Appendix A

Analytic formulation to calculate propagation of a charged particle in magnetic field

In the Kalman filter track fitting procedure (discussed in chapter 5), an extrapolation formula is used instead of the fourth-order Runge-Kutta method is implemented. The analytic formula for track extrapolation for a charged particle inside magnetic field is discussed here. This analytic formula expands the extrapolated track parameters in a power series of the magnetic field components. The position of a particle can be represented by its position coordinates (x, y), directions $t_x = dx/dz$, $t_y = dy/dz$, signed charge q, and momentum p. All these parameters form a state vector $\mathbf{r}(z) = (x, y, t_x, t_y, q/p)^T$. During particle transportation inside the detector volume, there is propagation of track parameters from one hit point $\mathbf{r}(z_0)$ to the new hit position $\mathbf{r}(z_p)$ i.e., $\mathbf{r}(z_0) \rightarrow \mathbf{r}(z_p)$. In addition to the state vector $\mathbf{r}$, the covariance matrix $C = \langle(\mathbf{r} - \langle\mathbf{r}\rangle).(\mathbf{r} - \langle\mathbf{r}\rangle)^T\rangle$ needs to be extrapolated in the fitting routine. To extrapolate the covariance matrix, it is only required to derive the extrapolated track parameters $\mathbf{r}(z_p)$ on the initial track parameters $\mathbf{r}(z_0)$. This derivative is calculated in the form of jacobian i.e.,

$$J = \frac{d\mathbf{r}(z_p)}{d\mathbf{r}(z_0)} \tag{A.1}$$

When the jacobian is known, the covariance is extrapolated to the next layer by matrix multiplication

$$C(z_p) = JC(z_0)J^T \tag{A.2}$$

In following section the extrapolation of both the state vector and its' covariance is discussed.

A.1 Equation of motion

The equation of motion for a charged particle moving inside the magnetic field is governed by Lorentz force F and the expression for the force is following:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} = \kappa.q.\mathbf{v} \times \mathbf{B} \tag{A.3}$$

with momentum $\mathbf{p}[\mathbf{GeV/c}]$, signed charge $q[e]$ i.e. $q = \pm 1$ for μ^+ and μ^- respectively, magnetic field $\mathbf{B}[\mathbf{kG}]$ and the coefficient $\kappa[(\text{GeV/c})\text{kG}^{-1}\text{cm}^{-1}] = 2.9979.10^{-4}$. Since the lorentz force is directed perpendicular to the direction of motion of the charged particle, so $v = |\mathbf{v}|$ and the momentum $p = |\mathbf{p}|$ are constants and the time can be replaced by the trajectory length s as $dt = ds/v$:

$$\mathbf{p} = \mathbf{F} = \kappa.q.\mathbf{v} \times \mathbf{B}\ ds/v \tag{A.4}$$

Introducing a unit vector $\mathbf{e} = \mathbf{v}/v = \mathbf{p}/p$ in the above equation, it becomes

$$d\mathbf{e} = \kappa.q.\mathbf{e} \times \mathbf{B}.ds = \kappa.(q/p).\begin{pmatrix} e_yB_z - e_zB_y \\ e_zB_x - e_xB_z \\ e_xB_y - e_yB_x \end{pmatrix} ds \tag{A.5}$$

The equation of motion(A.5) can be used only for particle extrapolation to a certain path length s. However we would like to do the extrapolation at every hit point of the particle track. For that the variable in equation A.5 have to be replaced by the track parameters i.e., by $\mathrm{r}(z) = \mathbf{r}(x, y, dx/dz, dy/dz, q/p)$.

During extrapolation, there is propagation of track parameters occur from one hit point to the next hit i.e., $\mathbf{r}(z_0) \rightarrow \mathbf{r}(z_p)$, for that we need evaluate the derivative of $\mathbf{r}(z)$ with respect to z.

The differentials of the track directions ($t_x = e_x/e_z = \frac{dx}{dz}, t_y = e_y/e_z = \frac{dy}{dz}$) are :

$$\begin{aligned} dt_x &= (de_x e_z - e_x de_z)/e_z^2) \\ &= \kappa.(q/p).(e_y e_z B_z - e_z^2 B_y - e_x^2 B_y + e_x e_y B_x)/e_z^2.ds \\ &= \kappa.(q/p).\left(t_y B_z - (1+t_x^2)B_y + t_x t_y B_x\right).ds, \end{aligned} \tag{A.6}$$

$$\begin{aligned} dt_y &= \ldots \\ &= \kappa.(q/p).\left((1+t_y^2).B_x - t_x t_y.B_y - t_x.B_z\right).ds \end{aligned}$$

