Elementary Excitations in Spin Ladder

2.1 Introduction

In one dimension (1d), the ground and low-lying excited states of the spin chain, in which nearest-neighbour (NN) spins of magnitude $\frac{1}{2}$ interact via the antiferromagnetic (AFM) Heisenberg exchange interaction, can be determined exactly using the Bethe Ansatz (BA) [1, 2]. The ground state has no long range order and the spin-spin correlations are characterized by a power-law decay. The elementary excitation is not the conventional spin-1 magnon but a pair of spin-$\frac{1}{2}$ excitations termed spinons. The physical origin of spinons can be best understood in the Ising limit of the exchange interaction Hamiltonian given by

$$H = \sum_{i=1}^{L} J_{x} S_{i}^{x} S_{i+1}^{x} + \sum_{i=1}^{L} \frac{J_{xy}}{2} \left( S_{i}^{+} S_{i+1}^{-} + S_{i}^{-} S_{i+1}^{+} \right) \tag{2.1}$$

with $J_{xy} = \varepsilon J_{x}$ as demonstrated in Section 1.2.2. In the Ising limit ($\varepsilon = 0$), the ground states of the Hamiltonian are the doubly degenerate Néel states, one of which is shown in Figure 2.1(a). An excited state is created by flipping a spin from its ground state arrangement, e.g., a down spin is flipped into an up spin in Figure 2.1(b). The flip gives rise to two domain walls (DWs) consisting of parallel spins and shown by dashed lines in the Figure [3, 4]. The transverse exchange interaction term in the Hamiltonian (2.1) interchanges the spins in an antiparallel spin pair and has the effect of making the DWs propagate independently (Figure 2.1(c)). Since the original spin flip carries spin-1,
each of the DWs or spinons has spin-$\frac{1}{2}$ associated with it. Spinons are thus examples of fractional excitations in an interacting spin system.

Spinons can be detected through inelastic neutron scattering in which neutrons scatter against spins to create spin flips. Due to energy and momentum conservation, the energy absorption spectrum for spin flips at different wave vectors can be measured. One observes a peak at a well-defined energy if the spin flip creates a single particle excitation. In the case of a pair of spinons, the total energy $\epsilon$ and momentum $k$ of the spin excitation are given by $\epsilon(k) = \epsilon_1(k_1) + \epsilon_2(k_2)$ and $k = k_1 + k_2$ where $\epsilon_i, k_i (i = 1, 2)$ denote individual spinon energy and momentum [3, 4]. The total momentum $k$ of the spin flip can be distributed in a continuum of ways among the spinons giving rise to a continuous absorption spectrum. For a single particle excitation, the energy versus momentum relation defines a single branch of excitations whereas for spinons a continuum of excitations with well-defined lower and upper boundaries is obtained. The compounds CsCoCl$_3$ and CsCoBr$_3$ are good examples of Ising-like Heisenberg antiferromagnets in 1d above the Néel temperature and provide evidence of the two-spinon continuum in neutron scattering experiments [4-6]. In the case of the isotropic Heisenberg Hamiltonian ($\varepsilon = 1$ in Equation (2.1)), the spinon spectrum has been clearly observed in the linear chain compound KCuF$_3$ [7] though a physical interpretation of spinons is not as straightforward as in the Ising-like case.

The existence of spinons, with fractional quantum number spin-$\frac{1}{2}$, is well established in 1d Heisenberg-type antiferromagnets. In higher dimensions, there are theoretical suggestions that quantum antiferromagnets with spin-liquid (SL) (no magnetic long range order and without broken symmetry) ground states may support elementary spinon-like excitations with fractional quantum numbers [8-10]. A well-known example is that of a resonating valence bond (RVB) state, a linear superposition of valence bond (VB) states, in which the spins are paired in singlet (VB) configurations. A broken VB gives rise to a pair of free spins which may propagate independently to give rise to spinon excitations. If the energetic cost of deconfinement is high, the spinons propagate as a bound pair (confinement) so that the elementary excitation has spin-1.

