1.1 INTRODUCTION

In chapter - I we have seen that in a low-β plasma if kinetic effects like electron inertia are included shear Alfvén wave becomes dispersive and this has been called the shear kinetic Alfvén wave. But for a perturbation whose perpendicular wave-length $2\pi/k_\perp$ is comparable to ion-gyro-radius $\rho_i$, not only Alfvén wave but also slow magneto-acoustic wave (ion-acoustic wave) can be excited in a low-β plasma. Since for electromagnetic perturbations at a frequency much below ion-cyclotron frequency three waves can be expected to exist in this frequency range, namely the Alfvén waves and slow and fast magneto-acoustic waves. Of these three waves the fast magnetosonic mode, which is the isotropic mode, has a frequency comparable to or larger than the ion-cyclotron frequency for such a short wavelength (Hasegawa 1977).

If we start from the governing equations consisting of the ion continuity equation, the parallel component of ion equation of motion, the Boltzmann distribution for electron number density, the electron continuity equation, the quasi-neutrality condition and Ampere’s law in parallel direction, in which we keep both $\mathbf{E} \times \mathbf{B}$ and polarization drift terms in the expression for ion drift velocity and use two potentials to represent the electric field, we arrive at the linear dispersion relation (Hasegawa 1977)

\[
(1 - \frac{\omega^2}{k_x^2 c_s^2}) (1 - \frac{\omega^2}{k_x^2 V_A^2}) = \frac{\omega^2 k_\perp^2}{k_x^2 \omega_{ci}^2},
\]

where $c_s$ is the ion-acoustic speed, $V_A$ is the Alfvén velocity, $\omega_{ci}$ is the ion-cyclotron frequency and $(k_x, k_\perp)$ are the parallel and perpendicular components of the wave vector. As expected, this dispersion relation shows a coupling between ion-acoustic and kinetic Alfvén waves and they

become dispersive due to inclusion of the kinetic effects - the finite ion Larmor radius effect. However, certain different assumptions simplify the governing equations to produce a linear dispersion relation only for kinetic Alfvén waves:
\[ \omega^2 = k_z V_A^2 \left(1 + k_z^2 \rho_s^2 \right) \]  
where \( \rho_s = c_s / \omega_{ci} \) is the equivalent ion gyroradius. Equation (2) is the low-\( \beta \) approximation of (1). Such assumptions, as has already been mentioned in the introduction of chapter-I, have been made by many authors: Hasegawa (1977), Hasegawa and Mima (1976), Yu and Shukla (1978), Shukla, Rahman and Sharma (1982), Kalita and Kalita (1986).

In this part of the present chapter we investigate the stability of kinetic Alfvén wave and ion-acoustic solitons starting from the governing equations which produce the linear dispersion relation (1). We first derive KdV equations in three dimensions for both kinetic Alfvén wave and ion-acoustic wave, which consist of two coupled equations for kinetic Alfvén wave and a single equation for ion-acoustic wave. The KdV equations are found to have soliton solutions for both kinetic Alfvén and ion-acoustic modes for oblique propagation. The stability of these soliton solutions are then investigated by the small-\( k \) perturbation expansion method of Rowlands and Infeld [Rowlands (1969), Infeld (1972), Infeld and Rowlands (1973)]. For kinetic Alfvén waves it is found that there is instability if the direction of the plane-wave perturbation lies inside a cone, and the growth rate of instability attains maximum when the direction of perturbation lies in the plane containing the external magnetic field and the direction of propagation of the solitary wave. For the ion-acoustic wave there is instability if the perturbation is given in a direction lying in the region bounded by two planes intersecting along the direction of propagation of the solitary wave, and the growth rate of instability attains a maximum for perturbations given in a plane perpendicular to the direction of propagation of the solitary wave. Graphs are drawn showing the variations of the square of normalized growth rate of instability against the direction of perturbation given in the plane in which the growth rate of instability attains maximum.

1.2 DERIVATION OF KdV EQUATIONS

The governing equations comprise the ion continuity equation, the parallel component of ion equation of motion, the Boltzmann distribution for electron number density, the electron
equation of continuity, the quasineutrality condition and Ampere's law in parallel direction
(Hasegawa 1977):

\[ \frac{\partial n_i}{\partial t} + \nabla_\perp \cdot (\mathbf{v}_{id} n_i) + \frac{\partial}{\partial z} (v_{iz} n_i) = 0 \]  

\[ \frac{\partial v_{iz}}{\partial t} + (\mathbf{v}_{id} \cdot \nabla_\perp) v_{iz} + \nu_{iz} + v_{iz} \frac{\partial v_{iz}}{\partial z} = - \frac{e}{m_i} \frac{\partial \psi}{\partial z} \]  

\[ n_e = n_o \exp \left( \frac{\psi}{T_e} \right) \]  

\[ \frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z} (n_e v_{ez}) = 0 \]  

\[ n_i = n_e \]  

\[ \frac{\partial}{\partial z} \nabla_\perp^2 (\phi - \psi) = \frac{4\pi}{c^2} \frac{\partial j_z}{\partial t} \]  

where the ion drift velocity \( v_{id} \) and parallel current density \( j_z \) are given by

\[ \mathbf{v}_{id} = \frac{e}{m_i} \mathbf{E}_\perp \times \mathbf{B} + \frac{m_i c^2}{eB_0^2} \frac{d\mathbf{E}_\perp}{dt} \]  

\[ j_z = -en_e v_{ez} + en_i v_{iz} \]

The electric field intensity components are related to two potentials \( \phi \) and \( \Psi \) through

\[ \mathbf{E}_\perp = -\nabla_\perp \phi, \quad E_z = -\frac{\partial \psi}{\partial z} \]  

(Kadomtsev 1965), where \( \perp \) and \( z \) indicate components perpendicular and parallel to the ambient magnetic field. In the above \( n_i \) and \( n_e \) are the ion and electron number densities, \( v_{iz} \) and \( v_{ez} \) are the parallel ion and electron velocities, \( E_z \) and \( E_\perp \) are the parallel and perpendicular components of the electric field intensity vector, \( B_0 \) is the external uniform magnetic field (directed along \( z \)-axis), and \( n_o \) is the unperturbed number density of electrons and ions.
To derive KdV equations in three dimensions, we give stretchings of spatial co-ordinates and time as follows.

\[ \zeta = e^{1/2}(z-V_0t), \quad \xi = e^{1/2}x, \quad \eta = e^{1/2}y, \quad \tau = e^{3/2}t \]  

where \( \varepsilon \) is a small parameter, which is a measure of weakness of dispersion, and \( V_o \) is constant.

We shall now make the following perturbation expansions of the field quantities,

\[
\begin{align*}
n &= n_0 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \ldots \\
\psi &= \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \ldots \\
\varphi &= \varphi^{(1)} + \varepsilon\varphi^{(2)} + \ldots \\
v_{iz} &= \varepsilon v_{iz}^{(1)} + \varepsilon^2 v_{iz}^{(2)} + \ldots \\
v_{ez} &= \varepsilon v_{ez}^{(1)} + \varepsilon^2 v_{ez}^{(2)} + \ldots 
\end{align*}
\]

(13)

where

\[ n = n_0 = n_i \]  

(14)

according to the charge neutrality condition (7).

After substituting the stretchings (12) and the perturbation expansions (13) into the equations (3) - (6), (8) and then equating coefficients different powers of \( \varepsilon \) on both sides of each equation, we get a sequence of equations for \( n^{(\nu)} \) \((\nu=1, 2, \ldots)\) etc.

From the lowest - order equations we get

\[
\begin{align*}
\psi^{(1)} &= \frac{T_e}{e n_0} n^{(1)}, \quad v_{ez}^{(1)} = \frac{V_o}{n_0} n^{(1)}, \quad v_{iz}^{(1)} = \frac{T_e}{n_0 V_o n_i} n^{(1)} \\
\frac{\partial^2 \varphi^{(1)}}{\partial \zeta^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2} &= \frac{e^2 B_0^2}{m_i n_0 e^2} (1 - \frac{c_s^2}{V_o^2}) n^{(1)} 
\end{align*}
\]

and also the equation

\[
(V_o^2 - c_s^2) (V_o^2 - V_A^2) = 0 \]  

(16)

which determines the constant \( V_o \), where

\[ c_s = \left( \frac{T_e}{m_i} \right)^{1/2} \]  

(17)

is the ion-acoustic speed and

\[ V_A = \frac{B_s}{(4\pi m_i n_0)^{1/2}} \]  

(18)
is the Alfvén velocity.

The equation (16) gives two values of $V_0^2$,

\[ V_0^2 = C_s^2, \quad V_o^2 = V_A^2 \]

which correspond respectively to the ion-acoustic mode and Alfvén mode.

