2}^{t} x^T(s) R_3 x(s) ds, \\
V_4(t, x(t)) = \int_{t-\tau_m}^{t} \gamma_3^T(s) Q_1 \gamma_3(s) ds + \int_{t-\tau(t)}^{t} x^T(s) Q_2 x(s) ds \\
+ \int_{t-\tau_2}^{t} x^T(s) Q_3 x(s) ds, \\
V_5(t, x(t)) = \int_{t-\tau_m}^{t} \gamma_4^T(s) R_1 \gamma_4(s) ds + \int_{t-\tau_2}^{t} \int_{t+\theta}^{t} x^T(s) R_2 x(s) d\theta ds \\
+ \int_{t-\tau_2}^{t} x^T(s) R_3 x(s) ds.
\]
\( V_i(t, x(t)) = \int_{t_1^0}^{t} x^T(s) R_{3i} x(s) ds d\theta \)
\[ + \int_{t_1^0}^{t} x^T(s) R_{4i} x(s) ds d\theta, \]
\( V_i(t, x(t)) = \int_{t_1^m}^{t} x^T(s) E^T S_{1i} E x(s) ds d\theta \)
\[ + \int_{t_1^m}^{t} \dot{x}^T(s) E^T S_{2i} E \dot{x}(s) ds d\theta \]
\[ + \int_{t_1^m}^{t} \dot{x}^T(s) E^T S_{3i} E \dot{x}(s) ds d\theta, \]

with \( \gamma_1(t) = \left[ x^T(t), x^T \left( t - \frac{\tau_1}{m} \right), \ldots, x^T \left( t - \frac{m-1}{m} \tau_1 \right) \right], \)
\( \gamma_2(t) = \left[ \int_{t-rac{h}{l}}^{t} x^T(s) ds, \ldots, \int_{t-rac{(i-2)h}{l}}^{t-rac{(i-1)h}{l}} x^T(s) ds, \int_{t-rac{(i-1)h}{l}}^{t-rac{h}{l}} x^T(s) ds \right], \)

here \( l, m \geq 1 \) are number of fractions. Calculate the time derivatives \( \dot{V}_i(t, x(t)), i = 1, 2, 3, 4, 5 \) along the trajectories of the singular system (5.2.6), then we have

\begin{align*}
\dot{V}_1(t, x(t)) &= 2x^T(t) E^T P_{1i} E x(t), & (5.2.14) \\
\dot{V}_2(t, x(t)) &= \gamma_1^T(t) Q_{1i} \gamma_1(t) - \gamma_1^T \left( t - \frac{\tau_1}{m} \right) Q_{1i} \gamma_1 \left( t - \frac{\tau_1}{m} \right) \\
&\quad + x^T(t)(Q_{2i} + Q_{3i}) x(t) - (1 - \dot{\tau}(t)) x^T(t - \tau(t)) Q_{2i} x(t - \tau(t)) \\
&\quad - x^T(t - \tau_2) Q_{3i} x(t - \tau_2), & (5.2.15) \\
\dot{V}_3(t, x(t)) &= \gamma_2^T(t) R_{1i} \gamma_2(t) - \gamma_2^T \left( t - \frac{h}{l} \right) R_{1i} \gamma_2 \left( t - \frac{h}{l} \right) \\
&\quad + \frac{h}{l} \dot{x}^T(t) R_{2i} x(t) - \int_{t-rac{h}{l}}^{t} x^T(s) R_{2i} x(s) ds, & (5.2.16) \\
\dot{V}_4(t, x(t)) &= (\tau_2 - \tau_1) x^T(t) R_{3i} x(t) - \int_{t-	au_1}^{t} x^T(s) R_{3i} x(s) ds \\
&\quad + \tau_2 \dot{x}^T(t) R_{4i} x(t) - \int_{t-	au_2}^{t} x^T(s) R_{4i} x(s) ds, & (5.2.17) \\
\dot{V}_5(t, x(t)) &= \dot{x}^T(t) E^T \left( \frac{\tau_1}{m} S_{1i} + (\tau_2 - \tau_1) S_{2i} + \frac{h}{l} S_{3i} \right) E \dot{x}(t) \\
&\quad - \int_{t-rac{h}{l}}^{t} \dot{x}^T(s) E^T S_{1i} E \dot{x}(s) ds \\
&\quad - \int_{t-	au_1}^{t} \dot{x}^T(s) E^T S_{2i} E \dot{x}(s) ds - \int_{t-rac{h}{l}}^{t} \dot{x}^T(s) E^T S_{3i} E \dot{x}(s) ds. & (5.2.18)
\end{align*}
By applying Lemma 1.5.2, we can get the following inequalities

\[ - \int_{t-\frac{\tau}{2}}^{t} x^T(s) R_2 x(s) ds \leq -\frac{1}{h} \left[ \int_{t-\frac{\tau}{2}}^{t} x(s) ds \right]^T R_{2i} \left[ \int_{t-\frac{\tau}{2}}^{t} x(s) ds \right], \tag{5.2.19} \]

\[ - \int_{t-\tau(t)}^{t-\tau_1} x^T(s) R_3 x(s) ds = - \int_{t-\tau_2}^{t-\tau(t)} x^T(s) R_{3i} x(s) ds - \int_{t-\tau(t)}^{t-\tau_2} x^T(s) R_{3i} x(s) ds, \tag{5.2.20} \]

\[ - \int_{t-\tau(t)}^{t-\tau_1} x^T(s) R_3 i x(s) ds \leq -\frac{1}{\tau_2 - \tau_1} \left[ \int_{t-\tau_2}^{t-\tau(t)} x(s) ds \right]^T R_{3i} \left[ \int_{t-\tau_2}^{t-\tau(t)} x(s) ds \right], \tag{5.2.21} \]

\[ - \int_{t-\tau(t)}^{t} x^T(s) R_4 x(s) ds = - \int_{t-\tau_2}^{t-\tau(t)} x^T(s) R_{4i} x(s) ds - \int_{t-\tau(t)}^{t} x^T(s) R_{4i} x(s) ds, \tag{5.2.22} \]