Despite considerable effort, there are few experimental evidences of spinon-like excitations in dimension $d > 1$ [11, 12]. One strong candidate is Cs$_2$CuCl$_4$, a spin-$\frac{1}{2}$ Heisenberg antiferromagnet defined on a spatially anisotropic triangular lattice [11]. The dynamical structure factor $S(k, \omega)$ for Cs$_2$CuCl$_4$, where $k$ and $\omega$ are the momentum and energy transfers in the neutron scattering experiment, is dominated by a broad continuum which has been cited as evidence for fractionalized excitations. Kohno et al [13] reanalyzed the neutron scattering data to show that the spinons are not characteristic of some exotic two-dimensional (2d) state but are descendants of the weakly-coupled excitations of individual chains in the material. The spectrum also has a sharp dispersing peak attributed to triplon bound states of the spinons. The bound pair lowers its kinetic energy through
2.1. Introduction

Figure 2.1: (a) Néel state; (b) A spin flip in the Néel state creates a pair of DWs or spinons the locations of which are shown by dashed lines; (c) The transverse exchange interaction term in the Hamiltonian (Equation (2.1)) gives rise to the propagation of independent spinons.

propagation between chains. The issue of fractional versus integer excitations has been extensively investigated in AFM spin-$\frac{1}{2}$ ladders [14–18]. A regular two-chain ladder consists of two AFM chains coupled by rung exchange interactions (as described by the Hamiltonian (1.8) with $J_{d,1} = J_{d,2} = 0$). The spinons of individual chains are confined even if the rung exchange interaction strength is infinitesimal. Ladders with strong rung exchange couplings suppress spinon excitations at all energy scales. Recently, Lake et al [19] have carried out neutron scattering experiments on a weakly-coupled ladder material, CaCu$_2$O$_3$, and shown that deconfined spinons at high energies evolve into $S = 1$ excitations at lower energies. The spinons are associated with individual chains whereas the $S = 1$ excitations are the triplon excitations, i.e., bound states of spinons. Two approaches are usually adopted in probing the nature of excitations in spin ladders: (i) the rung coupling strength dominates and (ii) the rung coupling strength is weaker than the intra-chain coupling strengths [19]. In this chapter, we consider a different case, not studied earlier, in which two $S = \frac{1}{2}$ Ising-like Heisenberg AFM chains, each of which is described by a Hamiltonian of the type shown in Equation (2.1), are coupled by Ising or Ising-like Heisenberg AFM exchange interactions. We investigate the dynamical properties of these models using the low-energy effective Hamiltonian (LEH) technique [20–22].
2.2 The LEH Technique

We now briefly introduce the LEH technique [20–22] used in this chapter to study the dynamical properties of a two-chain spin ladder. Let us consider a system described by a Hamiltonian

\[ H = H_0 + V \]  

acting in a Hilbert space \( \mathcal{H} \). Let the ground state of \( H_0 \) be highly degenerate with the Hilbert space of the ground state manifold being denoted by \( \mathcal{H}_0 \). These states constitute the low-energy subspace of the Hamiltonian \( H_0 \) with energy \( E_0 \) of each state. The objective of the LEH technique is to determine an effective Hamiltonian \( H_E \) such that

\[ H_E|\phi\rangle = E|\phi\rangle \Rightarrow H|\psi\rangle = E|\psi\rangle \]  

where \( |\phi\rangle \in \mathcal{H}_0 \) and \( |\psi\rangle \in \mathcal{H} \). Let us denote the low-energy manifold of \( H_0 \) by \( \{|p_i\}\} \). For the eigenstates \( \{|q_\alpha\}\} \) other than the low-energy eigenstates of \( H_0 \), \( H_0|q_\alpha\rangle = E_\alpha|q_\alpha\rangle \) such that \( E_\alpha \neq E_0 \). In general, perturbation lifts the degeneracy of the low-energy manifold leading to an effective Hamiltonian operating in the space of states associated with the low-energy manifold. Diagonalization of the \( n \)-th order \( (n = 1, 2, \ldots) \) effective Hamiltonian in the low-energy subspace of states reproduces the \( n \)-th order energy corrections to the low-energy unperturbed states. Using standard results in degenerate perturbation theory [23], the LEH, upto the second order and an overall constant, can be written as [20, 22]

\[ H_E = H_0 + H_E^{(1)} + H_E^{(2)} \]  

where \( H_E^{(1)} \) and \( H_E^{(2)} \) represent the first and second order LEH, respectively. The Hamiltonians describing frustrated quantum magnets can often be divided into two parts - the non-perturbative part and the kinetic part treated as perturbation. The non-perturbative part usually has highly degenerate ground states. Techniques such as the LEH technique contribute significantly to the analytical treatments of these frustrated
2.3 Ladder with Ising Rung Exchange Interactions