At the next order we get the equations

\[
\frac{4 \pi e n_0 V_0}{c^2} \frac{\partial}{\partial \zeta} \left( \nu_i \nu_0 \nu_0 - v_o v_A \right) + \frac{\partial}{\partial \zeta} \left( \frac{\partial^2 \phi^{(2)}}{\partial \xi^2} + \frac{\partial^2 \phi^{(2)}}{\partial \eta^2} \right) = a^{(2)}
\]

(19)

\[
- V_0 \frac{\partial \nu_i}{\partial \zeta} + \frac{e}{m_i} \frac{\partial \psi^{(2)}}{\partial \zeta} = b^{(2)}
\]

(20)

\[
- V_0 \frac{\partial n}{\partial \zeta} + n_o \frac{\partial \nu_0^{(2)}}{\partial \zeta} = c^{(2)}
\]

(21)

\[
n^{(2)} - \frac{e n_o}{T_e} \psi^{(2)} = d^{(2)}
\]

(22)

\[
- V_0 \frac{\partial n^{(2)}}{\partial \zeta} + n_o \frac{\partial \nu_i^{(2)}}{\partial \zeta} + \frac{m_i e^2 c_s^2 V_0}{e B_0^2} \frac{\partial}{\partial \zeta} \left( \frac{\partial^2 \phi^{(2)}}{\partial \xi^2} + \frac{\partial^2 \phi^{(2)}}{\partial \eta^2} \right) = e^{(2)}
\]

(23)

where $a^{(2)}$, $b^{(2)}$ etc. are expressed in terms of $n^{(1)}$ and $\phi^{(1)}$ by the use of the relations (15) and they are given by

\[
a^{(2)} = \frac{8 \pi e V_0^2}{c^2 n_o} \left( 1 - \frac{c_s^2}{V_0^2} \right) n^{(1)} \frac{\partial n^{(1)}}{\partial \zeta}
\]

\[
b^{(2)} = - \frac{c_s^4}{n_o^2 V_0^2} n^{(1)} \frac{\partial n^{(1)}}{\partial \zeta} + \frac{m_i e^2 c_s^2}{e B_0^2 n_o} \left( \frac{\partial^2 \phi^{(2)}}{\partial \xi^2 \partial \zeta} - \frac{\partial^2 \phi^{(2)}}{\partial \eta^2 \partial \zeta} \right) + \frac{\partial^2 \phi^{(2)}}{\partial \xi^2 \partial \eta} \frac{\partial n^{(1)}}{\partial \eta}
\]

\[
c^{(2)} = - \frac{2 V_0}{n_o} n^{(1)} \frac{\partial n^{(1)}}{\partial \zeta}
\]

\[
d^{(2)} = \frac{1}{2 n_o} n^{(1)} \frac{\partial n^{(1)}}{\partial \zeta}
\]

\[
e^{(2)} = - \frac{m_i e^2 V_0}{e B_0^2} \left[ \frac{\partial}{\partial \zeta} \left( n^{(1)} \frac{\partial^2 \phi^{(1)}}{\partial \xi \partial \zeta} \right) + \frac{\partial}{\partial \eta} \left( n^{(1)} \frac{\partial^2 \phi^{(1)}}{\partial \eta \partial \zeta} \right) \right] - 2 c_s^2 \frac{n_o V_0}{n_o} n^{(1)} \frac{\partial n^{(1)}}{\partial \zeta}
\]

(24)

Solving the equations (20) - (23)
for
\[ \frac{\partial \psi}{\partial \xi} = \frac{\partial \psi}{\partial \eta} = \frac{\partial \psi}{\partial \zeta} \cdot \frac{\partial}{\partial \zeta} \left( \frac{\partial^2 \psi}{\partial \zeta^2} + \frac{\partial^2 \psi}{\partial \eta^2} \right) \]

to express them in terms of \( n^{(2)}, n^{(1)}, \phi^{(1)} \), we get

\[ \frac{\partial \psi}{\partial \zeta} = \frac{T_e}{e n_o} \frac{\partial n}{\partial \zeta} - \frac{T_e}{e n_o} \frac{\partial \psi}{\partial \zeta} \]

\[ \frac{\partial \psi}{\partial \zeta} = \frac{T_e}{m n_o V_o} \frac{\partial n}{\partial \zeta} - \frac{1}{n_o} b^{(2)} - \frac{T_e}{m n_o V_o} \frac{\partial \psi}{\partial \zeta} \]

\[ \frac{\partial \psi}{\partial \zeta} = \frac{V_o}{n_o} \frac{\partial n}{\partial \zeta} + \frac{1}{n_o} c^{(2)} \]

\[ \frac{\partial}{\partial \zeta} \left( \frac{\partial^2 \psi}{\partial \zeta^2} + \frac{\partial^2 \psi}{\partial \eta^2} \right) = \frac{e B_o^2}{m n_o c^2} (1 - \frac{c_s^2}{V_o^2}) \frac{\partial n}{\partial \zeta} + \frac{e B_o^2}{m n_o c^2 V_o^2} b^{(2)} \]

\[ + \frac{e B_o^2}{m n_o c^2 V_o^2} \frac{\partial \psi}{\partial \zeta} + \frac{e B_o^2}{m n_o c^2 V_o^2} c^{(2)} \]

Substituting these solutions in equation (19), we find that the coefficient of \( \frac{\partial n^{(2)}}{\partial \zeta} \) in the resulting equation vanishes because of (16) and consequently the resulting equation assumes the form:

\[ \frac{8 \pi e}{c^2 V_o^3} \left( V_o^4 - V_A^2 c_s^2 \right) \frac{\partial n^{(1)}}{\partial t} - \frac{T_e}{e n_o} \left( \frac{\partial^2 \psi}{\partial \zeta^2} + \frac{\partial^2 \psi}{\partial \eta^2} \right) \frac{\partial n^{(1)}}{\partial \zeta} \]

\[ + \frac{4 \pi e}{c^2 n_o} \left[ c_s^4 \left( 1 - \frac{V_A^2}{V_o^2} \right) + c_s^2 \left( 1 - \frac{V_A^2}{V_o^2} \right) - V_A^2 \left( 1 - \frac{c_s^2}{V_o^2} \right) \right] \frac{\partial n^{(1)}}{\partial \zeta} \]

\[ + \frac{4 \pi e}{c B_o \omega_{ci}} \left[ c_s^2 \left( 1 - \frac{V_A^2}{V_o^2} \right) - V_A^2 \left( \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} + \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} \right) \frac{\partial n^{(1)}}{\partial \zeta} + \frac{\partial \phi^{(1)}}{\partial \eta} \frac{\partial n^{(1)}}{\partial \eta} \right] = 0 \]  

(29)

where the ion-cyclotron frequency \( \omega_{ci} = e B_o / m c \).

The two coupled equations (29) and the last equation of (15) constitute KdV equations in three dimensions for kinetic Alfven waves or ion-acoustic waves according to whether we take \( V_o^2 = V_A^2 \) or \( V_o^2 = c_s^2 \). For kinetic Alfven waves the KdV equation in three dimensions actually comprises of the following two coupled equations, which are obtained from (29) and last equation of (13) by setting \( V_o^2 = V_A^2 \).
\[
\frac{\partial n^{(1)}}{\partial \tau} - \lambda_1 \frac{\partial}{\partial \xi} \left( \frac{\partial^2 n^{(1)}}{\partial \xi^2} + \frac{\partial n^{(1)}}{\partial \eta} \frac{\partial n^{(1)}}{\partial \xi} \right) - \lambda_2 n^{(1)} \frac{\partial n^{(1)}}{\partial \xi} \\
= \lambda_3 \left( \frac{\partial^2 \phi^{(1)}}{\partial \xi \partial \xi} + \frac{\partial n^{(1)}}{\partial \eta} \frac{\partial n^{(1)}}{\partial \xi} - \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} \right) = 0
\]  
(30)

\[
n^{(1)} = \lambda_0 \left( \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} \right)
\]  
(31)

where
\[
\lambda_0 = \frac{e n_0}{B_0 \omega_{ci}} \left( 1 - \frac{c_s^2}{V_A^2} \right)^{-1}, \quad \lambda_1 = \frac{T_e c^2 V_A}{8 \pi e^2 n_0} \left( V_A^2 - c_s^2 \right)^{-1}
\]
\[
\lambda_2 = \frac{V_A}{2 n_0}, \quad \lambda_3 = \frac{V_A^3}{2 B_0 \omega_{ci}} \left( V_A^2 - c_s^2 \right)^{-1}
\]  
(32)

Since for ion-acoustic wave \( V_o^2 = c_s^2, \phi^{(1)} = 0 \) according to the last equation of (15), and so in this case the coupled equations (29) and the last equation of (15) reduce to the following single equation, which is the KdV equation in three dimensions for ion-acoustic wave:

\[
\frac{\partial n^{(1)}}{\partial \tau} + \lambda_4 \frac{\partial}{\partial \xi} \left( \frac{\partial^2 n^{(1)}}{\partial \xi^2} + \frac{\partial n^{(1)}}{\partial \eta} \frac{\partial n^{(1)}}{\partial \xi} \right) + \lambda_5 n^{(1)} \frac{\partial n^{(1)}}{\partial \xi} = 0
\]  
(33)

where
\[
\lambda_4 = - \frac{m c_s^2}{2 B_0 \omega_{ci}} \left( c_s^2 - V_A^2 \right)^{-1}, \quad \lambda_5 = \frac{c_s}{n_0}
\]  
(34)

In order to study the stability of a solitary wave propagating in a direction making an angle \( \alpha \) with the z-axis (i.e. with the direction of the external uniform magnetic field) and lying in the xz-plane, we rotate the co-ordinate axes \( \xi, \zeta \) through an angle \( \alpha \), keeping \( \eta \) axis fixed. That is we make the following change of variables:

\[
\xi' = \xi \cos \alpha + \zeta \sin \alpha, \quad \zeta' = -\xi \sin \alpha + \zeta \cos \alpha
\]  
(35)

With this, the coupled equations (30), (31) constituting the KdV equation for kinetic Alfven waves becomes as follow, in which we drop primes:

\[
\frac{\partial n^{(1)}}{\partial \tau} - \beta n^{(1)} \frac{\partial n^{(1)}}{\partial \xi} \frac{\partial n^{(1)}}{\partial \zeta} - \gamma \frac{\partial^2 n^{(1)}}{\partial \xi^2} - \delta \frac{\partial^2 \phi^{(1)}}{\partial \xi \partial \zeta} + a_1 \frac{\partial^2 n^{(1)}}{\partial \zeta^2} + a_2 \frac{\partial^2 n^{(1)}}{\partial \xi^2} + a_3 \frac{\partial^2 n^{(1)}}{\partial \xi \partial \eta} + a_4 \frac{\partial^2 n^{(1)}}{\partial \xi^2} + b_1 n^{(1)} \frac{\partial n^{(1)}}{\partial \xi} \\
+ b_2 \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + b_3 \frac{\partial^2 \phi^{(1)}}{\partial \xi \partial \zeta} + b_4 \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + b_5 \frac{\partial n^{(1)}}{\partial \xi} \\
+ b_6 \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + b_7 \frac{\partial^2 \phi^{(1)}}{\partial \xi \partial \zeta} + b_8 \frac{\partial^2 \phi^{(1)}}{\partial \xi^2}
\]
The coefficients $\beta$, $\gamma$ etc, in these two equations are given in the appendix - 2A on page 80.

