Chapter - 6
CHAPTER 6
ON CERTAIN IDENTITIES INVOLVING q-SERIES

6.1 Introduction

In this chapter we establish certain Eta function identities which compliment the works of Berndt and Zhang [1] Bhargava and Somashekara [1]. Fine [1].

we shall use the following known transformation due to Bailey [6].

\[
\begin{align*}
2^{2y/2} & \begin{bmatrix} a,b;q;z \\ c,d \end{bmatrix} = \frac{[az,d/c,b,cd/abz;q]_\infty}{[z,d,q/b,cd/abz;q]_\infty} 2^{2y/2} \begin{bmatrix} a,abz/d;q,d/a \\ az,c \end{bmatrix}
\end{align*}
\]

(6.1.1)

to establish our results involving Eta-function.

Taking \( z = q/a \) and \( b = d/q \) in (6.1.1) we get after some simplification.

\[
\sum_{n=-\infty}^{\infty} \frac{[a,d/q;q]_n (q/a)^n}{[c,d;q]_n} = \frac{[q,d/a,cq/d,q;q]_\infty}{[q/a,d,q^2/d,c;q]_\infty}
\]

(6.1.2)

Now replacing \( d \) by \( dq \) in (6.1.2), we get

\[
\sum_{n=-\infty}^{\infty} \frac{[a;q]_n (q/a)^n}{[c;q]_n (1-dq^n)} = \frac{[q,dq/a,c/d,q;q]_\infty}{[q/a,d,q/d,c;q]_\infty}
\]

(6.1.3)

As \( a \to \infty \) in (6.1.3), we get

\[
\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^n (n+1/2) (1-dq^n)}{[c;q]_n} = \frac{[c/d;q]_\infty [q;q]_\infty^2}{[c,d,q/d;q]_\infty}
\]

(6.1.4)

Now, taking \( c = a \) in (6.1.3), we get
\[
\sum_{n=-\infty}^{\infty} \frac{q^n}{a^n(1-dq^n)} = \frac{[q, dq / a, a / d, q; q]_\infty}{[a, q / a, d, q / d; q]_\infty} 
\] (6.1.5)

Again, with \( c = aq \) in (6.1.3) we have
\[
\sum_{n=-\infty}^{\infty} \frac{(q / a)^n}{(1-aq^n)(1-dq^n)} = \frac{[q, dq / a, aq / d, q; q]_\infty}{[a, q / a, d, q / d; q]_\infty} 
\] (6.1.6)

If we set \( a = d \) in (6.1.6) we get
\[
\sum_{n=-\infty}^{\infty} \frac{(q / a)^n}{(1-aq^n)^2} = \frac{[q; q]_\infty^4}{[a, q / a; q]_\infty^2} 
\] (6.1.7)

If we replace \( q \) by \( q^2 \) and set \( a = q \) in (6.1.7), we get
\[
\sum_{n=-\infty}^{\infty} \frac{q^n}{(1-q^{2n+1})^2} = \frac{[q^2; q^2]_\infty^{-8}}{[q; q]_\infty^4} = \frac{\eta^8(2\tau)}{q^{1/2}\eta^4(\tau)} 
\] (6.1.8)

Again, replacing \( q \) by \( q^3 \) in (6.1.7) and then setting \( a = q \), we get
\[
\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{3n+1})^2} = \frac{[q^3; q^3]_\infty^{-6}}{[q; q]_\infty^2} = \frac{\eta^6(3\tau)}{q^{2/3}\eta^2(\tau)} 
\] (6.1.9)

Further, if we replace \( q \) by \( q^3 \) in (6.1.7) and take \( a = q \) and \( a = q^2 \), respectively in it. The resulting equations lead to the following interesting relation, after some simplification.
Next, if we replace $q$ by $q^2$ and then set $c = q^2$ and $d = q$ in (6.1.4), we get
\[
\left\{ \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{(1 - q^{5n+1})^2} \right\} \left\{ \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1 - q^{5n+2})^2} \right\} = \frac{\eta^{10}(5\tau)}{q^2 \eta^2(\tau)}
\] (6.1.10)