We consider an AFM two-chain spin ladder with the spins of magnitude $\frac{1}{2}$. The individual chains of the ladder are described by the Ising-Heisenberg Hamiltonian (Equation (2.1)). The chains are coupled by rungs with the corresponding exchange interactions being of the Ising-type. The ladder Hamiltonian $H^I$ is given by

$$H^I = \left( J_Z \sum_{\alpha=1}^{2} \sum_{i=1}^{L} S_{i,\alpha}^z S_{i+1,\alpha}^z + J_Z \sum_{i=1}^{L} S_{i,1}^z S_{i,2}^z \right) + \frac{J_{XY}}{2} \sum_{\alpha=1}^{2} \sum_{i=1}^{L} \left( S_{i,\alpha}^+ S_{i+1,\alpha}^- + S_{i,\alpha}^- S_{i+1,\alpha}^+ \right)$$

where the index $\alpha = 1(2)$ refers to the top (bottom) chain of the ladder, $i$ denotes the site index, $L$ is the total number of rungs and the superscript ‘I’ indicates that the rung interaction is of the Ising type. We also assume that the anisotropy constant $\varepsilon = \frac{J_{XY}}{J_Z}$ is $<< 1$. Hence, the Ising part of the Hamiltonian, $H_Z$, can be considered to be the unperturbed Hamiltonian with $H_{XY}$, containing the transverse exchange interactions, providing the perturbation. Since $H_Z$ is AFM in nature, the lowest energy states are the Néel states with NN spin pairs antiparallel.

2.3.1 Case I: Odd Number of Rungs

In the spirit of Villain [4], we first consider a ladder with an odd number of rungs, i.e., $L = \text{odd}$ and periodic boundary conditions (PBCs). Energies are measured w.r.t. that of a Néel configuration of spins. Since there are $3L$ NN spin pairs, the energy of a configuration in which all such pairs are antiparallel is $E_N = -3L \frac{J_{XY}}{4}$. Since $L$ is odd, a perfect Néel configuration is not possible and the lowest energy states of $H_Z$ contain a pair of parallel spin pairs which define the DWs or spinons (Figure 2.2(a)). The DWs form a bound pair to ensure minimal energy loss. Any other arrangement of DWs in the individual chains gives rise to higher energy states. The lowest energy states are $L$-fold degenerate as there are $L$ possibilities for the location of the bound pair which is an $S^z = 0$ object.

We next consider the effect of the perturbing Hamiltonian on the minimum energy states. The transverse exchange interaction can give rise to independent DW motion in the chains which, however, costs energy as a propagating DW leaves in its wake ferromagnetically aligned rung spins (Figure 2.2(b)). The energy cost increases as the distance between the DWs increases resulting in confinement of the DW pair. We investigate the dynamics of the DW pair perturbatively using the LEH technique [21, 22] as described.
Figure 2.2: (a) A pair of DWs in the minimum energy configuration of a ladder with an odd number $L$ of rungs; (b) Motion of the DW in the top chain leaves in its wake ferromagnetically aligned rung spins which raises the energy of the system.

in Section 2.2 with $V \equiv H_{XY}$. The $L$-fold degenerate DW pair states $|p_i\rangle$, $i = 1, ..., L$ (Figure 2.2(a)) constitute the low-energy manifold and have energy $E_0 = -\frac{3LJ_z}{4} + J_Z$. The higher energy states of $H_Z$ are denoted by $|q_\alpha\rangle$ with energy $E_\alpha$. The perturbing Hamiltonian $H_{XY}$ connects the low-energy manifold to the manifold of higher-energy states. Since the matrix element $\langle p_i | H_{XY} | p_j \rangle = 0$, the LEH $H_E^{(1)}$ is determined using the second-order expression in Equation (2.6). Two types of processes contribute to $H_E^{(2)}$. In diagonal processes, the spins in an antiparallel pair exchange and then re-exchange back to the original configuration ($|p_i\rangle = |p_j\rangle$). Such processes do not lift the degeneracy and give rise to a constant energy shift. We neglect this contribution in deriving the effective Hamiltonian. In the off-diagonal processes, the spins in two antiparallel pairs belonging to the four-spin plaquettes bordering the bound DW pair are interchanged. For the DW pair state shown in Figure 2.2(a), the intermediate states $|q_i\rangle$, $i = 1, ..., 4$ are given by

\begin{align}
|q_1\rangle &= \left| \uparrow\uparrow\downarrow\downarrow \uparrow\uparrow \downarrow\downarrow \right> \\
|q_2\rangle &= \left| \uparrow\downarrow\uparrow\downarrow \uparrow\uparrow \downarrow\uparrow \right> \\
|q_3\rangle &= \left| \uparrow\downarrow\uparrow\downarrow \uparrow\downarrow \uparrow\downarrow \right> \\
|q_4\rangle &= \left| \uparrow\downarrow\uparrow\downarrow \uparrow\downarrow \downarrow\uparrow \right> \\
\end{align}

(2.8)
Figure 2.3: Two successive applications of $H_{XY}$ on low-energy states shifts the location of the bound DW pair by two lattice constants.