The same transformation (35) brings (33), which is the KdV equation in three dimensions on-acoustic waves, into the following form, where as before we drop the primes on the variables (but not the coefficients):

\[
\begin{align*}
\frac{\partial n^{(1)}}{\partial \tau} + \beta' n^{(1)} \frac{\partial n^{(1)}}{\partial \zeta} + \gamma' \frac{\partial^3 n^{(1)}}{\partial \zeta^3} + a_1 \frac{\partial^3 n^{(1)}}{\partial \zeta^2 \partial \xi} + a_2 \frac{\partial^3 n^{(1)}}{\partial \zeta^2 \partial \eta} + a_3 \frac{\partial^3 n^{(1)}}{\partial \xi \partial \eta^2} + a_4 \frac{\partial^3 n^{(1)}}{\partial \xi^2 \partial \eta} + b_1' n^{(1)} \frac{\partial n^{(1)}}{\partial \zeta} &= 0
\end{align*}
\]

(38)

where the coefficients $\beta'$, $\gamma'$ etc. are given in the appendix - 2B on page 80.

1.3 SOLITARY - WAVE SOLUTION

(A) Kinetic Alfvén waves

We seek a solitary wave solution of the coupled equations (36) and (37) propagating along $\zeta$- axis, which implies propagation in a direction making an angle $\alpha$ with the direction of the external uniform magnetic field and lying in $\xi \zeta$-plane. Consequently we take the solution of (36) and (37) in the form

\[
\begin{align*}
\xi^{(1)} &= N_0(Z), \phi^{(1)} = \Phi_0(Z)
\end{align*}
\]

(39)

where

\[
Z = \zeta + u_\tau \tau
\]

(40)

Substituting (39) into (36) and (37), we get the following two equations for $N_0(Z)$ and $\Phi_0(Z)$.
Eliminating $\Phi_0$ between these two equations, we find for $N_0$ the equation,

$$-u_0 \frac{dN_0}{dZ} + \kappa N_0 \frac{dN_0}{dZ} + \gamma \frac{d^3N_0}{dZ^3} = 0 \quad (43)$$

where $\kappa = \beta + \delta/\lambda$. The solitary wave solution of (43) is

$$N_0 = a \text{ sech}^2 \beta Z \quad (44)$$

where $a = \frac{3u_0}{\kappa}$, $p = \left(\frac{u_0}{4\gamma}\right)^{1/2}$ \quad (45)

\section{1.4 STABILITY ANALYSIS}

We now perform a stability analysis of the solitary waves obtained in sec. 1.3 with respect to long-wavelength perturbations given in an arbitrary direction. This analysis is done by the

(A) Kinetic Alfvén wave on setting

\[ n^{(1)} = N_0(Z) + q(Z, \xi, \eta, \tau) \]

\[ \phi^{(1)} = \Phi_0(Z) + \psi'(Z, \xi, \eta, \tau) \]  

in the equations (36) and (37) and then linearizing with respect to \( q \) and \( \psi' \), the following two equations are obtained.

\[
\begin{align*}
    \frac{u_0}{dZ} \frac{\partial q}{\partial \tau} + \frac{\partial q}{\partial \tau} - \beta N_0 \frac{\partial q}{\partial Z} - \beta \frac{dN_0}{dZ} q - \gamma \frac{\partial^3 q}{\partial Z^3} - \delta \frac{d^2 \Phi_0}{dZ^2} \frac{\partial q}{\partial Z} \\
    - \delta \frac{dN_0}{dZ} \frac{\partial^3 \psi'}{\partial Z^3} + a_1 \frac{\partial^3 q}{\partial Z \partial \xi \partial \xi} + a_2 \frac{\partial^3 q}{\partial Z \partial \zeta \partial Z} + a_3 \frac{\partial^3 q}{\partial \zeta \partial \zeta} + a_4 \frac{\partial^3 q}{\partial Z \partial \eta \partial \eta} + a_5 \frac{\partial^3 q}{\partial \xi \partial \eta \partial \eta} + b_1 \frac{\partial dN_0}{dZ \partial \xi} + b_2 \frac{dN_0}{dZ} \frac{\partial^2 \psi'}{\partial \xi \partial Z} + b_3 \frac{dN_0}{dZ} \frac{\partial^2 \psi'}{\partial \xi \partial \xi} + b_4 \frac{d^2 \Phi_0}{dZ^2} \frac{\partial q}{\partial \zeta} = 0 \\
    q = \lambda \frac{\partial^3 \psi'}{\partial Z^3} + c_1 \frac{\partial^3 \psi'}{\partial \xi \partial \xi} + c_2 \frac{\partial^3 \psi'}{\partial \xi \partial \zeta} + c_3 \frac{\partial^3 \psi'}{\partial \eta \partial \eta} 
\end{align*}
\]

For a long- wavelength plane-wave perturbation given in a direction with direction cosines \((l,m,n)\) we can take

\[
\begin{align*}
    q &= \tilde{q}(Z) \exp \left[ ik(l \xi + m \eta + nZ) - i \omega \tau \right] \\
    \psi' &= \tilde{\psi}(Z) \exp \left[ ik(l \xi + m \eta + nZ) - i \omega \tau \right] 
\end{align*}
\]

where \( k \) is small and \( l^2 + m^2 + n^2 = 1 \). For small \( k \) we expand \( \tilde{q}(Z) \), \( \tilde{\psi}(Z) \) and \( \omega \) as

\[
\begin{align*}
    \tilde{q}(Z) &= q_0(Z) + kq_1(Z) + k^2 q_2(Z) + ... \\
    \tilde{\psi}(Z) &= \psi_0(Z) + k\psi_1(Z) + k^2 \psi_2(Z) + ... \\
    \omega &= k\omega_1 + k^2 \omega_2 + ... 
\end{align*}
\]

Substituting the expressions (54) for $q$ and $\psi'$ and then cancelling out the factor $\exp [ ik(14+m\eta+nZ) - i\omega t]$, we get two equations for $\bar{q}(Z)$ and $\bar{\psi}(Z)$. Substituting the expansions (55) for $q$, $\psi$, $\omega$ in these two equations and then equating coefficients of different powers of $k$ on both sides of each equation, we get a sequence of equations for $q_j$, $\psi_j$, ($j = 0, 1, 2, ...$).

**Zeroth-order equations**

At the lowest (Zeroth) order we get the following two equations.

$$u_o \frac{dq_o}{dZ} - \beta \frac{d}{dZ}(N_0q_o) - \gamma \frac{d^3q_o}{dZ^3} - \frac{\delta}{\lambda} N_0 \frac{dq_o}{dZ} - \delta \frac{dN_0}{dZ} \frac{d^2\psi_o}{dZ^2} = 0$$

$$q_o = \lambda \frac{d^2\psi_o}{dZ^2}$$

Eliminating $\psi_o$ between these two equations, we get for $q_o$ the equation,

$$-u_o \frac{dq_o}{dZ} + \kappa \frac{d}{dZ}(N_0q_o) + \gamma \frac{d^3q_o}{dZ^3} = 0$$

Integrating this equation once and then substituting for $N_0$ from (44), the equation for $q_o$ is obtained as

$$\frac{1}{4p^2} \frac{d^2q_o}{dZ^2} + (3\text{sech}^2 pZ - 1) q_o = A$$

where $A$ is a constant.

Two linearly independent solutions of the homogeneous part of (59) are (Infeld 1985)

$$f = N_{oz} = Rs^2$$

$$g = N_{oz} \int \frac{Z \ dZ}{N_{oz}^2} = \frac{2}{15} \frac{s^2}{2} + \frac{1}{3}$$

where $R = \tanh pZ$ and $s = \text{sech} pZ$, and where constant factors have been omitted. Hence the general solution of (59) is
\( q_0 = A_1 f + A_2 g - f \int \frac{Z \, Ag}{W/4p^2} \, dZ + g \int \frac{Z \, Af}{W/4p^2} \, dZ \)  

(61)

where \( A_1 \) and \( A_2 \) are two arbitrary constants and

\[
W = f \frac{dg}{dZ} - g \frac{df}{dZ} = \frac{8}{15} \, p
\]

After evaluating the integrals in (61), the solution for \( q_0 \) can be put into the form

\[
q_0 = A_1 R s^2 + A_2 \left( p Z R s^2 + \frac{2}{15} s^2 + \frac{1}{3} - s^2 \right) + \frac{15}{2} \, A \left( \frac{1}{15} s^2 - \frac{1}{30} \, \frac{1}{10} s^2 - \frac{1}{10} \, p Z R s^2 \right)
\]

(62)

This solution for \( q_0 \) must satisfy the condition that \( q_0 \) must remain finite as \(|Z| \to \infty\) and \( q_0 \to 0 \) as \(|Z| \to \infty\). These two conditions give

\[
A_2 + \frac{15}{4} \, A = 0, \quad \frac{1}{3} \, A_2 + \frac{1}{4} \, A = 0
\]

(63)

Solving these two equations we find

\( A_2 = A = 0 \)

(64)

Hence the solution for \( q_0 \) becomes

\[
q_0 = A_1 R s^2 = \frac{A_1}{2ap} \frac{d N_o}{dZ}
\]

(65)

and consequently the solution for \( \psi_o \) given by (57) becomes

\[
\psi_o = - \frac{A_1}{2p^3 \lambda} \tanh pZ
\]

(66)