\[ - \int_{t-\tau_2}^{t} x^T(s) R_{4i} x(s) ds \leq -\frac{1}{\tau_2 - \tau_1} \left[ \int_{t-\tau_2}^{t} x(s) ds \right]^T R_{4i} \left[ \int_{t-\tau_2}^{t} x(s) ds \right], \tag{5.2.23} \]

\[ - \int_{t-\tau(t)}^{t} x^T(s) R_4 i x(s) ds \leq -\frac{1}{\tau_2} \left[ \int_{t-\tau(t)}^{t} x(s) ds \right]^T R_{4i} \left[ \int_{t-\tau(t)}^{t} x(s) ds \right], \tag{5.2.24} \]

\[ - \int_{t-\tau(t)}^{t} x^T(s) R_4 i x(s) ds \leq -\frac{1}{\tau_2} \left[ \int_{t-\tau(t)}^{t} x(s) ds \right]^T R_{4i} \left[ \int_{t-\tau(t)}^{t} x(s) ds \right], \tag{5.2.25} \]

\[ - \int_{t-\tau(t)}^{t} \dot{x}^T(s) S_{3i} \dot{x}(s) ds \leq -\frac{1}{h} \left[ \int_{t-\tau(t)}^{t} \dot{x}(s) ds \right]^T S_{3i} \left[ \int_{t-\tau(t)}^{t} \dot{x}(s) ds \right]. \tag{5.2.26} \]

From (5.2.13) - (5.2.26), we have

\[
\dot{V}(t, x(t)) \leq W^T_{P_1} P_1 W_{P_1} + W^T_{Q} Q W_{Q} + W^T_{R_{11}} R_{11} W_{R_{11}} + W^T_{R_{12}} R_{12} W_{R_{12}} + W^T_{S_{1i}} S_{1i} W_{S_{1i}} - \int_{t-\tau(t)}^{t} \dot{x}^T(s) E^T S_{1i} E \dot{x}(s) ds - \int_{t-\tau_2}^{t} \dot{x}^T(s) E^T S_{2i} E \dot{x}(s) ds - \int_{t-\frac{\tau}{2}}^{t} \dot{x}^T(s) E^T S_{3i} E \dot{x}(s) ds. \tag{5.2.27}
\]
On the other hand, by the Newton–Leibniz formula, for any arbitrary matrices $T_{1i}$, $T_{2i}$, $T_{3i}$ and $U_i$ with compatible dimensions, we have

$$2\alpha^T(t)T_{1i} \left[ E(x(t - \tau(t)) - E(x(t - \tau_2)) - \int_{t-\tau_2}^{t-\tau(t)} E\dot{x}(s)ds \right] = 0,$$

$$2\alpha^T(t)T_{2i} \left[ E(x(t - \tau_1) - E(x(t - \tau(t))) - \int_{t-\tau_2}^{t-\tau(t)} E\dot{x}(s)ds \right] = 0,$$

$$2\alpha^T(t)T_{3i} \left[ E(x(t) - E(x(t - \frac{\tau_1}{m})) - \int_{t-\frac{\tau_1}{m}}^{t} E\dot{x}(s)ds \right] = 0,$$

$$(\tau_2 - \tau(t))\alpha^T(t)T_{4i}S_{2i}^{-1}T_{4i}^T \alpha(t) - \int_{t-\tau_2}^{t-\tau(t)} \alpha^T(t)T_{4i}S_{2i}^{-1}T_{4i}^T \alpha(t)ds = 0,$$

$$(\tau(t) - \tau_1)\alpha^T(t)T_{2i}S_{2i}^{-1}T_{2i}^T \alpha(t) - \int_{t-\tau(t)}^{t-\tau_1} \alpha^T(t)T_{2i}S_{2i}^{-1}T_{2i}^T \alpha(t)ds = 0,$$

$$(\frac{\tau_1}{m})\alpha^T(t)T_{3i}S_{3i}^{-1}T_{3i}^T \alpha(t) - \int_{t-\frac{\tau_1}{m}}^{t} \alpha^T(t)T_{3i}S_{3i}^{-1}T_{3i}^T \alpha(t)ds = 0.$$

Noting that $E^T R = 0$, we can deduce that

$$2\alpha^T(t)U_i^T R^T E\dot{x}(t) = 0.$$  \hspace{1cm} (5.2.34)

Also, for any matrix $V_i$ of appropriate dimensions the following inequality holds

$$2\alpha^T(t)V_i^T \left[ A_i x(t) + B_i x(t - \tau(t)) + C_i \int_{t-h}^{t} x(s)ds + D_i u(t) + B_{wi} w(t) - E\dot{x}(t) \right] = 0.$$  \hspace{1cm} (5.2.35)

Combining (5.2.27) - (5.2.35), we obtain,

$$\dot{V}(t, x(t)) \leq \zeta^T(t) \left\{ W_{P_i}^T \overline{P}_1 W_{P_i} + W_{Q}^T \overline{Q} W_{Q} + W_{R_{11}}^T \overline{R}_1 W_{R_{11}} + W_{R_{12}}^T \overline{R}_2 W_{R_{12}} + W_{S_{1}}^T \overline{S}_1 W_{S_{1}} + \text{sym}(W_{\alpha} \nabla^T V + W_{\alpha} T E W_T) + W_{\alpha} U^T R^T W_u \right\} \zeta(t) + (\tau_2 - \tau(t))\alpha^T(t)T_{1i}S_{2i}^{-1}T_{1i}^T \alpha(t) + (\tau(t) - \tau_1)\alpha^T(t)T_{2i}S_{2i}^{-1}T_{2i}^T \alpha(t) + \left( \frac{\tau_1}{m} \right) \alpha^T(t)T_{3i}S_{3i}^{-1}T_{3i}^T \alpha(t)$$
\[- \int_{t - \tau_2}^{t - \rho(t)} \{ T_{1i}\alpha(t) + S_{2i}E\dot{x}(s) \}^T S_{2i}^{-1} \{ T_{1i}\alpha(t) + S_{2i}E\dot{x}(s) \} \, ds \]

\[- \int_{t - \tau(t)}^{t - \tau} \{ T_{2i}\alpha(t) + S_{2i}E\dot{x}(s) \}^T S_{2i}^{-1} \{ T_{2i}\alpha(t) + S_{2i}E\dot{x}(s) \} \, ds \]

\[- \int_{t - \frac{1}{m}}^{t} \{ T_{3i}\alpha(t) + S_{1i}E\dot{x}(s) \}^T S_{1i}^{-1} \{ T_{3i}\alpha(t) + S_{1i}E\dot{x}(s) \} \, ds, \]