Next, if we replace $q$ by $q^2$ and then set $c = q^2$ and $d = q$ in (6.1.4), we get
\[
\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{q^2; q^2, (1 - q^{2n+1})} = \frac{\eta^2(2\tau)}{q^{1/8} \eta(\tau)}
\] (6.1.11)

Again, replacing $q$ by $q^2$ and taking $c = q$ and $d = q^{-1}$ in (6.1.4), we get
\[
\sum_{n=-\infty}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{q; q^2, (1 - q^{2n-1})} = -q^{5/8} \frac{\eta^6(2\tau)}{\eta(\tau)}
\] (6.1.12)

Now, if we replace $q$ by $q^3$ and then take $a = q$ and $d = q^2$ in (6.1.6), we get
\[
(1 - q) \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1 - q^{3n+1})(1 - q^{3n+2})} = \frac{\eta^3(3\tau)}{q^{1/3} \eta(\tau)}
\] (6.1.13)

Next, if we replace $q$ by $q^4$ and then take $a = q^2$ in (6.1.7), we get
\[
\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1 - q^{4n+2})^2} = \left[ \frac{q^4; q^4}{q^2; q^2} \right]_{\infty}^{8} = \frac{\eta^8(4\tau)}{q^4 \eta^4(2\tau)}
\] (6.1.14)

which could also be obtained by replacing $q$ by $q^2$ in (6.1.8).

Further, replacing $q$ by $q^4$ and then taking $a = q$ and $d = q^2$ in (6.1.6), we get
Next, if we replace $q$ by $q^4$ in (6.1.5) and then put $a = q$ and $d = q^2$ in it, we have

$$\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1 - q^{4n+1})(1 - q^{4n+2})} = \frac{\eta^4(4\tau)}{q^{1/2}(1 - q)\eta^2(2\tau)}$$

(6.1.16)

Again, replacing $q$ by $q^4$ and setting $a = q^2$ and $d = q$ in (6.1.5), we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1 - q^{4n+1})} = \frac{\eta^2(4\tau)}{q^{1/6}\eta^2(2\tau)}$$

(6.1.17)

Now, if we write $q^3$ for $q$ and set $a = q^2$ and $d = q$ in (6.1.5), we get

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{(1 - q^{3n+1})} = \frac{\eta^3(3\tau)}{q^{1/3}\eta(\tau)}$$

(6.1.18)

If we replace $c$ by 0 in (6.1.4), we get

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n q^n(n+1)/2}{(1 - dq^n)} = \frac{[q; q]^2_\infty}{[d, q / d; q]_\infty}$$

(6.1.19)

Which is a known result (cf. Andrews and Berndt [1:12.2.9.p.264]).

Now, if we write $q^2$ for $q$ in the above result with $d$ replaced by $q$, we get

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{(1 - q^{2n+1})} = \frac{[q^2; q^2]^4_\infty}{[q; q]^2_\infty} = \frac{\eta^4(2\tau)}{q^{1/4}\eta^2(\tau)}$$

(6.1.20)
Again, replacing $q$ by $q^3$ and then taking $d = q^1$ in (6.1.19), we get

\[
\frac{\sum_{n=-\infty}^{\infty} (-)^n q^{3(n+1)/2}}{1-q^{3n+2}} = \frac{\left[ q^3; q^3 \right]_\infty}{\left[ q; q \right]_\infty} = \frac{\eta^3(3\tau)}{q^{1/3}\eta(\tau)} \tag{6.1.21}
\]

Next, replacing $q$ by $q^5$ and setting $d = q$ in (6.1.19), we get

\[
\frac{\sum_{n=-\infty}^{\infty} (-)^n q^{5(n+1)/2}}{1-q^{5n+1}} = \frac{\left[ q^5; q^5 \right]_\infty}{\left[ q^4, q^4, q^4 \right]_\infty} \tag{6.1.22}
\]