The spin pairs deviated from the arrangement shown in Figure 2.2(a) are marked by double arrows. The perturbing Hamiltonian, $H_{XY}$, acting on the intermediate $|q_\alpha\rangle$ states shifts the location of the bound DW pair by two lattice constants either towards the left or the right. This is illustrated in Figure 2.3 for the state $|q_1\rangle$. The energy of the states $|q_\alpha\rangle$ ($\alpha = 1, \ldots, 4$) is $E_\alpha = -\frac{3J_{xy}}{4} + 2J_Z$. The second-order LEH is thus given by

$$H^{I,(2)}_E = -\frac{\varepsilon^2 J_Z}{2} \sum_i \left( |p_{i+2}\rangle \langle p_i| + |p_{i-2}\rangle \langle p_i| \right)$$

(2.9)

where $\varepsilon = \frac{J_{xy}}{J_Z}$. The off-diagonal processes are equivalent to ring exchanges involving four spins. The full second-order Hamiltonian, defined in the low-energy manifold, is thus given by

$$H'_E = H_Z + H^{I,(2)}_E = H_Z + H_R^I$$

(2.10)

where

$$H_R^I = J_R^I \sum_\square \left( S_{1}^+ S_{2}^- S_{3}^+ S_{4}^- + h.c. \right)$$

(2.11)

with $J_R^I = -\frac{\varepsilon^2 J_Z}{2}$ and the sum over all elementary plaquettes of the ladder. Ring or cyclic exchange interactions (Equation (2.11)) also appear in the perturbative effective Hamiltonian theories developed for the XXZ Heisenberg model on the checkerboard lattice [24] and in the case of an easy-axis Kagomé antiferromagnet [25]. In the ladder model, the ring exchange interaction has the effect of deconfining the bound DW pair. In the low-energy subspace, the dispersion relation of the bound pair can be determined.
in a straightforward manner. The eigenstate \( |\Psi\rangle \) of the pair can be written as a linear combination of states \( |\psi_j\rangle (j = 1, 2, \ldots, L) \) where \( j \) denotes the location of the bound DW pair,

\[
|\Psi\rangle = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{ikj}|\psi_j\rangle \tag{2.12}
\]

\( H^I_\perp \) (Equation (2.10)) operating on \( |\Psi\rangle \) yields the eigenvalue

\[
E_0(k) = J_Z(1 - e^2 \cos 2k) \tag{2.13}
\]

The subscript ‘\( b \)’ in \( E_b \) denotes that the dispersion relation is that of a bound DW pair.

### 2.3.2 Case II: Even Number of Rungs

We next consider a two-chain spin ladder with an even number \( L \) of rungs and described by the Hamiltonian in Equation (2.7) satisfying PBCs. One can show that the LEH in this case also is given by Equation (2.10). The low-lying excitation spectrum is obtained in the subspace of degenerate eigenstates of \( H_Z \) which are generated by flipping all the spins in a block of \( \mu \) (\( \mu \) may be odd/even with \( \mu < L \)) adjacent rungs in the ground state (Neel state) of \( H_Z \). Each of the states contains two bound DW pairs (Figure 2.4) and has energy

\[
E_{DW} = -\frac{3LJ_Z}{4} + 2J_Z \tag{2.14}
\]

The \( z \)-component of the total spin of each state, \( S^z = 0 \). We consider \( \mu \) to be odd with the degenerate eigenstates of \( H_Z \) given by

\[
|\psi_1\rangle = \sqrt{\frac{2}{L}} \sum_j e^{ikj} S_j |\psi_N\rangle
\]

\[
|\psi_3\rangle = \sqrt{\frac{2}{L}} \sum_j e^{ikj} S_j S_{j+1} |\psi_N\rangle
\]

\[
|\psi_5\rangle = \sqrt{\frac{2}{L}} \sum_j e^{ikj} S_j S_{j+1} S_{j+2} |\psi_N\rangle
\]

\[
|\psi_{L-1}\rangle = \sqrt{\frac{2}{L}} \sum_j e^{ikj} S_j \prod_{\mu=1}^{L-2} S_{j+2\mu-1} S_{j+2\mu} |\psi_N\rangle \tag{2.15}
\]