The path length s is replaced by $ds = \sqrt{1+t_x^2+t_y^2}.dz$ in equation A.6 and the derivatives[1] of the track parameters with respect to z will be:

$$\begin{aligned} x' &= t_x \\ y' &= t_y \\ t_x' &= \kappa.(q/p).\sqrt{1+t_x^2+t_y^2}.\left(t_x t_y.B_x - (1+t_x^2).B_y + t_y.B_z\right) \\ t_y' &= \kappa.(q/p).\sqrt{1+t_x^2+t_y^2}.\left((1+t_y^2).B_x - t_x t_y.B_y - t_x.B_z\right) \\ (q/p)' &= 0 \end{aligned} \tag{A.7}$$

[1]prime denotes derivative with respect to z

So the equation of motion becomes:

$$\frac{d\mathbf{r}(z)}{dz} = \begin{pmatrix} t_x \\ t_y \\ \kappa.(q/p).\sqrt{1+t_x^2+t_y^2}.\left(t_x t_y.B_x - (1+t_x^2).B_y + t_y.B_z\right) \\ \kappa.(q/p).\sqrt{1+t_x^2+t_y^2}.\left((1+t_y^2).B_x - t_x t_y.B_y - t_x.B_z\right) \\ 0 \end{pmatrix} = f(z,\mathbf{r}) \quad \text{(A.8)}$$

The differential equation is solved by the Runge-Kutta method, by solving the following differential equation:

$$\frac{d\mathbf{r}(z)}{dz} = \mathbf{f}(z,\mathbf{r}) \qquad \text{(A.9)}$$

The propagated track coordinates $x(z_p), y(z_p)$ can be calculated from the following expression:

$$\begin{aligned} x(z_p) &= x(z_0) + \int_{z_0}^{z_p} t_x(z)dz, \\ y(z_p) &= y(z_0) + \int_{z_0}^{z_p} t_y(z)dz \end{aligned} \qquad \text{(A.10)}$$

Hence to obtain the extrapolated track parameters, we need to focus only on the extrapolation of the directions t_x, t_y. From equation A.7, we can write:

$$\begin{aligned} t'_x &= \sum_{i_1=x,y,z} \mathbf{B}_{i_1}(z).a_{i_1}(z), \\ t'_y &= \sum_{i_1=x,y,z} \mathbf{B}_{i_1}(z).b_{i_1}(z) \end{aligned} \qquad \text{(A.11)}$$

Hence the derivatives of track directions are linearly dependent on the magnetic field and the multipliers $a_{i_1}(z), b_{i_1}(z)$ depend on the track directions t_x, t_y only can be represented by:

$$\begin{aligned} \mathbf{a}(z) &\equiv \kappa.(q/p).\sqrt{1+t_x^2+t_y^2}.\left(\ t_x t_y, \quad -(1+t_x^2), \quad t_y\right), \\ \mathbf{b}(z) &\equiv \kappa.(q/p).\sqrt{1+t_x^2+t_y^2}.\left(\ (1+t_y^2), \quad -t_x t_y, \quad -t_x\right) \end{aligned} \qquad \text{(A.12)}$$

and the magnetic field $\mathbf{B}(z)$ along the particle trajectory is:

$$\mathbf{B}(z) \equiv \mathbf{B}(x_{\text{track}}(z), y_{\text{track}}(z), z_{\text{track}}(z)) \equiv (B_x(z), B_y(z), B_z(z)). \quad \text{(A.13)}$$

The extrapolated track parameters can be represented by the following form:

$$\begin{aligned}
x(z_p) &= x(z_0) + \int_{z_C}^{z_p} t_x(z)dz, \\
y(z_p) &= y(z_0) + \int_{z_0}^{z_p} t_y(z)dz, \\
t_x(z_p) &= t_x(z_0) + \sum_{k=1}^{n} \sum_{i_1,\dots,i_k=x,y,z} t_{x_{i_1 \dots i_k}}(z_0) \cdot \left(\int_{z_0}^{z_p} B_{i_1}(z_1) \dots \int_{z_0}^{z_{k-1}} B_{i_k}(z_k) dz_k \dots dz_1 \right) \\
t_y(z_p) &= t_y(z_0) + \sum_{k=1}^{n} \sum_{i_1,\dots,i_k=x,y,z} t_{y_{i_1 \dots i_k}}(z_0) \cdot \left(\int_{z_0}^{z_p} B_{i_1}(z_1) \dots \int_{z_0}^{z_{k-1}} B_{i_k}(z_k) dz_k \dots dz_1 \right)
\end{aligned} \quad \text{(A.14}$$

The details about the intermediate steps will be available in Ref. [103]. The important thing to note here that in the above formulae the track directions are at initial position z_0 and the magnetic field components are integrated along the particle trajectory.