(5.2.36)

where

\[
\zeta^T(t) = \begin{bmatrix}
\gamma_1^T(t) x^T(t - \tau_1) x^T(t - \tau(t)) x^T(t - \tau_2) (E\dot{x}(t))^T \int_{t - \frac{1}{m}}^{t} \dot{x}^T(s) \, ds \\
\gamma_2^T(t) \int_{t - 2\tau}^{t} x^T(s) \, ds \int_{t - \tau(t)}^{t} x^T(s) \, ds \int_{t - \tau(t)}^{t - \tau} x^T(s) \, ds \int_{t - \tau(t)}^{t - \tau} x^T(s) \, ds \, w^T(t) \\
\alpha^T(t) = \begin{bmatrix}
x^T(t) x^T(t - \tau(t)) (E\dot{x}(t))^T
\end{bmatrix}.
\]

To discuss the \( H_\infty \) performance of the singular system (5.2.6), we introduce

\[
J_n = \int_{0}^{\infty} \left[ z^T(t)z(t) - \gamma^2 w^T(t)w(t) \right] \, dt. \tag{5.2.37}
\]

Under the zero initial condition we have \( V(0) = 0 \) and \( V(\infty) \geq 0 \),

\[
J_n \leq \int_{0}^{\infty} \left[ z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{V}(t, x(t)) \right] \, dt \leq \int_{0}^{\infty} \zeta^T(t) \Omega_1 \zeta(t) \, dt, \tag{5.2.38}
\]

where

\[
\Omega_1 = \theta + \theta_1 + \frac{\tau_1}{m} W_{a1}^T T_{3i} S_{1i}^{-1} T_{3i}^T W_{a1} + (\tau(t) - \tau_1) W_{a1}^T T_{2i} S_{2i}^{-1} T_{2i}^T W_{a1} + (\tau_2 - \tau(t)) W_{a1}^T T_{1i} S_{2i}^{-1} T_{1i}^T W_{a1}. \tag{5.2.39}
\]

By applying the idea of convex combination and (5.2.2), the term on the right-hand side of (5.2.39)
\[ \theta + \theta_1 + \frac{T_1}{m} W^T_{\alpha 1} T_{31} S_{-1}^{-1} T_{31}^T W_{\alpha 1} + (\tau(t) - \tau_1) W^T_{\alpha 1} T_{21} S_{-1}^{-1} T_{21}^T W_{\alpha 1} \\
+ (\tau_2 - \tau(t)) W^T_{\alpha 1} T_{11} S_{-1}^{-1} T_{11}^T W_{\alpha 1} < 0 \]

can be changed equivalently into the form

\[ \begin{align*}
\theta + \theta_1 + m W^T_{\alpha 1} T_{31} S_{-1}^{-1} T_{31}^T W_{\alpha 1} + (\tau_2 - \tau_1) W^T_{\alpha 1} T_{21} S_{-1}^{-1} T_{21}^T W_{\alpha 1} & < 0, \\
\theta + \theta_1 + m W^T_{\alpha 1} T_{31} S_{-1}^{-1} T_{31}^T W_{\alpha 1} + (\tau_2 - \tau_1) W^T_{\alpha 1} T_{11} S_{-1}^{-1} T_{11}^T W_{\alpha 1} & < 0. 
\end{align*} \] (5.2.40)

By Lemma 1.5.1, it is easy to see that (5.2.40) and LMIs (5.2.7) - (5.2.8) are equivalent, then we can obtain \( J_n \leq 0 \), that is

\[ \int_0^\infty \left[ z^T(t) z(t) - \gamma^2 w^T(t) w(t) \right] dt \leq 0 \]

\[ \int_0^\infty z^T(t) z(t) dt \leq \gamma^2 \int_0^\infty w^T(t) w(t) dt. \] (5.2.41)

Then based on the above discussion and Definition 5.2.3, we conclude that the switched singular system (5.2.6) with given performance attenuation level \( \gamma > 0 \) is stable. Hence the system (5.2.6) is admissible, which completes the proof of the Theorem 5.2.5.

Next, we extend the results obtained in Theorem 5.2.5 to uncertain singular system (5.2.5) when \( u(t) = 0 \).

**Theorem 5.2.6.** Given an integers \( k, m \geq 1 \), for scalars \( \mu > 1 \) and \( 0 \leq \eta < 1 \), the singular system (5.2.5) when \( u(t) = 0 \) is said to be admissible with the robust \( H_\infty \) performance level \( \gamma > 0 \), if there exists symmetric positive definite matrices \( P_{ii}, R_{ui}, u = 1, 2, 3, 4, Q_{vi}, S_{vi}, v = 1, 2, 3 \) and any appropriately dimensioned matrices \( V_{vi}, T_{vi}, U_{vi}, v = 1, 2, 3 \), constant matrix \( R \in \mathbb{R}^{n \times (n-r)} \) satisfying \( E^T R = 0 \) with \( \text{rank}(R) = (n-r) \) and positive scalars \( \epsilon \geq 0 \) such that the following LMIs hold.
for $i, j \in S, i \neq j$:

$$
\begin{bmatrix}
\Theta_3 & \Theta_1^T & \Theta_2^T & \sqrt{\frac{E}{m} W_{\alpha_1} T_{3i}^T} & \sqrt{\tau_2 - \tau_1 W_{\alpha_1} T_{2i}^T} \\
- I & 0 & 0 & 0 & 0 \\
* & * & - \epsilon & 0 & 0 \\
* & * & * & S_{1i} & 0 \\
* & * & * & * & S_{2i}
\end{bmatrix} < 0, \quad (5.2.42)
$$

$$
\begin{bmatrix}
\Theta_3 & \Theta_1^T & \Theta_2^T & \sqrt{\frac{E}{m} W_{\alpha_1} T_{3i}^T} & \sqrt{\tau_2 - \tau_1 W_{\alpha_1} T_{1i}^T} \\
- I & 0 & 0 & 0 & 0 \\
* & * & - \epsilon & 0 & 0 \\
* & * & * & S_{1i} & 0 \\
* & * & * & * & S_{2i}
\end{bmatrix} < 0, \quad (5.2.43)
$$

$$
P_{3i} \leq \mu P_{1j}, \quad R_{ui} \leq \mu R_{uj}, \quad Q_{ui} \leq \mu Q_{vj}, \quad S_{ui} \leq \mu S_{vj}, \quad (5.2.44)
$$

$$
\Theta_2 = \begin{bmatrix}
V_1 N_{1i} & 0 & V_2 N_{2i} & 0 & V_3 N_{3i} & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\Theta_3 = \theta + \theta_1 + \epsilon M_i M_i^T,
$$

and other parameters are defined as in Theorem 5.2.5.