**First order equations**

At the order \( k \) we get the following two equations for \( q_1 \) and \( \psi_1 \).
\[ u_0 \frac{dq_1}{dZ} - \beta N_0 \frac{dq_1}{dZ} - \beta \frac{dN_0}{dZ} q_1 - \gamma \frac{d^3 q_1}{dZ^3} - \delta \frac{dN_0}{dZ} - \frac{\delta}{\lambda} \frac{dN_0}{dZ} = \] 
\[ \frac{d^2 \psi_1}{dZ^2} + i \omega q_0 + i \beta n N_0 q_0 + 3i \gamma m \frac{d^2 q_0}{dZ^2} + \frac{\text{in} \delta}{\lambda} N_0 q_0 + 2i \gamma n \frac{dN_0}{dZ} + \frac{1}{\lambda} N_0 q_0 \] 

(67)

\[ q_1 = \lambda \frac{d^2 \psi_1}{dZ^2} + 2i \omega \frac{d\psi_0}{dZ} + i c_1 \frac{d\psi_0}{dZ} \] 

(68)

Eliminating \( \frac{d^2 \psi_1}{dZ^2} \) between (67) and (68), we find that \( q_1 \) satisfies the equation

\[ -u_0 \frac{dq_1}{dZ} + \mathcal{K} \frac{d}{dZ} (N_0 q_1) + \gamma \frac{d^3 q_1}{dZ^3} = Q_1 \] 

(69)

where \( Q_1 \) is given by

\[ \frac{d^2 q_1}{dZ^2} + \frac{\text{in} \delta}{\lambda} N_0 \frac{dN_0}{dZ} + \frac{1}{\lambda} N_0 q_0 \] 

\[ + \frac{i A_1}{2 \text{ap}^2} (\omega_1 - nu_0) \frac{dN_0}{dZ} + \frac{i A_1}{2 \text{ap}} (\beta n + \frac{n \delta}{\lambda} - b_1 l) \] 

\[ - \frac{b_1 l}{\lambda} \frac{dN_0}{dZ} \frac{dN_0}{dZ} + \frac{i A_1}{2 \text{ap}} (3 \gamma n - a_1 l) \frac{d^3 N_0}{dZ^3} \] 

(70)

The equation (69) can be put in the following form after integrating it once.

\[ (-1 + 3 \text{sech}^2 pZ) q_1 + \frac{1}{4p^2} \frac{d^3 q_1}{dZ^3} = B + i A_1 a_2 \text{sech}^2 pZ \] 

\[ + i A_1 b_2 \text{sech}^2 pZ \tanh^2 pZ \] 

(71)

where \( B \) is an arbitrary constant and

\[ a_2 = \frac{1}{2 \text{pu}_0} [(\omega_1 - nu_0) + \frac{a}{2} (\beta n + \frac{n \delta}{\lambda} - b_1 l) \]
Proceeding is the same way as in deriving the solution (62) of the equation (59), we arrive at the following general solution of the equation (71).

\[ q_1 = B_1 R s^2 + B_2 (p Z R s^2 + \frac{2}{15} s^2 + \frac{1}{3} \cdot s^2) \]

\[ + \frac{15}{2} B_2 \left( \frac{1}{15} s^2 - \frac{1}{30} - \frac{1}{10} s^2 + \frac{1}{10} p Z R s^2 \right) \]

\[ + \frac{15}{2} i A_1 \tilde{a}_2 \left( \frac{1}{30} s^2 + \frac{1}{12} - \frac{7}{60} s^2 + \frac{7}{60} p Z R s^2 \right) \]

\[ + \frac{15}{2} i A_1 \tilde{b}_2 \left( \frac{1}{90} s^2 + \frac{1}{36} - \frac{7}{180} s^2 - \frac{1}{20} p Z R s^2 \right) \]  

(73)

where \( B_1 \) and \( B_2 \) are two arbitrary constants.

Since \( q_1 \) must remain finite at \(|Z| \to \infty\), the coefficient of \( s^2 \) in the above solution for \( q_1 \) must be equal to zero. This gives

\[ \frac{2}{15} B_2 + \frac{1}{30} B + \frac{1}{12} i A_1 \tilde{a}_2 + \frac{1}{36} i A_1 \tilde{b}_2 = 0 \]  

(74)

With this condition fulfilled, \( q_1 \) must also satisfy the condition that \( q_1 \to 0 \) as \(|Z| \to \infty\). This condition leads to

\[ \frac{1}{3} B_2 + \frac{1}{4} B + \frac{1}{8} i A_1 \tilde{a}_2 + \frac{1}{3} i A_1 \tilde{b}_2 = 0 \]  

(75)
Solving (74) and (75) for $B_2$ and $B$, we find

$$B_2 = \frac{-iA_1(3a_2 + b_2)}{8}, \quad B = 0$$

Therefore the solution for $q_1$ becomes

$$q_1 = B_1Rs^2 - iA_1ps^2 (a_2 + b_2) + \frac{1}{3} iA_1(3a_2 + b_2)s^2$$

Substituting this solution for $q_1$ in (68) and then integrating once, we find

$$\lambda \frac{d\psi_1}{dZ} = \frac{B_1}{2p} R^2 - \frac{iA_1}{2p}(a_2 + b_2) (-pZ^2 + R)$$

$$+ iR \frac{A_1}{3p} (3a_2 + b_2) + \frac{A_1}{2p^2\lambda} (2n\lambda + l_0) + D$$

where $D$ is an arbitrary constant.

**Second order equations**

At the order $k^2$ we get the following two equations for $q_2$ and $\psi_2$.

$$u_0 \frac{d^2q_2}{dZ^2} - \beta N_0 \frac{dq_2}{dZ} - \beta \frac{dN_0}{dZ} q_2 - \gamma \frac{d^2q_2}{dZ^2} - \frac{\delta}{\lambda} N_0 \frac{dq_2}{dZ}$$

$$- \beta N_0 \frac{d^2\psi_2}{dZ^2} = -i\omega_1q_1 + i\omega_2q_0 + i\beta N_0 q_1$$

$$+ 3i\gamma n \frac{d^2q_1}{dZ^2} - 3\gamma n^2 \frac{dq_0}{dZ} + \frac{in\delta}{\lambda} N_0 q_1 + 2in\delta \frac{dN_0}{dZ} \frac{dq_1}{dZ}$$

$$- n^2\delta \frac{dN_0}{dZ} \psi_0 - ia_1 \frac{d^2q_2}{dZ^2} + 2a_1 \psi_0 \frac{dq_0}{dZ} + a_2 \frac{dq_0}{dZ}$$

$$+ a_4 m^2 \frac{dq_0}{dZ} - ib_1 \frac{dN_0}{dZ} q_1 - ib_2 \frac{dN_0}{dZ} \frac{dq_0}{dZ} + b_2 \ln - \frac{dN_0}{dZ} \psi_0$$

$$+ b_3 \frac{dN_0}{dZ} \psi_0 - ib_4 \frac{1}{\lambda} N_0 q_1$$

(78)
\[ q_2 = \kappa \frac{d^2 \psi_2}{dZ^2} + 2i\kappa \lambda \frac{d\psi_1}{dZ} - n^2 \lambda \psi_0 + ilc_1 \frac{d\psi_1}{dZ} \]

\[ -nlc_1 \psi_0 - l^2 c_2 \psi_0 - m^2 c_3 \psi_0 \quad (79) \]

Eliminating \( d^2 \psi_2 / dZ^2 \) between (78) and (79), we find the following equation for \( q_2 \),

\[ -u_0 \frac{dq_2}{dZ} + \kappa - \frac{d}{dZ}(N_0 q_2) + \gamma \frac{d^3 q_2}{dZ^3} = Q_1 \quad (80) \]

where \( Q_2 \) is given by

\[ Q_2 = -i\omega_0 q_0 + i\alpha_3 q_1 + b_3 \frac{dq_0}{dZ} + ic_3 N_0 q_1 \]

\[ \frac{d^3 q_1}{dZ^3} + \frac{dN_0}{dZ} \psi_0 + i\beta_3 \frac{dN_0}{dZ} \frac{d\psi_1}{dZ} \]

The quantities \( \alpha_3, \beta_3 \) etc. appearing in this expression are given by

\[ \alpha_3 = n u_0 - \omega_1, \quad \beta_3 = 3\gamma n^2 - 2a_1 n - a_2 l^2 - a_4 m^2 \]

\[ \gamma = -n m + b_1 l + \frac{b_4 l}{\lambda}, \quad d_3 = -3\gamma n + a_1 l \]

\[ e_3 = -n^2 \delta - \frac{nlc_1 \delta}{\lambda} - \frac{l^2 c_2 \delta}{\lambda} - \frac{m^2 c_3 \delta}{\lambda} + n^2 \delta - b_2 l n - b_3 l^2 \]

\[ f_3 = 2\delta n + \frac{lc_1 \delta}{\lambda} - 2n \delta + b_2 l \quad (82) \]

The adjoint of the homogeneous part,
of (80), is
\[
\frac{dF}{dZ} + \frac{d^2F}{dZ^2} = 0
\]

This equation being same as (43), the solution of equation (83), that satisfies the same boundary condition as (80) for \( q_2 \) (i.e. that satisfies the condition \( F \to 0 \) as \( |Z| \to \infty \)) is
\[
F = c q_2^2
\]
where \( c \) is an arbitrary constant. Hence for the solution of (80) to exist, its right hand side should be orthogonal to the kernel \( s^2 \) of the adjoint operator
\[
\frac{d}{dZ} - \frac{d^3}{dZ^3} \left( -u_0 + \kappa N_0 \right) + \gamma \frac{dF}{dZ} = 0
\]

This condition gives the following equation, which determines the desired \( \omega_1 \).
\[
\int_{-\infty}^{\infty} Q q_2^2 dZ = 0
\]