where \( S_j = S_{j,1}^+ S_{j,2}^- \) and \( S'_j = S_{j,1}^- S_{j,2}^+ \). \( |\psi_N\rangle \) is the ground state of \( H_Z \) shown in Figure 2.4(a). The Hamiltonian \( H^I_\perp \) in (2.10) has the following matrix elements:

\[
\langle \psi_1|H^I_\perp|\psi_0\rangle = \langle \psi_3|H^I_\perp|\psi_0\rangle = \cdots = J^I_{\perp} (1 + e^{-2ik}) \equiv v \tag{2.16}
\]
where $J'^{+}_{H} = -\frac{J Z}{2}$. The low-lying excited state of the Hamiltonian $H_{E}^{L}$ (Equation (2.10)) is given by

$$|\Psi\rangle_{DW} = \sum_{\nu=1}^{L/2} C\nu|\psi_{2\nu-1}\rangle$$  \hspace{1cm} (2.17)

From the eigenvalue equation $H_{E}^{L}|\psi\rangle_{DW} = \lambda|\Psi\rangle_{DW}$, one obtains

$$\sum_{\nu'=1}^{L/2} \langle\psi_{2\nu'-1}|H_{E}^{L}|\psi_{2\nu'-1}\rangle C_{\nu'} = \lambda C_{\nu}$$  \hspace{1cm} (2.18)

where

$$\langle\psi_{2\nu'-1}|H_{E}^{L}|\psi_{2\nu'-1}\rangle =
\begin{cases}
2J Z & \text{for } \nu' = \nu \\
\nu & \text{for } \nu' = \nu + 1 \\
\nu^* & \text{for } \nu' = \nu - 1 \\
0 & \text{otherwise}
\end{cases}$$  \hspace{1cm} (2.19)
The diagonal matrix element is $2J_z$ as energies are measured with respect to the Néel state energy $-\frac{3J_z^2}{4}$ (see Equation (2.14)). We choose the coefficients $C_\nu$'s to be $C_\nu = e^{-i\nu\alpha}$. The eigenvalues constitute an excitation continuum given by

$$\lambda = 2J_z \left[ 1 - \varepsilon^2 \cos k \cos (k + \phi) \right]$$  \hspace{1cm} (2.20)

where $-\pi < \phi \leq \pi$. Figure 2.5 shows the excitation continuum with upper and lower bounds given by $2J_z(1 \pm \varepsilon^2 \cos k)$. The degenerate eigenstates of $H_Z$ defined in (2.15) correspond to the top chain of the spin ladder being in a $S_{T,1}^z = +1$ and the bottom chain being in an $S_{T,2}^z = -1$ state. One can construct a set of degenerate eigenstates with the situation reversed. Also, $\mu$, the number of adjacent rungs constituting the block of flipped spins can be even ($\mu = 2, 4, 6, \ldots$ etc.). There are two distinct sets of such states [6, 26]. All these subspaces of states give rise to the same excitation continuum (Figure 2.5). The excitation continuum arises due to the motion of two walls each of which consists of a bound pair of DWs.

One notes that the effects of $H_{H_{1}}^{z}$, for the two-chain ladder, and $H_{XY}$, for the 1d chain, on the DW states are similar. In the first case, a bound pair of DWs shifts by two lattice constants and in the second case a single DW shifts by the same distance. In the case of a single chain (Hamiltonian (2.1)), the eigenvalues $\lambda_{1d}$ constituting the excitation continuum are

$$\lambda_{1d} = J_z \left[ 1 + 2\varepsilon \cos k \cos (k + \phi) \right]$$  \hspace{1cm} (2.21)

as demonstrated in Section 1.2.2. Comparing Equations (2.20) and (2.21), one finds that the spread of the continuum around the unperturbed level is less in the case of the spin.

---

Figure 2.5: Excitation continuum with energies given in Equation (2.20).
The dynamic form factor, \( S_b(k, \omega) \), associated with the bound DW pair is defined (at \( T = 0 \)) by

\[
S_b(k, \omega) = \sum_f |\langle f | A(k) | g \rangle|^2 \delta(\omega - E_f + E_g)
\]  

(2.22)

where

\[
A(k) = \frac{1}{\sqrt{L}} \sum_l A_l e^{i k l}; \quad A_l = (S^+_{l,1}S^-_{l,2} + S^-_{l,1}S^+_{l,2})
\]  

(2.23)