**Proof.** The proof immediately follows from Theorem 5.2.5 by replacing $A_i, B_i$ and $C_i$ with $A_i + M_i F_i(t) N_{1i}, B_i + M_i F_i(t) N_{2i}$ and $C_i + M_i F_i(t) N_{3i}$ in Theorem 5.2.5, then applying Lemma 1.5.1 and Lemma 5.2.4, we can obtain (5.2.42) - (5.2.43). Hence, switched singular system is regular, impulse free and robustly stable. Hence the proof.

### 5.2.2 Robust $H_{\infty}$ Controller Design

In this section, we shall implement LMI approach to solve the robust $H_{\infty}$ control problem formulated in the previous section. In particular, we design a state feedback controller for the nominal form singular system (5.2.5) with disturbance attenuation level $\gamma > 0$. For this, we consider
\begin{align}
E\dot{x}(t) &= (A_i + D_i K_i)x(t) + B_i x(t - \tau(t)) + C_i \int_{t-h}^{t} x(s) ds \\
&\quad + B_{w_i} w(t),
\end{align}
\begin{align}
z(t) &= (G_i + H_i K_i)x(t) + J_i(t).
\end{align}

**Theorem 5.2.7.** Given an integers \( l, m \geq 1 \), for scalars \( \mu > 1 \), \( 0 \leq \eta < 1 \) and \( \lambda_{vi}, \; v = 1, 2, 3 \), the singular system (5.2.45) with the control (5.2.4) is said to be admissible with the disturbance attenuation level \( \gamma > 0 \), if there exists symmetric positive definite matrices \( P_{ii}, R_{ui}, u = 1, 2, 3, 4 \), \( Q_{vi}, S_{vi}, v = 1, 2, 3 \) and any matrices \( V_i, T_{vi}, U_{vi}, v = 1, 2, 3 \) and constant matrix \( R \in \mathbb{R}^{n \times (n-r)} \) satisfying \( E^T R = 0 \) with \( \text{rank}(R) = (n-r) \), such that the following LMIs hold for \( i, j \in S, \; i \neq j \):

\[
\begin{bmatrix}
\theta + \Theta_1 & \sqrt{\frac{\mu}{m}} W_{\alpha_1}^T T_{3i}^T & \sqrt{\gamma_2 - \tau_i} W_{\alpha_1}^T T_{2i}^T \\
* & -I & 0 \\
* & * & S_{1i} \\
* & * & * & S_{2i}
\end{bmatrix} < 0,
\tag{5.2.46}
\]

\[
\begin{bmatrix}
\theta + \Theta_1 & \sqrt{\frac{\mu}{m}} W_{\alpha_1}^T T_{3i}^T & \sqrt{\gamma_2 - \tau_i} W_{\alpha_1}^T T_{1i}^T \\
* & -I & 0 \\
* & * & S_{1i} \\
* & * & * & S_{2i}
\end{bmatrix} < 0,
\tag{5.2.47}
\]

\[
P_{ii} \leq \mu P_{jj}, \; R_{ui} \leq \mu R_{uj}, \; Q_{vi} \leq \mu Q_{vj}, \; S_{vi} \leq \mu S_{vj},
\tag{5.2.48}
\]

\[
W_V = \begin{bmatrix}
(A_i + D_i K_i)^T 0_{n, mn} B_i^T 0_n - I_n & 0_{n \times (l+3)n} (G_i + H_i K_i)^T \\
\end{bmatrix},
\]

\[
\bar{V} = [\lambda_1 V_i, \lambda_2 V_i, \lambda_3 V_i], \; \Theta_1 = [B_{w_i}^T, 0_{n, (m+l+8)n}, J_i^T].
\]

and other parameters are defined as in Theorem 5.2.5.

**Proof.** Since \( \det(sE - (A_i + B_i K_i)) = \det(sE^T - (A_i + B_i K_i)^T) \), the pair \( (E, (A_i + B_i K_i)) \) is regular and impulse free if and only if \( (E^T, (A_i + B_i K_i)^T) \) is regular and impulse free. Therefore the system (5.2.45) is equivalent to the system

\begin{align}
E^T \dot{x}(t) &= (A_i + D_i K_i)^T x(t) + B_i^T x(t - \tau(t)) + C_i^T \int_{t-h}^{t} x(s) ds \\
&\quad +(G_i + H_i K_i)^T w(t),
\end{align}
\begin{align}
z(t) &= B_{w_i}^T x(t) + J_i^T(t).
\end{align}
Then by replacing \( E, (A_i + D_i K_i), B_i, C_i, B_{ui}, (G_i + H_i K_i) \) and \( J_i \) in LMI (5.2.7) and (5.2.8), with \( E^T, (A_i + D_i K_i)^T, B_i^T, C_i^T, B_{ui}^T, (G_i + H_i K_i)^T \) and \( J_i^T \) respectively and setting \( V_{1i} = \lambda_{1i} V_i, V_{2i} = \lambda_{2i} V_i \) and \( V_{3i} = \lambda_{3i} V_i, \) and \( W_i = K_i V_i, \) we can obtain the LMIs (5.2.46) - (5.2.47) easily. This complete the proof.

Next, the robust \( H_{\infty} \) state feedback controller for the singular system (5.2.5) is designed.