Now, the field integrals in the analytic expressions A.14 are calculated along the true particle trajectory, which remains unknown initially during the extrapolation process. It is assumed that the field derivatives along x and y-direction i.e., $\partial\mathbf{B}/\partial x, \partial\mathbf{B}/\partial y$ are negligible in the region around the particle trajectory and magnetic field change only along z-direction. And under this consideration the magnetic field in equation A.13 becomes:

$$(B_x(z), B_y(z), B_z(z)) \equiv \mathbf{B}(x_{true}(z), y_{true}(z), z) = \mathbf{B}(x_{approx}(z), y_{approx}(z), z) \quad \text{(A.15)}$$

i.e., the field integrals can be calculated along the approximate particle trajectory. Following new variables are introduced:

$$h = \kappa.(q/p).\sqrt{1 + t_x^2(z_0) + t_y^2(z_0)}, \quad \text{(A.16)}$$

$$
\begin{aligned}
A_{i_1...i_k} &= t_{x_{i_1,...,i_k}}(z_0)/h^k, & \text{(A.17)}\\
B_{i_1...i_k} &= t_{y_{i_1,...,i_k}}(z_0)/h^k, & \text{(A.18)}\\
s_{i_1...i_k} &= \int_{z_0}^{z_p} B_{i_1}(z_1)... \int_{z_0}^{z_{k-1}} B_{i_k}(z_k)dz_k...dz_1, & \text{(A.19)}\\
S_{i_1...i_k} &= \int_{z_0}^{z_p} B_{i_1}(z_1)... \int_{z_0}^{z_{k-1}} B_{i_k}(z_k)dz_k...dz_1. & \text{(A.20)}
\end{aligned}
$$

Among the above expressions, equations A.20 & A.20 represents the field integrals and other three variables will be used in the jacobian calculation. Introducing the above mentioned notations, the extrapolated track parameters become:

$$
\begin{aligned}
t_x(z_p) &= x(z_0) + t_x(z_0)(z_p - z_0) + \sum_{k=1}^{n} \sum_{i_1,...,i_k=x,y,z} h^k A_{i_1...i_k} S_{i_1...i_k},\\
t_y(z_p) &= y(z_0) + t_y(z_0)(z_p - z_0) + \sum_{k=1}^{n} \sum_{i_1,...,i_k=x,y,z} h^k A_{i_1...i_k} S_{i_1...i_k},\\
t_x(z_p) &= t_x(z_0) + \sum_{k=1}^{n} \sum_{i_1,...,i_k=x,y,z} h^k A_{i_1...i_k} s_{i_1...i_k},\\
t_x(z_p) &= t_x(z_0) + \sum_{k=1}^{n} \sum_{i_1,...,i_k=x,y,z} h^k B_{i_1...i_k} s_{i_1...i_k} & \text{(A.21)}
\end{aligned}
$$

During the extrapolation of track using the analytic expression A.21, one needs to calculate the field integrals $s_{i_1...i_k}, S_{i_1...i_k}$ along the particle trajectory.

For the extrapolation of the covariance matrix, one needs to calculate the jacobian. The extrapolation jacobian is:

$$
J = \begin{pmatrix}
1 & 0 & \partial x(z_p)/\partial t_x(z_0) & \partial x(z_p)/\partial t_y(z_0) & \partial x(z_p)/\partial(q/p)\\
0 & 1 & \partial y(z_p)/\partial t_x(z_0) & \partial y(z_p)/\partial t_y(z_0) & \partial y(z_p)/\partial(q/p)\\
0 & 0 & \partial t_x(z_p)/\partial t_x(z_0) & \partial t_x(z_p)/\partial t_y(z_0) & \partial t_x(z_p)/\partial(q/p)\\
0 & 0 & \partial t_y(z_p)/\partial t_x(z_0) & \partial t_y(z_p)/\partial t_y(z_0) & \partial t_y(z_p)/\partial(q/p)\\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad \text{(A.22)}
$$

Hence to calculate the jacobian from equation A.21, the derivatives of $A...$ and $B...$ with respect to $t_x(z_0)$, $t_y(z_0)$ are evaluated.

The coefficients $A_{i_1...i_k}, B_{i_1...i_k}$ and their derivatives are listed below, using the notations $t_x \equiv t_x(z_0)$, $t_y \equiv t_y(z_0)$, $A^{(x)} \equiv \partial A.../\partial t_x$, $A^{(y)} \equiv \partial A.../\partial t_y$, $B^{(x)} \equiv \partial B.../\partial t_x$, and $B^{(y)} \equiv \partial B.../\partial t_y$.