In Equation (2.22), \(|g\rangle\) and \(|f\rangle\) are the ground and excited states of \( H'_R \) connected by \( A(k) \), with energies \( E_g \) and \( E_f \) respectively, \( \omega \) and \( k \) are the frequency and wave number of the excitation. \( S_b(k, \omega) \), involving a pair of spin deviations, could be probed by the light-scattering techniques [27, 28]. Upto the first order of \( J_{11} \sim J_Z \) in Equation (2.10),

\[
|g\rangle \simeq |\psi^1_N\rangle + \frac{1}{E_0 - H_Z} H'_R |\psi^1_N\rangle
\]  

(2.24)

where \( E_0 \) is the energy of \( |\psi^1_N\rangle\). Since \( H'_R \) acting on \( |\psi^1_N\rangle\) creates two bound DW pairs, \( \frac{1}{E_0 - H_Z} = -\frac{1}{2J_Z} \). Thus,

\[
A(k)|g\rangle \simeq \frac{1}{\sqrt{2}} \left( 1 + \frac{\epsilon^2}{2} \cos k \right) |\psi_1\rangle + \frac{\nu^*}{\sqrt{2} - 2J_Z} |\psi_3\rangle
\]  

(2.25)

where \(|\psi_1\rangle\) and \(|\psi_3\rangle\) are as defined in Equation (2.15) and \( \nu^* = -\frac{\epsilon^2 J_Z}{2} \left( 1 + e^{2i k} \right) \). Using Equation (2.25) and (2.22) and the expression (2.17) for \(|f\rangle = |\Psi\rangle_{DW}\), one gets [5, 26]

\[
S_b(k, \omega) \simeq \frac{\sqrt{|\nu|^2 - \Omega^2 / 2}}{2\pi |\nu|^2} \left( 1 + \frac{\epsilon^2 \cos k - \Omega}{J_Z} \right) \quad \text{for} \quad \Omega < 2|\nu| \\
= 0 \quad \text{otherwise}
\]  

(2.26)

with \( \Omega = \omega - 2J_Z \). The expression (2.26) is similar to that for the dynamic structure factor \( S_{xx}(k, \omega) \) of the 1d chain (described by the Hamiltonian (2.1)) obtained in first order perturbation theory [5] (\( A_l \equiv S^\tau_1 \) in Equation (2.23)) except that in the latter case, \( \Omega = \omega - J_z \) and the contribution of the anisotropy term is to first order in \( \epsilon \). Figure 2.6 shows the plots of \( S_b(k, \omega) \times 2J_Z \cos k \) versus \( \frac{\nu}{J_Z} \) for \( \epsilon = 0.15 \) and for various values of the wave number \( k \). The lineshape is almost symmetric in contrast to the prominent asymmetry found in the 1d case [5].
2.4 Ladder with Ising-Heisenberg Rung Exchange Interactions

We now consider the case in which the rung exchange interactions of the two-chain spin ladder are Ising-like Heisenberg-type. The Hamiltonian is given by

\[
H^{IH} = \left\{ J_Z \sum_{\alpha=1}^{2} \sum_{i=1}^{L} S_{1,\alpha}^z S_{i+1,\alpha}^z + J_Z \sum_{i=1}^{L} S_{i,1}^z S_{i,2}^z \right\} \\
+ \left\{ \frac{J_{XY}}{2} \sum_{\alpha=1}^{2} \sum_{i=1}^{L} \left( S_{i,\alpha}^+ S_{i+1,\alpha}^- + S_{i,\alpha}^- S_{i+1,\alpha}^+ \right) + \frac{J_{XY}}{2} \sum_{i=1}^{L} \left( S_{i,1}^+ S_{i,2}^- + S_{i,1}^- S_{i,2}^+ \right) \right\} \\
= H_Z + H_{XY}
\]

The Hamiltonian (2.27) differs from \( H^I \) is Equation (2.7) by the addition of the last term and the superscript ‘\( IH \)’ indicates that the rung interaction is Ising-Heisenberg type. The ground states of \( H_Z \) are the doubly degenerate Néel states. We consider the Néel state \( |\psi_N^1\rangle \) shown in Figure 2.4(a). The ladder can be divided into two sublattices \( A \) and \( B \) such that in \( |\psi_N^1\rangle \) the \( A \) (\( B \)) sublattice spins are pointing up (down). The ground state energy \( E_0 = -\frac{3J_ZL}{4} \).