**Theorem 5.2.8.** Given an integers \( l, m \geq 1, \) for scalars \( \mu > 1, 0 \leq \eta < 1 \) and \( \lambda_{vi}, \mu = 1, 2, 3, \) the singular system (5.2.5) with the control (5.2.4) is said to be admissible with the disturbance attenuation level \( \gamma > 0. \) If there exists symmetric positive definite matrices \( P_{1i}, R_{ui}, i = 1, 2, 3, 4, Q_{vi}, S_{vi}, \mu = 1, 2, 3, \) and any matrices \( V_i, T_{vi}, U_{vi}, \mu = 1, 2, 3, \) constant matrix \( R \in \mathbb{R}^{n \times (n-r)} \) satisfying \( E^T R = 0 \) with rank(\( R \) = (n - r)) and positive scalars \( \epsilon_{vi}, \mu = 1, 2, 3, \) such that the following LMIs hold for \( i, j \in \mathbb{S}, i \neq j:\)

\[
\begin{bmatrix}
\Theta_1 & \Theta_2^T & \Theta_3^T & \Theta_4^T & \Theta_5^T & \sqrt{\eta^2} \sqrt{\tau_2 - \tau_1} W_{vi}^T T_{vi}^T & \sqrt{\tau_2 - \tau_1} W_{vi}^T T_{vi}^T \\
* & -I & 0 & 0 & 0 & 0 & 0 \\
* & * & -\epsilon_{1i} & 0 & 0 & 0 & 0 \\
* & * & * & -\epsilon_{2i} & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_{3i} & 0 & 0 \\
* & * & * & * & * & S_{1i} & 0 \\
* & * & * & * & * & * & S_{2i}
\end{bmatrix} < 0, \quad (5.2.50)
\]

\[
\begin{bmatrix}
\Theta_1 & \Theta_2^T & \Theta_3^T & \Theta_4^T & \Theta_5^T & \sqrt{\eta^2} \sqrt{\tau_2 - \tau_1} W_{vi}^T T_{vi}^T & \sqrt{\tau_2 - \tau_1} W_{vi}^T T_{vi}^T \\
* & -I & 0 & 0 & 0 & 0 & 0 \\
* & * & -\epsilon_{1i} & 0 & 0 & 0 & 0 \\
* & * & * & -\epsilon_{2i} & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_{3i} & 0 & 0 \\
* & * & * & * & * & S_{1i} & 0 \\
* & * & * & * & * & * & S_{2i}
\end{bmatrix} < 0, \quad (5.2.51)
\]

\( P_{1i} \leq \mu P_{1j}, R_{ui} \leq \mu R_{uj}, Q_{vi} \leq \mu Q_{vj}, S_{vi} \leq \mu S_{vj}, \) \quad (5.2.52)

\[
\begin{align*}
\Theta_1 &= \theta + \theta_1 + \epsilon_{1i} M_i M_i^T + \epsilon_{2i} M_i M_i^T + \epsilon_{3i} M_i M_i^T, \\
\varsigma_1 &= N_{1i} V_i + N_{4i} W_i, \quad \Theta_3 = \left[ \lambda_{1i} \varsigma_1 \quad 0_{n, mn} \quad \lambda_{2i} \varsigma_1 \quad 0_{3n} \quad \lambda_{3i} \varsigma_1 \quad 0_{n,(l+4)n} \right],
\end{align*}
\]
$$
\Theta_4 = \begin{bmatrix}
\lambda_1 N_{2i} & 0_{n,mn} & \lambda_2 N_{2i} & 0_{3n} & \lambda_3 N_{2i} & 0_{n,(l+4)n}
\end{bmatrix},
$$

$$
\Theta_5 = \begin{bmatrix}
\lambda_1 N_{3i} & 0_{n,mn} & \lambda_2 N_{3i} & 0_{3n} & \lambda_3 N_{3i} & 0_{n,(l+4)n}
\end{bmatrix},
$$

$$
\nabla = [\lambda_1 V_i, \lambda_2 V_i, \lambda_3 V_i] \Theta_2 = \begin{bmatrix} B_w & 0_{n,(m+i+8)n} & J_i \end{bmatrix},
$$

$$
W_\nabla = \begin{bmatrix} A_i^T + K_i^T D_i^T & 0_{n,mn} & B_i^T & 0_n & C_i^T & 0_{n,(l+3)n} (G_i + H_i K_i)^T \end{bmatrix},
$$

and other parameters are defined as in Theorem 5.2.6.

**Proof.** The proof of this Theorem 5.2.8 immediately follows from Theorem 5.2.7 replacing $A_i^T, B_i^T, C_i^T$ and $D_i^T$ by $(A_i + M_i F_i(t) N_{1i})^T, (B_i + M_i F_i(t) N_{2i})^T, (C_i + M_i F_i(t) N_{3i})^T$ and $(D_i + M_i F_i(t) N_{4i})^T$ together with Lemma 1.5.1 and 5.2.4. The subsequent proof of this theorem is similar to Theorem 5.2.8 and we omit it.

**Remark 5.2.9.** In case of constant discrete time delay, the singular system (5.2.1) without switching phenomenon, control input and distributed time delays can be written as

$$
Ex(t) = (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - \tau) + B_w w(t) \quad (5.2.53)
$$

$$
z(t) = Cx(t),
$$

where the parameter uncertainties are defined as $[\Delta A(t) \Delta A_d(t)] = MF(t) [N_{0} \ N_{d}]$, in which $M, N_{0}$ and $N_{d}$ are known real constant matrices of appropriate dimensions with $F^T(t)F(t) \leq I$.