Substituting here the expressions for \( q_0, q_1, \psi_0 \) and \( d\psi_1 / dZ \) given by (65), (76), (66), and (77) respectively, and then performing the integration, we arrive at the following equation, determining \( \omega_1 \).
\[
\Omega^2 + D_1 \Omega + D_2 = 0
\]

where
\[
\Omega = \frac{(nu_0 - \omega_1) \sin \alpha \cos \alpha}{u_0}
\]

and
\[
D_1 = -\frac{2}{3} \cos \alpha (n \sin \alpha + \cos \alpha)
\]
\[
D_2 = -\frac{1}{5} \left[ \sin^2 \alpha \left( 19 \sin^2 \alpha - \frac{82}{3} \right) - \frac{28}{15} \right] - \frac{8m^2}{15} \cos^2 \alpha
\]
\[
- \frac{19}{5} n^2 \sin^2 \alpha \cos^2 \alpha + \frac{2}{5} \ln \sin \alpha \cos \alpha (19 \sin^2 \alpha - 11)
\]
Fig. 4: Squared normalized growth rate of the instability of Kinetic Alfvén waves versus the angle θ for various values of the angle α.
The discriminant of the quadratic equation (86) in $\Omega$ in

$$\Delta = Al^2 + Bm^2 + Cn^2 + Dln$$

where

$$A = \frac{1}{45} (704 \sin^4 \alpha - 1024 \sin^2 \alpha + 356)$$

$$B = \frac{32}{15} \cos^2 \alpha, \quad C = \frac{704}{45} \sin^2 \alpha \cos^2 \alpha$$

$$D = - \frac{128}{45} \sin \alpha \cos \alpha (11 \sin^2 \alpha - 8)$$

Therefore there is instability if the direction cosines $l$, $m$ and $n$ of the long wave length plane-wave perturbation satisfy the condition,

$$Al^2 + Bm^2 + Cn^2 + Dln < 0$$

Since $B$, $C > 0$, it can be shown that (91) is equivalent to the condition that for instability the direction along which perturbation is given should lie inside the cone

$$A\xi^2 + B\eta^2 + C\zeta^2 + D\xi \zeta = 0$$

The growth rate $g_R$ of this instability is given by

$$g_R = \frac{u_0^2 k^2}{\sin^2 \alpha \cos^2 \alpha} |Al^2 + Bm^2 + Cn^2 + Dln|$$

Since $l^2 + m^2 + n^2 = 1$, by the Lagrange's method of undetermined multipliers, it can be shown that $g_R^2$ attains an extremum for the values of $l$, $m$ and $n$ given by

$$m = 0, \quad \frac{1}{-D} = \frac{n}{(A-C) \pm [(A-C)^2 + D^2]^{1/2}}$$

Therefore the growth rate of instability attains maximum for perturbations given in the $\xi \zeta$ - plane. If perturbations given in this plane makes an angle $\theta$ with $\xi$-axis, then the growth rate of instability, $g_R$ is given by

$$\frac{g_R^2}{u_0 k^2} = \frac{1}{\sin^2 \alpha \cos^2 \alpha} \left[ \frac{128}{25} \sin \alpha \cos \alpha (11 \sin^2 \alpha - 8) \sin \theta \cos \theta \right]$$
This square of dimensionless growth rate of instability is plotted against $\theta$ for some different values of $\alpha$ and is shown in fig. 4.

(B) Ion-acoustic waves

Setting $n^{(1)} = N_0(Z) + q'(Z, \xi, \eta, \tau)$ in (38) and then linearizing with respect to $q'$, we find that $q'$ satisfies the equation,

$$- u_0 \frac{\partial q'}{\partial Z} + \frac{\partial q'}{\partial \tau} + B'N'_0 \frac{\partial q'}{\partial Z} + B' \frac{dN'_0}{dZ} q' + \gamma' \frac{\partial^2 q'}{\partial Z^2}$$

$$+ a'_1 \frac{\partial^3 q'}{\partial Z^2 \partial \xi} + a'_2 \frac{\partial^3 q'}{\partial Z^2 \partial \eta} + a'_3 \frac{\partial^3 q'}{\partial \xi^3} + a'_4 \frac{\partial^3 q'}{\partial Z \partial \xi \partial \eta}$$

$$+ a'_5 \frac{\partial^3 q'}{\partial Z \partial \xi^2} + b'_o N'_o \frac{\partial q'}{\partial \xi} = 0$$

(96)

For a long wavelength plane-wave perturbation given in a direction with direction cosines $l, m, n$, we take

$$q' = q(Z) \exp \left[ i k (l \xi + m \eta + nZ) - i \omega \tau \right]$$

(97)

Substituting this in (96) and then cancelling out the factor $\exp \left[ i k (l \xi + m \eta + nZ) - i \omega \tau \right]$ we get an equation for $q(Z)$. Substituting the same expansions for $q$ and $\omega$ as given in (55) in the equation for $q(Z)$, and then equating coefficients of different powers of $k$ on both sides, we get a sequence of equations for $q_j$ ($j = 0, 1, 2, .......$).

Proceeding in the same way as in the case of kinetic Alfven waves, we find the following solutions for the first two members of the sequence.

$$q_0 = - \frac{A'_1}{2a'_1 p'} \frac{dN'_0}{dZ}$$

$$q_1 = B'_1 R s^2 - i A'_1 p' Z R s^2 (a'_2 + b'_2) + \frac{1}{3} i A'_1 (3a'_2 + b'_2) s^2$$

(98)
where

\[ a_2' = \frac{1}{2u_o p'} \left[ \left( \omega_0 + \nu_0 \right) - a' \left( \beta n + b' l \right) + 2p'' \left( 3\gamma' n + a'_ l \right) \right] \]

\[ b_2' = \frac{1}{2u_o p'} \left[ -a' \left( \beta n + b' l \right) - 6p'' \left( 3\gamma' n + a'_ l \right) \right] \] (99)

At the order \( k^2 \) we find the equation for \( q_2 \), which is

\[ \frac{d^2 q_2}{dZ^2} - \beta' \left( N_0 q_2 \right) - \gamma' = Q_2' \] (100)

where

\[ Q_2' = i \omega_2 q_0 + i a'_3 q_1 + b'_3 - i c'_3 N_0 q_1 - i d'_3 \] (101)

and the constants \( a'_3, b'_3, d'_3 \) appearing in this expression are given by

\[ a'_3 = n \nu_0 - \omega_1, \quad b'_3 = 3\gamma' n^2 - 2a'_1 l^2 - a'_4 m^2 \]

\[ c'_3 = -n \beta' - b'_1 l, \quad d'_3 = -3\gamma' n - a'_4 l \] (102)

Now following the same procedure as in the derivation of (86) from (80) for kinetic Alfvén wave, we obtain the following equation for the determination of \( \omega_1 \) for the ion-acoustic wave,

\[ \Omega^2 + D_1' \Omega + D_2' = 0 \] (103)

where

\[ \Omega = -\frac{\left( nu_0 + \omega_1 \right) \sin \alpha \cos \alpha}{u_o} \] (104)

and

\[ D_1' = -2n \sin \alpha \cos \alpha + \frac{2}{3} \left( 5 \sin^2 \alpha + 2 \cos^2 \alpha \right) \]
Fig. 5: Squared normalized growth rate of the instability of ion-acoustic waves versus the angle $\theta'$ for various values of the angle $\alpha$. 
\[ D_2' = \frac{1}{3} \left( 4\sin^2 \alpha + 7 \cos^2 \alpha \right) + \frac{4}{15} m^2 \cos^2 \alpha \]

\[ + n^2 \sin^2 \alpha \cos^2 \alpha - \frac{2}{3} \ln \left( 5 \sin^2 \alpha + 2 \cos^2 \alpha \right) \quad (105) \]

The discriminant \( \Delta' \) of equation (103) is \( \Omega \) is

\[ \Delta' = \frac{1}{15} \left( 5l^2 - 3m^2 \cos^2 \alpha \right) \quad (106) \]

Hence there is instability if the direction cosines \( l, m \) and \( n \) of a long-wavelength plane-wave perturbation satisfy the condition

\[ \frac{m^2}{l^2} > \frac{5}{3 \cos^2 \alpha} \quad (107) \]

The growth rate \( g_R \) of this instability is given by

\[ g_R^2 = \frac{u_0^2 k^2}{15 \sin^2 \alpha \cos^2 \alpha} \left( 3m^2 \cos^2 \alpha - 5l^2 \right) \quad (108) \]

For a perturbation in a plane through \( \zeta \)-axis making an angle \( \theta' \) with \( \xi \zeta \)-plane this expression for the square of the growth rate of instability becomes

\[ g_R^2 = \frac{u_0^2 (1-n^2) k^2}{15 \sin^2 \alpha \cos^2 \alpha} \left( 3\sin^2 \theta' \cos^2 \alpha - 5 \cos^2 \theta' \right) \quad (109) \]

This expression shows that the growth rate attains maximum for \( n = 0 \). Hence the maximum growth rate of instability is attained for perturbations given in this \( \xi \eta \)-plane, which is a plane perpendicular to the direction of propagation of the solitary wave. If the direction of perturbation in this plane makes an angle \( \theta' \) with \( \xi \)-axis, then the growth rate of instability is given by

\[ \frac{g_R^2}{u_0^2 k^2} = \frac{1}{15 \sin^2 \alpha \cos^2 \alpha} \left( 3\sin^2 \theta' \cos^2 \alpha - 5 \cos^2 \theta' \right) \quad (110) \]

This square of the dimensionless fees growth rate of instability is plotted against \( \theta' \) for some different values of \( \alpha \) in fig. 5.