The lowest energy excitation of the unperturbed Hamiltonian is obtained by flipping
a single spin in either the \( A \) \((S^z_F = -1)\) or the \( B \) \((S^z_F = +1)\) sublattice. We consider the latter case with \(|i\rangle\) denoting the state in which the flipped spin is located in the \(i\)th rung (Figure 2.7(a)). These excited states are \(L\)-fold degenerate with the energy

\[
E' = -\frac{3LJ_z}{4} + \frac{3J_z}{2}
\]  

(2.28)

The perturbing Hamiltonian acting on the state \(|i\rangle\) generates the following states

\[
H_{XY}|i\rangle = \frac{\varepsilon J_z}{2} (|1\rangle + |2\rangle + |3\rangle + |4\rangle + \cdots)
\]  

(2.29)

where \(\varepsilon = \frac{J_{XX}}{J_z}\). The states \(|m\rangle (m = 1, \ldots, 4)\) are shown in Figure 2.7(b) with energy

\[
E_m = -\frac{3LJ_z}{4} + \frac{5J_z}{2}
\]  

(2.30)

The other states which are generated when \(H_{XY}\) acts on the state \(|i\rangle\) have higher energies and are hence not considered. \(H_{XY}\) acting on the states \(|m\rangle\) gives

\[
H_{XY}|m\rangle = \frac{\varepsilon J_z}{2} (|1\rangle + |2\rangle), \quad m = 1, 2
\]  

(2.31)

\[
H_{XY}|m\rangle = \frac{\varepsilon J_z}{2} (|3\rangle + |4\rangle), \quad m = 3, 4
\]

In first order perturbation theory there is no energy correction. A finite energy correction is obtained in the second order perturbation theory. The effective LEH (Equation (2.6)) with \(|p_i = |i\rangle\), \(|p_j = |j\rangle\), \(|q_m = |m\rangle\) and \(E_\alpha = E_m\) in the same order is given by

\[
H^{I.H.}_{E} = H_Z + \frac{\varepsilon^2 J_z}{2} \sum_i (|i\rangle \langle i-1| + |i\rangle \langle i+1|)
\]  

(2.32)

\[
= \frac{\varepsilon^2 J_z}{2} \sum_i \sum_{\delta = -1, 1} (S^+_{i,\delta} S^-_{i+\delta,2} + S^-_{i,\delta} S^+_{i+\delta,2})
\]  

(2.33)

The effect of this Hamiltonian on the low-energy excited state \(|i\rangle\) (Figure 2.7(a)) is to shift the flipped spin from the \(i\)th to the \((i + 1)\)th or the \((i - 1)\)th rungs. Since the flipped spins are located in the \(B\) sublattice, the shift is in the diagonal direction. The flipped spin is associated with a bound pair of DWs in a chain. The bound pair lowers its kinetic energy by propagating between chains. The full second-order Hamiltonian, defined in the low-energy manifold of states with single spin flips, is

\[
H^{I.H.}_E = H_Z + H^{I.H.}_{E,2}
\]  

(2.34)

As before, we have not included the terms arising from the diagonal processes in Equation (2.6) as they give rise to a constant energy shift. The low energy excited state with
2.4 Ladder with Ising-Heisenberg Rung Exchange Interactions

Figure 2.7: (a) The lowest energy excitation ($S_i^z = +1$) of the unperturbed Hamiltonian $H_Z$ in Equation (2.27). The deviated spin is represented by an encircled arrow; (b) The perturbing Hamiltonian, $H_{XY}$, acting on the state $|i⟩$ generates the states $|m⟩$ ($m = 1, 2, 3, 4$). The spin deviations from the state $|i⟩$ are shown by encircled arrows.

$S_i^z = +1$ can be constructed as

$$|Ψ⟩ = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{i k j} S_{j,B}^z |ψ^1_N⟩$$  \hspace{1cm} (2.35)$$

where ‘B’ denotes the $B$ sublattice. The dispersion relation for the propagation of the flipped spin or equivalently the bound DW pair is given by

$$E^{IH}(k) = J_Z (1 - ε^2 \cos k)$$  \hspace{1cm} (2.36)$$

where the energy is measured w.r.t. the Néel state energy. The bound DW pair moves diagonally across the spin ladder. The unperturbed Hamiltonian, $H_Z$, is the same irrespective of whether the rung exchange interactions are the Ising-type or the Ising-Heisenberg-type. Thus, the lowest unperturbed excited state is the single flip state in both the cases. The excitation has a localized character when the rung exchange interactions are of the Ising-type. The bound DW pair associated with the single spin-flip can not propagate between the chains as the inter-chain interactions are Ising-like and propagation of the DWs in a single chain is, as pointed out before, energetically prohibited. The energy, $-\frac{3J_Z}{2}$, of the localized excitation is lower than that of the propagating excitations involving two bound DW pairs when the rung exchange interactions are of the Ising-type. In this case, propagating excitations with the lowest energy involve two bound DW pairs rather than one.
2.5 Summary