$$
\begin{pmatrix} A_x, & A_y, & A_z \\ A_{xx}, & A_{xy}, & A_{xz} \\ A_{yx}, & A_{yy}, & A_{yz} \\ A_{zx}, & A_{zy}, & A_{zz} \end{pmatrix} = \begin{pmatrix} t_x t_y, & -t_x^2 - 1, & t_y \\ t_x(3t_y^2+1), & -t_y(3t_x^2+1), & 2t_y^2+1 \\ -t_y(3t_x^2+1), & t_x(3t_x^2+3), & -2t_x t_y \\ (t_y^2 - t_x^2), & -2t_x t_y, & -t_x \end{pmatrix},
$$

$$
\begin{pmatrix} B_x, & B_y, & B_z \\ B_{xx}, & B_{xy}, & B_{xz} \\ B_{yx}, & B_{yy}, & B_{yz} \\ B_{zx}, & B_{zy}, & B_{zz} \end{pmatrix} = \begin{pmatrix} t_y^2+1, & -t_x t_y, & -t_x \\ t_y(3t_y^2+3), & -t_x(3t_y^2+1), & -2t_x t_y \\ -t_x(3t_y^2+1), & t_y(3t_x^2+1), & 2t_x^2+1 \\ -2t_x t_y, & (t_x^2 - t_y^2), & -t_y \end{pmatrix},
$$

$$
\begin{pmatrix} A_x^{(x)}, & A_y^{(x)}, & A_z^{(x)} \\ A_{xx}^{(x)}, & A_{xy}^{(x)}, & A_{xz}^{(x)} \\ A_{yx}^{(x)}, & A_{yy}^{(x)}, & A_{yz}^{(x)} \\ A_{zx}^{(x)}, & A_{zy}^{(x)}, & A_{zz}^{(x)} \end{pmatrix} = \begin{pmatrix} t_x y, & -2t_x, & 0 \\ (3t_y^2+1), & -6t_x t_y, & 0 \\ -6t_x t_y, & (9t_x^2+3), & -2t_y \\ -2t_x, & -2t_y, & -1 \end{pmatrix}, \qquad \text{(A.23)}
$$

$$
\begin{pmatrix} A_x^{(y)}, & A_y^{(y)}, & A_z^{(y)} \\ A_{xx}^{(y)}, & A_{xy}^{(y)}, & A_{xz}^{(y)} \\ A_{yx}^{(y)}, & A_{yy}^{(y)}, & A_{yz}^{(y)} \\ A_{zx}^{(y)}, & A_{zy}^{(y)}, & A_{zz}^{(y)} \end{pmatrix} = \begin{pmatrix} t_x, & 0, & 1 \\ 6t_x t_y, & -(3t_x^2+1), & 4t_y \\ -(3t_x^2+1), & 0, & -2t_x \\ 2t_y & -2t_x, & 0 \end{pmatrix},
$$

$$
\begin{pmatrix} B_x^{(x)}, & B_y^{(x)}, & B_z^{(x)} \\ B_{xx}^{(x)}, & B_{xy}^{(x)}, & B_{xz}^{(x)} \\ B_{yx}^{(x)}, & B_{yy}^{(x)}, & B_{yz}^{(x)} \\ B_{zx}^{(x)}, & B_{zy}^{(x)}, & B_{zz}^{(x)} \end{pmatrix} = \begin{pmatrix} 0, & -t_y, & -1 \\ 0, & -(3t_y^2+1), & -2t_y \\ -(3t_y^2+1), & 6t_x t_y, & 4t_x \\ -2t_y, & 2t_x, & 0 \end{pmatrix},
$$

$$\begin{pmatrix} B_x^{(y)}, & B_y^{(y)}, & B_z^{(y)} \\ B_{xx}^{(y)}, & B_{xy}^{(y)}, & B_{xz}^{(y)} \\ B_{yx}^{(y)}, & B_{yy}^{(y)}, & B_{yz}^{(y)} \\ B_{zx}^{(y)}, & B_{zy}^{(y)}, & B_{zz}^{(y)} \end{pmatrix} = \begin{pmatrix} 2t_y, & -t_x, & 0 \\ (9t_y^2+3), & -6t_xt_y, & -2t_x \\ -6t_xt_y, & (3t_x^2+1), & 0 \\ -2t_x, & -2t_y, & -1 \end{pmatrix}.$$

The analytic formula discussed here, is independent of the shape of the magnetic field. The general formula for track extrapolation is a bit cumbersome, however it becomes simple when a particular magnetic field makes many coefficients negligible. In our code we have used the following parameters:

$$s_y = \int_{z_0}^{z_p} B_y(0,0,z1)dz_1, \tag{A.24}$$

$$S_y = \int_{z_0}^{z_p}\int_{z_0}^{z} B_y(0,0,z1)dz_1dz, \tag{A.25}$$

$$\kappa = 2.99792458.10^{-4} \tag{A.26}$$

Here s_y, S_y are the field integrals and it is considered that the B_y component of magnetic field is most dominating and hence influence the particle trajectory.