The condition (107) shows that there is instability if the perturbations are given in a direction lying in the region bounded by the planes.
\[ \xi \pm \frac{2}{3} \eta \cos \alpha = 0 \]
that does not contain the plane \( \eta = 0 \)
2.1 INTRODUCTION

In part-I of the present chapter we have investigated the stability of kinetic Alfvén soliton and ion-acoustic soliton starting from a set of equations which produce linear dispersion relation for both kinetic Alfvén wave and ion-acoustic wave. Beginning from the same set of equations we determine in this part of the present chapter the higher order nonlinear and dispersive effects for both kinetic Alfvén and ion-acoustic solitary waves by applying the method developed by Das and Majumder (1991). Assuming a solitary wave propagation a single equation is obtained for the perturbed electron or ion number density from the set of governing equations. Perturbation solution of this equation is obtained proceeding in the same way as in part-II of chapter-I. In the lowest order KdV soliton solutions are obtained for both kinetic Alfvén wave and ion-acoustic wave. Perturbation solutions are obtained for the next two higher orders. Solitary wave profiles are found to be same for both kinetic Alfvén mode and ion-acoustic mode. Each of the three quantities, amplitude, width and Mach number of the solitary wave for kinetic Alfvén and ion-acoustic mode have been expressed as functions of a parameter. Hence eliminating this parameter the width and Mach number of solitary waves can be expressed as functions of amplitude of the solitary waves. Graphs are plotted showing their variations with amplitude. It is found that the solitary kinetic Alfvén wave moves with a faster speed and solitary ion-acoustic wave with a slower speed than the speeds predicted by KdV theory. The width, which is same for both the modes, is found to be greater than that of KdV solitons.

2.2 BASIC EQUATIONS

The governing equations are same as the equations (3) - (11) of part - I of the present chapter.

For solitary waves propagating obliquely to the external magnetic field we assume that all the dependent variables depend on a single independent variable,

$$\xi = z \cos \alpha + x \sin \alpha - Mt$$  \hspace{1cm} (111)

Since the dependent variables depend on the independent variable $\xi$ only given by (111), the equations (3), (4), (6), (8) become as follows.

$$-M \frac{dn}{d\xi} + \frac{me^2}{cB_0} M \sin^2 \alpha \frac{d}{d\xi} (n \frac{d^2 \phi}{d\xi^2}) + \cos \alpha \frac{d}{d\xi} (n \omega_n \psi_k) = 0$$  \hspace{1cm} (112)

$$-M \frac{dv_{iz}}{d\xi} + \frac{me^2}{cB_0} M \sin^2 \alpha \frac{d^2 \psi}{d\xi^2} \frac{dv_{iz}}{d\xi} + \cos \alpha \frac{d}{d\xi} (n \psi_{iz}) = 0 \hspace{1cm} (113)$$

$$\cos \alpha \sin^2 \alpha \frac{d^3 \psi}{d\xi^3} (\varphi - \psi) = -\frac{4\pi eM}{c^2} \frac{d}{d\xi} (n \psi_{iz} - n_v \psi_{iz})$$  \hspace{1cm} (114)

Due to the change neutrality condition (7), we take

$$n_i = n_e = n'$$  \hspace{1cm} (116)

For solitary wave solutions we take the following condition to be satisfied by the dependent variables at $|\xi| \to \infty$.

$$n_e, n_i \to n_0, \quad \varphi, \psi \to 0,$$

$$v_{ez}, v_{iz} \to 0, \quad \frac{dn}{d\xi} \to 0$$

as $\xi \to \pm \infty$.  \hspace{1cm} (117)

Integrating each of the equations (115), (112), (114) once and using the conditions (117), we get respectively the equations

$$\cos \alpha \sin^2 \alpha \frac{d^2 \psi}{d\xi^2} (\varphi - \psi) = -\frac{4\pi eM}{c^2} \psi_{iz} - n'(v_{iz} - v_{ez})$$  \hspace{1cm} (118)
\[-M(n' - n_o) + \frac{m_c^2 M}{e B_o^2} \sin^2 \alpha \frac{d^2 \phi}{d \xi^2} n' + v_{ix} n' \cos \alpha = 0 \] (119)
\[-M(n' - n) + n' v_{ex} \cos \alpha = 0 \] (120)

Multiplying (119) by
\[\frac{1}{n'} \frac{dv_{ix}}{d \xi} \]
and then subtracting this from (113) we get
\[-M \frac{dv_{ix}}{d \xi} + \frac{M}{n'} (n' - n_o) \frac{dv_{ix}}{d \xi} = -\frac{e}{m_i} \cos \alpha \frac{dv}{d \xi} \] (121)

From (5) we get
\[\frac{dv}{d \xi} = \frac{T_e}{m_i n_o} \frac{dn'}{d \xi} \] (122)

Substituting this expression for \( \frac{dv}{d \xi} \) in equation (121) and then integrating, we get
\[v_{ix} = \frac{T_e \cos \alpha}{m_i n_o} (n' - n_o) \] (123)

Substituting this expression for \( v_{ix} \) in equation (119) we get
\[\frac{m_c^2 M}{e B_o^2} \sin^2 \alpha \frac{d^2 \phi}{d \xi^2} = (n' - n_o) \left( M^2 - \frac{T_e \cos^2 \alpha}{m_i n_o} n' \right) \] (124)

Finally substituting for
\[\frac{dv}{d \xi}, \quad v_{ex}, \quad v_{ix}\]
respectively from (122), (120), (123) in equation (118) and then eliminating \( d^2 \phi/d \xi^2 \) from the resulting equation by the use of (124), we get the following equation for \( n' \).
\[M \rho_s^2 \frac{d}{d \xi} \left( \frac{1}{n} \frac{dn}{d \xi} \right) \cos \alpha \sin^2 \alpha = \frac{M_c^2}{V_A^2} \cos \alpha (n-1)n \]
\[\frac{M^3}{V_A^2 \cos \alpha} (n-1) + \frac{\cos \alpha}{Mn} (M^2 - n_c^2 \cos^2 \alpha) (n-1) \] (125)
where \( n = n' / n_0 \), and
\[
V_A = B_0 / (4\pi m_e n_0)^{1/2}, \quad c_s = (T_e / m_i)^{1/2}, \quad \rho_s = c_s / \omega_{ci}
\]
which are respectively the Alfven velocity, ion-acoustic speed and equivalent ion-gyroradius.

Multiplying (125) by
\[
\frac{1}{n} \frac{dn}{d\xi}
\]
and then integrating we get the following desired equation for \( n \).
\[
\begin{align*}
M^2 \cos^2 \alpha \sin^2 \alpha \left(-\frac{dn}{d\xi}\right)^2 &\cos^2 \alpha n^2 (n^2 - 1) M^2 \chi \\
- 2 (M'^4 + M'^2 \chi \cos^2 \alpha + \chi \cos^4 \alpha) n^2 (n-1) \\
+ 2 (M'^4 + M'^2 \cos^2 \alpha + \chi \cos^4 \alpha) n^2 \log n - 2M'^2 \cos^2 \alpha n (n-1)
\end{align*}
\]
where
\[
M' = \frac{M^2 \cos^2 \alpha \sin^2 \alpha (-\frac{dn}{d\xi})^2 \cos^2 \alpha n^2 (n^2 - 1) M^2 \chi}{V_A^2}, \quad \xi = \frac{\xi}{\rho_s}, \quad \chi = \frac{c_s^2}{V_A^2}
\]
which are dimensionless quantities.

2.3 SOLITARY WAVE SOLUTIONS WITH HIGHER ORDER CORRECTION

If we set \( n = 1 + N \) and stretch \( \bar{\xi} \) - co-ordinate according to the relation,
\[
X = e^{1/2 \bar{\xi}}
\]
and then expand r.h.s. of (127) in ascending powers of \( N \) we get the following equation, where \( \varepsilon \) is a smallness parameter and is a measure of weakness of dispersion.
\[
\frac{\varepsilon}{2} \left( \frac{dN}{dX} \right)^2 = \frac{1}{2} \Delta_1 N^2 + \frac{1}{3} \Delta_2 N^3 + \frac{1}{4} \Delta_3 N^4 + \frac{1}{5} \Delta_4 N^5 + \ldots
\]
where

\[ \Delta_1 = \frac{1}{k_1^2 k_3^2} \left[ -M'^2 + (k_1^2 + k_2^2) - k_1^2 k_2^2 M'^2 \right], \]

\[ \Delta_2 = \frac{1}{k_1^2 k_3^2} \left[ -2M'^2 + (k_1^2 + 3k_2^2) - 2k_1^2 k_2^2 M'^2 \right], \]

\[ \Delta_3 = \frac{1}{3k_1^2 k_3^2} \left[ M'^2 + (k_1^2 - 6k_2^2) + k_1^2 k_2^2 M'^2 \right] \]

\[ \Delta_4 = \frac{1}{6k_1^2 k_3^2} \left( M'^2 + k_1^2 + k_1^2 k_2^2 M'^2 \right) \]  

(131)

In these expressions, \( k_1, k_2, k_3 \) are given by

\[ k_1 = \cos \alpha, \quad k_2 = \sqrt{\chi} \cos \alpha, \quad k_3 = \sin \alpha \]  

(132)

To solve equation (130), we make the following perturbation expansions of \( N \) and \( M' \).

\[ N = \epsilon N^{(1)} + \epsilon^2 N^{(2)} + \ldots \]

\[ M' = M'^{(0)} + \epsilon M'^{(1)} + \ldots \]  

(133)

where \( M'^{(0)} \) is given by the equation

\[ (M'^{(0)} - k_1^2) (M'^{(0)}^2 - k_2^2) = 0 \]  

(134)

This gives two values of \( M'^{(0)} \):

\[ M'^{(0)} = k_1^2, \quad M'^{(0)} = k_2^2 \]  

(135)

which correspond respectively to the kinetic Alfvén mode and ion-acoustic mode.