Also, replacing $q$ by $q^5$ and taking $d = q^2$ in (6.1.19), we get

\[
\frac{\sum_{n=-\infty}^{\infty} (-)^n q^{5(n+1)/2}}{1-q^{5n+2}} = \frac{\left[ q^5; q^5 \right]_\infty}{\left[ q^2, q^3, q^4 \right]_\infty} \tag{6.1.23}
\]

Now, (6.1.22) and (6.1.23) yield the following interesting result involving $q$-series and continued fraction

\[
\frac{\sum_{n=-\infty}^{\infty} (-)^n q^{5n(n+1)/2}}{1-q^{5n+1}} = 1 + \frac{q}{1 + \frac{q}{1 + \frac{q}{1 + \ddots}}} \tag{6.1.24}
\]

Also (6.1.22) and (6.1.23) lead to

\[
\left\{ \frac{\sum_{n=-\infty}^{\infty} (-)^n q^{5n(n+1)/2}}{1-q^{5n+1}} \right\} \left\{ \frac{\sum_{n=-\infty}^{\infty} (-)^n q^{5n(n+1)/2}}{1-q^{5n+2}} \right\} = \frac{\left[ q^5; q^5 \right]_\infty}{\left[ q; q \right]_\infty} = \frac{\eta^5(5\tau)}{q\eta(\tau)} \tag{6.1.25}
\]

Now, $q$ replaced by $q^4$ and $d$ by $q$ in (6.1.19) yield
\[
\sum_{n=-\infty}^{\infty} \frac{(-)^n q^{2n(n+1)}}{1-q^{4n+1}} = \frac{[q^4; q^4]_\infty^2 [q^2; q^2]_\infty}{[q^2; q^2]_\infty^2} = \frac{\eta^2 (4\tau) \eta(2\tau)}{q^{3/8} \eta(\tau)} \quad (6.1.26)
\]

Now, \( q \) replaced by \( q^4 \) and \( d \) by \( q^2 \) in (6.1.19) lead to

\[
\sum_{n=-\infty}^{\infty} \frac{(-)^n q^{2n(n+1)}}{1-q^{4n+2}} = \frac{[q^4; q^4]_\infty^4}{[q^2; q^2]_\infty^2} = \frac{\eta^4 (4\tau)}{q^{1/2} \eta^2 (2\tau)} \quad (6.1.27)
\]

Finally, if \( q \) and \( c \) both replaced by \( q^2 \) and \( a \) and \( d \) each by \( q \), we get,

\[
\sum_{n=-\infty}^{\infty} \frac{\left[ q; q^2 \right]_n q^n}{q^2; q^2 \left( 1 - q^{2n+1} \right)} = \frac{\left[ q^2; q^2 \right]_\infty^4}{[q; q]_\infty} = \frac{\eta^4 (2\tau)}{q^{1/4} \eta^2 (\tau)} \quad (6.1.28)
\]

### 6.2 q-series transformations

In this section we shall discuss certain q-series transformations arising out
of the results of the section (6.1).

If we take into account the relations (6.1.9) and (6.1.18), we get

\[
\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{\left( 1 - q^{3n+1} \right)^2} = \left\{ \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{3n+1}} \right\}^2 \quad (6.2.1)
\]

Again, (6.1.8) and (6.1.11) lead to

\[
\sum_{n=-\infty}^{\infty} \frac{q^n}{\left( 1 - q^{2n+1} \right)^2} = \left\{ \sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{q^2; q^2 \left( 1 - q^{2n+1} \right)} \right\}^4 \quad (6.2.2)
\]

Also, (6.1.14) and (6.1.15) yield

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Further, comparing (6.1.14) and (6.1.27), we get

\[
\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{4n+2})^2} = \left\{ \frac{1-q}{1-q^{4n+1}} \frac{q^{3n}}{1-q^{4n+2}} \right\}^2 \quad (6.2.3)
\]

It will be worthwhile providing the direct proof of the identities established in Section (6.2).

It is evident that we can establish a large number of relations involving Eta-functions and q-series leading to q-series identities.