**Corollary 5.2.10.** Given an integer $l \geq 1$, the singular system (5.2.53) is said to be admissible, if there exists symmetric positive definite matrices $P, Q, R, S, Z$ and any matrices $U_n, V_n, T_n, n = 1, 2, \ldots, 5$ with appropriate dimensions, such that the following LMI holds,

$$
\Omega_5 = \begin{bmatrix}
\Theta & \sqrt{\frac{1}{\alpha_{1}T}} W_m T & \Theta_1^T \\
* & -S & 0 \\
* & * & -I
\end{bmatrix} < 0, \quad (5.2.54)
$$
where $R \in \mathbb{R}^{n \times (n-r)}$ is any matrix with full column and satisfy $ER = 0$ and

$$
\Theta = W^T_P \overline{P}_1 W_P + W^T_Q \overline{Q} W_Q + W^T_R \overline{R} W_R + W^T_S SW_S + \gamma^2 W^T_\gamma W_\gamma + \text{sym}(TW_T + VW_V + W_{\alpha_1} U^T R^T W_u),
$$

$$
\overline{P} = \begin{bmatrix}
0 & E^T P \\
E^T P & 0
\end{bmatrix},
\overline{Q} = \text{diag} \{Q, -Q\},
\overline{R}_1 = \text{diag} \{R, -R\},
$$

$$
W_S = \begin{bmatrix}
0_{n,(m+1)n} & \sqrt{\frac{n}{m}} I_n & 0_{n,2n}
\end{bmatrix},
W_P = \begin{bmatrix}
I_n & 0_{n,(m+3)n} \\
0_{n,(m+1)n} & I_n & 0_{2n}
\end{bmatrix},
$$

$$
W_Q = \begin{bmatrix}
I_{mn} & 0_{mn,(m+3)n} \\
0_{mn} & -I_{mn} & 0_{mn,(m+2)n}
\end{bmatrix},
W_\gamma = \begin{bmatrix}
0_{n,(m+3)n}, I_n
\end{bmatrix},
$$

$$
W_{\alpha_1} = \begin{bmatrix}
I_n & 0_{n,(m+3)n} \\
0_{n,(m+1)n} & I_n & 0_{mn,2n}
\end{bmatrix},
W_T = \begin{bmatrix}
I_n & -I_{mn} & 0_{n,3n}
\end{bmatrix},
$$

$$
W_R = \begin{bmatrix}
\sqrt{\frac{1}{m}} I_n & 0_{n,(m+3)n} \\
0_{n} & \sqrt{\frac{m}{n}} I_n & 0_{n,(m+2)n}
\end{bmatrix},
\Theta_1 = \begin{bmatrix}
C & 0_{n,(m+3)}
\end{bmatrix},
$$

$$
W_V = \begin{bmatrix}
A & A_d & -I_n & 0_{mn} & B_w
\end{bmatrix},
W_u = \begin{bmatrix}
0_{n,(m+1)n} & I_n & 0_{n,2n}
\end{bmatrix}.
$$

**Proof.** We consider the following Lyapunov-Krasovskii functional based on the idea of delay fractioning,

$$
V(t, x(t)) = \sum_{i=1}^{3} V_i(t, x(t)),
$$

where

$$
V_1(t, x(t)) = x^T(t) E^T P E x(t),
$$

$$
V_2(t, x(t)) = \int_{t-\frac{\tau}{m}}^{t} \gamma_1^T(s) Q \gamma_1(s) ds + \int_{t-\theta}^{t} \int_{t+i+\theta}^{t} x^T(s) R x(s) dsqd\theta,
$$

$$
V_3(t, x(t)) = \int_{t-\frac{\tau}{m}}^{t} \int_{t+\theta}^{t} \hat{x}^T(s) E^T S E \hat{x}(s) dsqd\theta.
$$

The proof of this corollary is similar to that of Theorem 5.2.5 with some changes and hence it is omitted.
Now we extend the results in Corollary 5.2.10 to uncertain singular time delay system (5.2.53), which yields the following corollary.

**Corollary 5.2.11.** Given an integer \( l \geq 1 \) the singular system (5.2.53) is said to be admissible with disturbance attenuation level \( \gamma > 0 \) if there exists a scalar \( \epsilon > 0 \), symmetric positive definite matrices \( P, Q, R, S, Z \) and any matrices \( V_n, V_n, T_n, n = 1, 2, \ldots, 5 \) such that the following LMI holds:

\[
\begin{bmatrix}
\Theta_3 & \sqrt{\frac{2}{m}}W_{\alpha_2}^TW_{\alpha_1}\Theta_1^T & \Theta_1^T \\
\ast & -S & 0 & 0 \\
\ast & \ast & -I & 0 \\
\ast & \ast & 0 & -\epsilon
\end{bmatrix} < 0,
\]

\( (5.2.56) \)

\[
\Theta_2 = \begin{bmatrix}
N_a & N_d & 0_{n,(m+2)n}
\end{bmatrix}, \quad \Theta_3 = \epsilon M^T M,
\]

where \( R \in \mathbb{R}^{n \times (n-r)} \) is any matrix with full column and satisfying \( E^T R = 0 \) and the remaining parameters are share the same expressions as those in (5.2.54).

### 5.3 NUMERICAL SIMULATION

In this section, numerical examples are given to demonstrate the applicability and less conservativeness of the proposed approach.

**Example 5.3.1.** Consider the switched singular system (5.2.5) when \( u(t) = 0 \) consisting of two modes with the parameter of each mode is given as follows:

**Mode 1:**

\[
A_1 = \begin{bmatrix}
-3.5 & -1.0 \\
1.4 & 1.2
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
3.2 & 0.1 \\
3.1 & -1.7
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
1.2 & 0.7 \\
1.3 & 1.4
\end{bmatrix},
\]

\[
G_1 = \begin{bmatrix}
0.7 & 0.4 \\
0.4 & 1.2
\end{bmatrix}, \quad J_1 = \begin{bmatrix}
0.3 & 0.7 \\
1.1 & 1.3
\end{bmatrix}, \quad B_{w_1} = \begin{bmatrix}
1.2 & 0.3 \\
1.5 & 1.2
\end{bmatrix},
\]

\[
N_{11} = N_{21} = N_{31} = 0.2 \ast I.
\]
Mode 2:

\[ A_2 = \begin{bmatrix} -3.7 & -1.9 \\ 1.3 & 1.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2.8 & 0.4 \\ 2.5 & -1.7 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1.4 & 0.9 \\ 1.5 & 1.8 \end{bmatrix}, \]

\[ G_2 = \begin{bmatrix} 0.8 & 0.5 \\ 0.7 & 0.1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1.3 & 0.7 \\ 1.5 & 1.8 \end{bmatrix}, \quad B_{w2} = \begin{bmatrix} 0.5 & 0.4 \\ 0.6 & 0.7 \end{bmatrix}, \]

\[ N_{12} = N_{22} - N_{32} = 0.2 * I. \]