Substituting the expansion for \( M' \) given by (133) in the expressions for \( \Delta_j \)'s given by (131), we get the following expansions for \( \Delta_j \)'s, \( j=1,2,3,4 \) in powers of \( \epsilon \).

\[ \Delta_1 = \epsilon \delta_1^{(1)} + \epsilon^2 \delta_1^{(2)} + \epsilon^3 \delta_1^{(3)} + \ldots \]

\[ \Delta_2 = \delta_2^{(0)} + \epsilon \delta_2^{(1)} + \epsilon^2 \delta_2^{(2)} + \ldots \]

\[ \Delta_3 = \delta_3^{(0)} + \epsilon \delta_3^{(1)} + \ldots \]

\[ \Delta_4 = \delta_4^{(0)} + \ldots \]  

(136)

where the \( \delta_j^{(0)} \)'s are given in appendix-2C on page 80 .

Substituting the expansions (133) in equation (130) and then equating different powers of \( \epsilon \) on both sides, we get a sequence of equations for \( N^{(i)} \)'s. The equation for \( N^{(i)} \) for each \( i \geq 2 \) becomes a first order inhomogeneous differential equation.
(i) The equation at the lowest order

In the lowest order we get the following equation, which is the equation for \( N^{(1)} \):

\[
\frac{dN^{(1)}}{dX} \left( \frac{dN^{(1)}}{dX} \right)^2 = \delta_{1}^{(1)} N^{(1)} + \frac{2}{3} \delta_{2}^{(0)} N^{(1)} \tag{137}
\]

This equation has the solitary wave solution

\[
N^{(1)} = a_1 \text{sech}^2 \eta \tag{138}
\]

where

\[
a_1 = \frac{3 \delta_{1}^{(1)}}{2 |\delta_{2}^{(0)}|}
\]

and

\[
\eta = \sqrt{\lambda X}, \tag{139}
\]

\( \lambda \) being given by

\[
\lambda = \delta_{1}^{(1)} / 4 \tag{140}
\]

(ii) Solution for \( N^{(2)} \)

At the next order, which is of the order \( \varepsilon^4 \), we get the following equation for \( N^{(2)} \):

\[
\frac{dN^{(1)}}{dX} \frac{dN^{(2)}}{dX} - \left[ \delta_{1}^{(1)} N^{(1)} + \delta_{2}^{(0)} N^{(1)} \right] N^{(2)} = - \frac{1}{2} \delta_{1}^{(2)} N^{(1)} + \frac{1}{3} \delta_{2}^{(1)} N^{(1)} + \frac{1}{4} \delta_{3}^{(0)} N^{(1)} \tag{141}
\]

Taking derivative of (137) with respect to \( X \) we get

\[
\frac{d^2 N^{(1)}}{dX^2} = \delta_{1}^{(1)} N^{(1)} + \delta_{2}^{(0)} N^{(1)} \tag{142}
\]
In view of this relation we replace the coefficient of \( N^{(2)} \) on the left hand side of (141) by \( d^2 N^{(1)} / dX^2 \). Then multiplying both sides of this equation by \( 1/N\text{X}^{(1)} \) and introducing the variable \( \eta \) given by the second equation of (139), the equation (141) can be put in the following form.

\[
\frac{d}{d\eta} \left( \frac{N^{(2)}}{N^{(1)}} \right) = \frac{\delta_1^{(0)}}{2 N^{(1)^2}} + \frac{\delta_2^{(1)}}{3 N^{(1)^2}} + \frac{\delta_3^{(0)}}{4 N^{(1)^2}} + \frac{N^{(1)}}{N^{(1)^2}}
\]

(143)

where \( \delta_j^i = \delta_j^i / \lambda \)

(144)

Integrating (143) after substitution for \( N^{(1)} \) from (138), we get

\[
N^{(2)} = (a_2 + b_2 \tanh^2 \eta) \sech^2 \eta \frac{1}{4} \sum_{i=0}^{1} \delta_i^{(2)} \eta \sech^2 \eta \tanh \eta
\]

+ \( c^{(2)} \tanh \eta \sech^2 \eta \)

(145)

where

\[
a_2 = -\frac{a_1^2}{6} \delta_2^{(1)} + \frac{1}{8} \delta_3^{(0)}
\]

(146)

\[
b_2 = -\frac{1}{8} \delta_3^{(0)}
\]

Since \( N^{(2)}/N^{(1)} \) should remain finite at \( \eta \to \pm \infty \), we must set

\[
\frac{\delta_1^{(2)}}{} = 0
\]

(147)

This determines \( M^{(2)} \). Again for \( N^{(2)} \) to attain maximum at \( \eta = 0 \), we must set \( c^{(2)} = 0 \). Therefore the solution for \( N^{(2)} \) finally becomes

\[
N^{(2)} = (a_2 + b_2 \tanh^2 \eta) \sech^2 \eta
\]

(148)

where \( a_2 \) and \( b_2 \) are given by (146).
(iii) Solution for $N^{(3)}$

At the order $e^5$ we get the equation for $N^{(3)}$. Proceeding in the same way as in the derivation of the equation (143), we get the following similar equation for $N^{(3)}$.

$$
\frac{d}{d\eta} N^{(3)} = \frac{1}{N_n^{(1)}} \left[ \frac{\delta_1^{(1)} N^{(2)}}{2} + \frac{\delta_1^{(3)} N^{(1)^3}}{2} + \frac{\delta_2^{(2)} N^{(1)^2} N^{(2)}}{2} \right. \\
+ \frac{\delta_2^{(1)} N^{(1)^2} N^{(2)}}{3} + \frac{1}{5} \left. \delta_2^{(2)} N^{(1)^3} + \delta_3^{(1)} N^{(1)} N^{(2)} \right]
$$

Substituting here for $N^{(1)}$ and $N^{(2)}$ from (138) and (148) respectively and then integrating, we get

$$
N^{(3)} = (a_3 + b_3 \tanh^2 \eta + c_3 \tanh^4 \eta) \sech^2 \eta - \frac{a_1}{4} \delta_1^{(3)} \eta \tanh \eta \sech^2 \eta \\
+ c^{(3)} \tanh \eta \sech^2 \eta
$$

where $c^{(3)}$ is an arbitrary constant and $a_3, b_3, c_3$ are given by

$$
a_3 = \frac{1}{\delta_1^{(1)}} a_2^2 + \frac{1}{\delta_2^{(1)}} a_1 (a_2 + b_2) + \frac{1}{\delta_2^{(2)}} a_1^2 \\
+ \frac{1}{2} \delta_3^{(1)} a_1^2 a_2 + \frac{1}{8} \delta_3^{(1)} a_1^3 + \frac{1}{10} \delta_4^{(6)} a_1^4 + \frac{1}{2} \delta_2^{(6)} a_2^2 \\
- \frac{1}{2} \delta_2^{(1)} a_1 b_1 \\
b_3 = \frac{1}{\delta_1^{(1)}} b_2^2 - \delta_2^{(6)} b_2 a_2 - \frac{1}{2} \delta_2^{(1)} a_1 b_2 - \delta_3^{(6)} a_1^2 (a_2 - b_2)
$$
Since \( N^{(3)} / N^{(2)} \) should remain finite at \( \eta \to \infty \), we must have

\[
\overline{\delta_1}^{(3)} = 0
\]  

which determine \( M^{(3)} \). Further for \( N^{(3)} \) to attain maximum at \( \eta = 0 \), we must set \( c^{(3)} = 0 \). Hence, the final expression for \( N^{(3)} \) becomes

\[
N^{(3)} = (a_3 + b_3 \tanh^2 \eta + c_3 \tanh^4 \eta) \text{sech}^2 \eta
\]  

So we have the following expression for \( N \) correct to order \( \epsilon^2 \) terms as given by the first equation of (133),

\[
N = a(1 + \alpha_2 \tanh^2 \eta + \alpha_4 \tanh^4 \eta) \text{sech}^2 \eta
\]  

where

\[
a = a_1 + a_2 + a_3
\]

\[
\alpha_2 = (b_2 + b_3) / (a_1 + a_2 + a_3)
\]

\[
\alpha_4 = c_3 / (a_1 + a_2 + a_3)
\]  

The width \( D = X \) of the solitary wave (154) is determined by the equation

\[
N(X) = 0.42 a
\]  

Thus we find that the width of the solitary wave (154) is

\[
D = \eta_D / \sqrt{\lambda}
\]  

where \( \eta_D \) is the positive root of the following equation in \( \eta \).

\[
\alpha_4 \tanh^6 \eta + (\alpha_2 - \alpha_4) \tanh^4 \eta + (1 - \alpha_2) \tanh^2 \eta - 0.58 = 0
\]  

\[
(151)
\]  

\[
(152)
\]  

\[
(153)
\]  

\[
(154)
\]  

\[
(155)
\]  

\[
(156)
\]  

\[
(157)
\]
2.4 HIGHER ORDER SOLUTIONS FOR ALFVEN AND ION-ACOUSTIC MODES

The higher order solutions for kinetic Alfven and ion-acoustic solitary waves can now be obtained from the solutions obtained in the previous section by setting $M^{(0)2} = k_1^2$ for Alfven mode and $M^{(0)2} = k_2^2$ for ion-acoustic mode

(i) \textit{Alfven mode}

For the Alfven mode we have $M^{(0)2} = k_1^2$ and consequently $\delta_i^{(0)}$'s given in the Appendix-2C on page 80 become as follows.