AFM spin models in which the existence of spinons is well-established include the spin-$\frac{1}{2}$ Heisenberg AFM chain [3], the Majumdar-Ghosh (MG) model [29] and the Haldane-Shastry model [30, 31]. The physical picture of a spinon as a DW between two degenerate ground states emerges in the Ising-Heisenberg limit of the AFM Hamiltonian [3, 5, 6]. In the case of the MG model, the spin-$\frac{1}{2}$ excitation acts as a DW between the two dimerized ground states of the model [32, 33]. In a closed chain, the DWs occur in pairs so that the lowest-lying excitation is given by the two-spinon continuum. The spinons are deconfined in this case and can move away from each other. There is no energy cost in moving the spinons far apart. This is not so when two AFM chains are coupled in the form of a spin ladder described by the Hamiltonian (1.8) with $J_{d1} = J_{d2} = 0$. The spinon excitations of individual chains are confined by even an infinitesimal coupling strength $J_r$ [16, 19]. The two $S = \frac{1}{2}$ spinons form a bound state giving rise to singlet and triplet excitation branches. In the case of the strongly coupled ladder ($J_r >> J_i$), the elementary excitation is a triplet. Lake et al. [19] carried out neutron scattering experiments on the weakly-coupled ($J_r << J_i$) ladder material CaCu$_2$O$_3$ and obtained evidence of the singlet excitation mode. The spinon continuum was observed at high energies for which the chains are effectively decoupled. The spinons in a chain evolve into an $S = 1$ excitation at lower energies thereby confirming that the $S = 1$ triplon excitation is a bound state of two spinons and not a conventional magnon.

In this chapter, we have studied a two-chain $S = \frac{1}{2}$ AFM spin ladder in which the individual chains are described by the Ising-like Heisenberg Hamiltonian and the rung couplings are of the Ising-type. Using the LEH approach, we establish that in a ladder with an odd number of rungs the spinons (DWs) form a bound pair. A four-spin ring exchange interaction in the effective Hamiltonian is responsible for the delocalization of the bound pair. In the case of a ladder with an even number of rungs, the low-lying propagating excitation involves two bound pairs of DWs which can move away from each other giving rise to a continuum of excitations. The physical origin of the excitation continuum is similar to that in the case of the Ising-Heisenberg AFM chain in 1d except that in the former case the spinons form bound pairs and the dispersion of the excitation spectrum is a higher-order effect in perturbation theory. This results in an almost symmetric lineshape in the case of the dynamic structure factor $S_b(k, \omega)$ (Figure 2.6) in contrast to the asymmetry observed in the structure factor $S_{xx}(k, \omega)$ in 1d [5]. The delocalization of a bound spinon pair is brought about via a ring or a diagonal exchange interaction term in the effective Hamiltonian.

We have further considered a second model in which the rung exchange interactions are described by the Ising-like Heisenberg Hamiltonian. In this case also, the spinon pair in a single chain is bound and the bound pair lowers its kinetic energy by hopping
between chains. Kohno et al. [13] have studied a $S = \frac{1}{2}$ spatially anisotropic frustrated Heisenberg antiferromagnet in 2d in the weak interchain coupling regime. The model provides a good quantitative fit to the inelastic neutron scattering data of the triangular antiferromagnet Cs$_2$CuCl$_4$. The spectrum consists of a continuum arising from the deconfinement of spinons in individual chains and a sharp dispersing peak associated with the coherent propagation of a triplon bound state of two spinons between neighbouring chains. In the case of our model, the bound pair has $S^z = +1$. One can similarly construct an $S^z = -1$ excitation. In summary, we have studied ladder models with the Ising-like Heisenberg Hamiltonian describing the interactions in individual chains. The rung interactions may be pure Ising or Ising-like Heisenberg type. The ladder models studied in this chapter share some common features with models in which the exchange interactions are isotropic. In the latter case, two types of models have generally been considered: ladder models in which the rung exchange interactions are the most dominant and models which describe spin chains coupled by weak exchange interactions. The models considered in this chapter belong to a category not studied earlier and provide considerable physical insight on the origin of spinon confinement and how the bound spinon pairs delocalize. The isotropic models have, however, a richer dynamics with interactions generating cascades of virtual particles so that the two-body confinement problem becomes a many-body one [19]. The major common feature emerging from the study of ladder models with both Ising-like and isotropic exchange interactions appears to be the confinement of spinons in the form of bound states. The origin of excitation continua in specific cases lies in multi-triplon excitations rather than in the fractionalization of excitations [16].
Bibliography