**Table 5.1: Maximum allowable \( \tau_2 \)**

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 5.2.6 ( l = m = 1 )</td>
<td>1.02</td>
<td>1.01</td>
<td>1.00</td>
<td>0.99</td>
<td>0.97</td>
<td>0.95</td>
</tr>
<tr>
<td>Theorem 5.2.6 ( l = m = 2 )</td>
<td>1.25</td>
<td>1.21</td>
<td>1.17</td>
<td>1.12</td>
<td>1.07</td>
<td>1.01</td>
</tr>
</tbody>
</table>

**Table 5.2: Minimum allowable \( \gamma \)**

<table>
<thead>
<tr>
<th>( \tau_2 )</th>
<th>0.20</th>
<th>0.60</th>
<th>0.70</th>
<th>0.80</th>
<th>0.90</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 5.2.6 ( l = m = 1 )</td>
<td>2.7288</td>
<td>2.7291</td>
<td>2.7550</td>
<td>2.8014</td>
<td>2.8974</td>
<td>3.1846</td>
</tr>
<tr>
<td>Theorem 5.2.6 ( l = m = 2 )</td>
<td>2.0467</td>
<td>2.2772</td>
<td>2.3673</td>
<td>2.4839</td>
<td>2.6372</td>
<td>2.8440</td>
</tr>
</tbody>
</table>

If the fractioning numbers \( l \) and \( m \) are given, for scalars \( \tau_1, \mu, \eta \) and \( \gamma \), by solving the LMIs in Theorem 5.2.6 using MATLAB LMI tool box, one can easily obtain feasible solutions which are not given here due to the page constraint. If we set \( \mu = 1.05, \gamma = 5.0, \ h = 0.1 \) and \( \tau_1 = 0.1 \) the calculated maximum upper bound of discrete time delay \( \tau_2 \) for different values of \( \eta \) as shown in Table 5.1. The main purpose here is to obtain the minimum allowable \( \gamma \) for the stabilization of considered singular system. For this, if we set \( h = 0.1, \mu = 1.05 \) and \( \tau_1 = 0.1 \) the calculated minimum allowed \( \gamma \) for various value of \( \tau_2 \) are given in Table 5.2. It is clear that the calculated upper bound \( \tau_2 \) increases as the fractioning time \( l \) and \( m \) increases and also it is shown that the better disturbance attenuation effect is achieved (i.e. smaller values of \( \gamma \)) for fractioning time \( l = m = 2 \).
Example 5.3.2. Consider the uncertain switched singular system (5.2.5) consisting of two modes. The parameter of each mode are given as follows:

Mode 1:

\[
A_1 = \begin{bmatrix} 4.5 & -1.1 \\ 1.2 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1.1 & 0.5 \\ 1.2 & 1.5 \end{bmatrix}, \\
D_1 = \begin{bmatrix} -0.9 \\ 1.2 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.5 & 0.4 \\ 0.3 & 0.2 \end{bmatrix}, \quad H_1 = \begin{bmatrix} -0.9 \\ 1.2 \end{bmatrix}, \quad J_1 = \begin{bmatrix} -0.9 & 1.2 \end{bmatrix}, \\
B_{w_1} = \begin{bmatrix} -0.9 \\ 1.2 \end{bmatrix}, \quad N_{11} = N_{21} = N_{31} = 0.2 \times I, \quad N_{41} = \begin{bmatrix} -2.0 \\ 1.0 \end{bmatrix}.
\]

Mode 2:

\[
A_2 = \begin{bmatrix} 4.7 & -0.9 \\ 1.4 & 1.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2.5 & 0.1 \\ 2.8 & -1.3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1.3 & 0.7 \\ 1.3 & 1.6 \end{bmatrix}, \\
D_2 = \begin{bmatrix} -0.8 \\ 1.4 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.7 & 0.6 \\ 0.4 & 0.3 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.7 \\ 1.3 \end{bmatrix}, \quad J_2 = \begin{bmatrix} -0.8 & 1.2 \end{bmatrix}, \\
B_{w_2} = \begin{bmatrix} -0.6 \\ 1.6 \end{bmatrix}, \quad N_{12} = N_{22} = N_{32} = 0.2 \times I, \quad N_{42} = \begin{bmatrix} -2.0 \\ 1.0 \end{bmatrix}.
\]

For the given scalars $\tau_m$, $\mu$, $\eta$, $\gamma$, $l$ and $m$, by solving the LMIs in Theorem 5.2.8 using MATLAB LMI tool box, one can easily obtain feasible solutions which are not presented here due to the page constraint. If we set $\mu = 1.05$, $h = 0.1$ and $\tau_1 = 0.1$ the calculated maximum discrete time delay upper bound $\tau_2$ for different values of $\eta$ when $\gamma = 5.0$ and for different values of $\gamma$ when $\eta = 0.1$ are presented in Table 5.3 and Table 5.4. For the stability of the considered singular system, the obtained the

Table 5.3: Maximum $\tau_2$ for different values of $\eta$

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>0.1</th>
<th>0.15</th>
<th>0.2</th>
<th>0.25</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 5.2.8 $l = m = 1$</td>
<td>0.54</td>
<td>0.52</td>
<td>0.51</td>
<td>0.50</td>
<td>0.49</td>
</tr>
<tr>
<td>Theorem 5.2.8 $l = m = 2$</td>
<td>0.64</td>
<td>0.60</td>
<td>0.56</td>
<td>0.53</td>
<td>0.50</td>
</tr>
</tbody>
</table>
Figure 5.1: Switching signal for Example 5.3.2

minimum allowable $\gamma$ for various value of $\tau_2$ when $h = 0.1$, $\mu = 1.05$, $\eta = 0.1$ and $\tau_1 = 0.1$ are given in Table 5.5. It is clear that the calculated upper bound $\tau_2$ increases as the fractioning time $l$ and $m$ increases and also it is shown that the better disturbance attenuation effect is achieved for fractioning time $l = m = 2$. For instance, if we set

Table 5.4: Maximum $\tau_2$ for different values of $\gamma$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>13</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 5.2.8 $l = m = 1$</td>
<td>0.515</td>
<td>0.543</td>
<td>0.555</td>
<td>0.563</td>
<td>0.566</td>
<td>0.569</td>
</tr>
<tr>
<td>Theorem 5.2.8 $l = m = 2$</td>
<td>0.570</td>
<td>0.648</td>
<td>0.690</td>
<td>0.711</td>
<td>0.714</td>
<td>0.725</td>
</tr>
</tbody>
</table>

$\tau_2 = 0.64$, the obtained state feedback controller gains as follows,

$$K_1 = W_1 V_1^{-1} = \begin{bmatrix} -36.53 & -40.42 \end{bmatrix}, \quad K_2 = W_2 V_2^{-1} = \begin{bmatrix} -43.07 & -55.70 \end{bmatrix}.$$  

Table 5.5: Minimum allowable $\gamma$

<table>
<thead>
<tr>
<th>$\tau_2$</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
<th>0.35</th>
<th>0.40</th>
<th>0.45</th>
<th>0.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 5.2.8 $l = m = 1$</td>
<td>2.0201</td>
<td>2.2044</td>
<td>2.2044</td>
<td>2.2721</td>
<td>2.2721</td>
<td>2.3603</td>
<td>2.3603</td>
</tr>
<tr>
<td>Theorem 5.2.8 $l = m = 2$</td>
<td>1.8664</td>
<td>1.9249</td>
<td>1.9249</td>
<td>1.9787</td>
<td>1.9787</td>
<td>2.0563</td>
<td>2.0563</td>
</tr>
</tbody>
</table>
Figure 5.2: State responses of the switched singular system in Example 5.3.2 when $\tau_2 = 0.64$ without control variable

Figure 5.3: State responses of the switched singular system in Example 5.3.2 when $\tau_2 = 0.64$ with control variable
The simulation results are presented in Fig. 5.1, Fig. 5.2 and Fig. 5.3. Fig.5.1 depicts the simulated switching signal, Fig. 5.2 shows that state responses of the considered singular system in Example 5.3.2 without control for the initial condition $x(t) = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}^T$ while Fig. 5.3 depicts the state responses of the considered switched singular system with control. The simulation results reveals that the obtained controller gain makes the state $x(k)$ converging to the equilibrium point zero, so we conclude that the considered switched singular system in Example 5.3.2 is stable through the proposed controller gains.

Example 5.3.3. Consider singular time delay system (5.2.53) when $u(t) = 0$ with the parameters

$$E = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad A = \begin{bmatrix} -3.5 & -1 \\ -1 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 3 & 0 \\ 3 & -1.5 \end{bmatrix}, \quad C = \begin{bmatrix} 2.5 \\ 2 \end{bmatrix},$$

Case 1: When $B_w = 0$, if the fractioning number $l$ is given, by solving the LMI in Corollary 5.2.10 using MATLAB LMI tool box, one can easily obtain feasible solution which are not given here due to the page constraint. The calculated maximum discrete time delay upper bound $\tau$ for fractioning time $l = 2$ is presented in Table 5.6, which are compared with the results obtained in Mei (2009), Yang and Zhang (2005), Yue and Han (2004), Zhu et al (2005). It is observed that our results are much less conservative than the previous ones.

Case 2: When $B_w = \begin{bmatrix} 2 & 2.5 \end{bmatrix}^T$ and the parameter uncertainties are given as

<table>
<thead>
<tr>
<th>Table 5.6: Maximum allowable time delay $\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Methods</td>
</tr>
<tr>
<td>Yue and Han (2004) &amp; Zhu et al 2005</td>
</tr>
<tr>
<td>Yang and Zhang (2005)</td>
</tr>
<tr>
<td>Mei (2009)</td>
</tr>
<tr>
<td>Corollary 5.2.10 $l = 2$</td>
</tr>
</tbody>
</table>
\[
M = \begin{bmatrix}
0.1 & 0 \\
0 & 0.1 \\
\end{bmatrix}, \quad N_a = N_d = \begin{bmatrix}
0.02 & 0.02 \\
0.02 & 0.02 \\
\end{bmatrix}.
\]

If the fractioning number \( l \) is given, by solving the LMI in Corollary 5.2.11 using MATLAB LMI tool box, one can easily obtain feasible solutions. For the stability of the considered singular system, the calculated maximum discrete time delay upper bound \( \overline{\tau} \) and the obtained minimum allowable \( \gamma \) for fractioning number \( l = 1, 2 \) are presented in Table’s 5.7 & 5.8, which are compared with the results obtained in Zhong and Yang (2006), Mei (2009). It is observed that our results are much less conservative than Zhong and Yang (2006), Mei (2009).

**Table 5.7: Maximum allowable time delay \( \overline{\tau} \)**

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>3.5</th>
<th>4.0</th>
<th>4.5</th>
<th>5</th>
<th>5.5</th>
<th>6.0</th>
<th>6.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zhong and Yang (2006)</td>
<td>0.3009</td>
<td>0.3686</td>
<td>0.4067</td>
<td>0.4321</td>
<td>0.4504</td>
<td>0.4645</td>
<td>0.4756</td>
</tr>
<tr>
<td>Mei (2009)</td>
<td>0.7703</td>
<td>0.9035</td>
<td>0.9595</td>
<td>0.9894</td>
<td>1.0079</td>
<td>1.0207</td>
<td>1.0311</td>
</tr>
<tr>
<td>Corollary 5.2.11 ( l = 2 )</td>
<td>1.2459</td>
<td>1.2571</td>
<td>1.2658</td>
<td>1.2727</td>
<td>1.2783</td>
<td>1.2830</td>
<td>1.2869</td>
</tr>
</tbody>
</table>

**Table 5.8: Minimum allowable \( \gamma \)**

<table>
<thead>
<tr>
<th>( \overline{\tau} )</th>
<th>0.30</th>
<th>0.35</th>
<th>0.40</th>
<th>0.45</th>
<th>0.50</th>
<th>0.55</th>
<th>0.60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corollary 5.2.11 ( l = 1 )</td>
<td>0.7692</td>
<td>0.7692</td>
<td>0.8944</td>
<td>0.8944</td>
<td>1.0469</td>
<td>1.0469</td>
<td>1.2369</td>
</tr>
<tr>
<td>Corollary 5.2.11 ( l = 2 )</td>
<td>0.7257</td>
<td>0.7257</td>
<td>0.8264</td>
<td>0.8264</td>
<td>0.9479</td>
<td>0.9479</td>
<td>1.0980</td>
</tr>
</tbody>
</table>