\begin{align*}
\delta_1^{(0)} &= 0, \quad \delta_1^{(1)} = \frac{2(1-\chi)}{\sin^2 \alpha} M^{(1)} \\
\delta_1^{(2)} &= \frac{1}{\sin^2 \alpha} \left[ 2M^{(2)} (1 - \chi) + M^{(3)} (1 + 3\chi) \right] \\
\delta_1^{(3)} &= \frac{1}{\sin^2 \alpha} \left[ 2M^{(3)} (1 - \chi) - 2M^{(2)} M^{(1)} (1 + 3\chi) - 4\chi M^{(3)} \right] \\
\delta_2^{(0)} &= \frac{1}{\sin^2 \alpha} (1 - \chi), \quad \delta_2^{(1)} = \frac{4(1-\chi)}{\sin^2 \alpha} \frac{M^{(1)}}{M^{(1)}} \\
\delta_2^{(2)} &= \frac{2}{\sin^2 \alpha} \left[ 2M^{(2)} (1 - \chi) - M^{(3)} (1 + 3\chi) \right] \\
\delta_3^{(0)} &= \frac{1}{3\sin^2 \alpha} (2 - 5\chi), \quad \delta_3^{(1)} = \frac{2(1-\chi)}{3 \sin^2 \alpha} \frac{M^{(1)}}{M^{(1)}} \\
\delta_4^{(0)} &= \frac{1}{6\sin^2 \alpha} (2 + \chi) \quad (158)
\end{align*}
where
\[ \bar{M}^{(i)} = -\frac{M^{(i)}}{k_i}, \quad (i = 1, 2, 3) \] (159)

By the use of the relations (128) and (159), the expansion for \( M \) can be obtained from (133).
\[ \frac{M}{V_A \cos \alpha} = 1 - e - e^2 - e^3 M^{(3)} - \ldots \] (160)

With the expression for \( \delta_1^{(1)} \) given by (158), \( \lambda \) as given by (140) becomes
\[ \lambda = \frac{(1 - \chi) \bar{M}^{(1)}}{2 \sin^2 \alpha} \] (161)

and with the expressions for \( \delta_1^{(1)} \)'s given by (158) the quantities \( a_1, a_2, b_2, a_3, b_3, c_3 \) appearing in the expression for \( N^{(1)} , N^{(2)} , N^{(3)} \) become as follows.
\[ a_1 = 3 \bar{M}^{(1)} \]
\[ a_2 = \frac{3(10 - \chi)}{4(1 - \chi)} \bar{M}^{(1)} \]
\[ b_2 = \frac{9(2 - 5\chi)}{4(1 - \chi)} \bar{M}^{(1)} \]
\[ a_3 = \frac{3 \bar{M}^{(1)}}{40(1 - \chi)^2} (232 + 184\chi - 11\chi^2) \]
\[ b_3 = -\frac{9\bar{M}^{(1)}}{80(1 - \chi)^2} (164 - 652\chi + 83\chi^2) \]
\[ c_3 = \frac{9 \bar{M}^{(1)}}{40(1 - \chi)^2} (44 - 292\chi + 383\chi^2) \] (162)
The conditions (147) and (152) give for \( M^{(2)} \) and \( M^{(3)} \) the expressions:

\[
\begin{align*}
\bar{M}^{(2)} &= \frac{(1 + 3\chi)\bar{M}^{(1)}^2}{2(1 - \chi)} \\
\bar{M}^{(3)} &= \frac{\bar{M}^{(1)}^3}{2(1 - \chi)^2} (1 + 10\chi + 5\chi^2)
\end{align*}
\] (163)

With the expressions for \( a_j \)'s given by (162), the quantities \( a, a_2, a_4 \) appearing in the expression (154) for \( N \) become as follows for kinetic Alfvén mode,

\[
\begin{align*}
a &= 3\bar{M}^{(1)} P \\
\alpha_2 &= -\frac{3\bar{M}^{(1)} (2 - 5\chi)}{4P (1 - \chi)} [1 + \frac{\bar{M}^{(1)} (44 - 292\chi + 383\chi^2)}{10 (1 - \chi) (2 - 5\chi)}] \\
\alpha_4 &= \frac{3 \bar{M}^{(1)}^2}{40P (1 - \chi)^2} (44 - 292\chi + 383\chi^2)
\end{align*}
\] (164)

where

\[
P = 1 + \bar{M}^{(1)} \frac{(10 - \chi)}{4(1 - \chi)} + \bar{M}^{(1)} \frac{\bar{M}^{(1)}^2}{40(1 - \chi)^2} (232 - 184\chi - 11\chi^2)]
\] (165)

The expression for the Mach number \( M / (V_A \cos \alpha) \) given by (160) correct to order \( \epsilon^3 \) terms becomes

\[
\frac{M}{V_A \cos \alpha} = 1 - M^{(1)} - \bar{M}^{(1)} \frac{(1 + 3\chi)}{2(1 - \chi)} - \frac{\bar{M}^{(1)}^3}{2(1 - \chi)^2} (1 + 10\chi + 5\chi^2)
\] (166)
(ii) **Ion-acoustic mode**

For the ion-acoustic mode we have $M'^2 = k_2^2$ and consequently $\delta_i^{(0)}$'s given in the appendix - 2C on page 80 become

\[
\delta_1^{(0)} = 0, \quad \delta_1^{(1)} = \frac{2(1 - \chi)}{\sin^2 \alpha} \overline{M}^{(1)},
\]

\[
\delta_2^{(2)} = \frac{1}{\sin^2 \alpha} \left[ 2M^{(2)} (1 - \chi) - M^{(1)} (3 + \chi) \right],
\]

\[
\delta_3^{(3)} = \frac{1}{\sin^2 \alpha} \left[ 2M^{(3)} (1 - \chi) - 2M^{(2)} M^{(1)} (3 + \chi) + 4M^{(1)} \right],
\]

\[
\delta_4^{(6)} = \frac{1}{6 \sin^2 \alpha} (2 + \chi), \quad \delta_5^{(1)} = \frac{2 (1 - \chi)}{3 \sin^2 \alpha} \overline{M}^{(1)},
\]

where

\[
\overline{M}^{(i)} = \frac{M^{(i)}}{k_2}, \quad (i = 1,2,3)
\]

(167)

By the use of the relations (128) and (168) $M$ for ion-acoustic mode can be obtained from (133) as follows.
\[ \frac{M}{c_s \cos \alpha} = 1 + \varepsilon M^{(1)} + \varepsilon^2 M^{(2)} + \varepsilon^3 M^{(3)} + \ldots \] (169)

where \( M^{(2)} \) and \( M^{(3)} \) as determined from the conditions (147) and (152) become

\[
\begin{align*}
\bar{M}^{(2)} &= \frac{(3 + \chi)}{2(1 - \chi)} M^{(1)}, \\
\bar{M}^{(3)} &= \frac{M^{(1)}^3}{2(1 - \chi)} (5 + 10\chi + \chi^2)
\end{align*}
\] (170)

Therefore the expression for the Mach numbers for ion-acoustic solitary wave correct to order \( c^3 \) terms becomes

\[
\frac{M}{c_s \cos \alpha} = 1 + M^{(1)} + \frac{(3 + \chi)}{2(1 - \chi)} M^{(1)} + \frac{(5 + 10\chi + \chi^2)}{2(1 - \chi)} M^{(1)}^3
\] (171)

From (158) and (167) we find that the \( \delta_i^{(0)} \)'s, which are involved in the expression for \( N \) given by (154) have the same values for both Alfvén and ion-acoustic modes, where \( M^{(1)} \) appears as a parameter. Hence solitary wave profile for ion-acoustic mode has the same form as for the kinetic Alfvén mode.

The width \( D \) of both Alfvén and ion-acoustic solitary wave is given by (156), where \( \eta_D \) is the positive root of the equation (157), whose coefficients are expressed in terms of \( \alpha_2, \alpha_4 \). There two quantities have the same expression for both kinetic Alfvén and ion-acoustic mode.

The amplitude \( a \) for both Alfvén and ion-acoustic solitary wave is given by the first equation (164), which has been expressed as a function of the parameter \( M^{(1)} \). The Mach numbers \( M/V_A \cos \alpha \) and \( M/c_s \cos \alpha \) for Alfvén and ion-acoustic solitary waves respectively have also been expressed as functions of the parameter \( M^{(1)} \), and these are given respectively by (166) and (171). Therefore eliminating the parameter \( M^{(1)} \) between the expressions for \( M/V_A \cos \alpha \) and \( a \), the Mach number for kinetic Alfvén solitary wave can be expressed as a function of solitary wave amplitude. Similarly the Mach number of ion-acoustic solitary wave can be expressed as a function of solitary wave amplitude.
Fig. 6: Mach number as a function of soliton amplitude $a$

- KdV soliton; - higher order soliton.
Fig. 7: Soliton width $D$ as a function of soliton amplitude
- $\ldots$. KdV soliton; - higher order soliton.
In fig. 6 graphs are plotted showing the variations of the Mach numbers $M/V_A \cos \alpha$ and $M/c_s \cos \alpha$ for Alfvén and ion-acoustic solitary waves against their amplitudes $a$ for one particular value of $\chi(\chi=0.1)$. For comparison corresponding graphs are drawn (broken lines) for KdV soliton. From this graphs it is seen that solitary kinetic Alfvén wave moves with a faster speed and solitary ion-acoustic wave with a slower speed than the speeds predicted by KdV theory.

As the coefficients of equation (157) are functions of the parameter $\bar{M}^{(1)}$, its positive root $\eta_D$ is also a function of parameter $\bar{M}^{(1)}$. Consequently the width $D$ of both Alfvén and ion-acoustic solitary wave determined by (156) will also be a function of $\bar{M}^{(1)}$. Therefore following the same procedure as in the case of Mach numbers we can express soliton width $D$ as a function of soliton amplitude $a$.

In fig. 7 graphs are plotted showing the variation of soliton width $D$, which is same for both the modes, against solitary wave amplitude $a$ for $\chi=0.1$. Here also for comparison corresponding graphs are drawn for KdV soliton. From these graphs it is found that the width of a solitary kinetic Alfvén or ion-acoustic wave is greater than that of a KdV soliton.
SUMMARY AND CONCLUSION:

Starting from a set of governing equations, which produce linear dispersion relation coupling Kinetic Alfvén wave and ion-acoustic wave, we have made in this chapter a study of the stability and higher order effects for kinetic Alfvén and ion-acoustic solitary waves. For stability analysis KdV equations in three dimensions are derived for both the waves and the small-k perturbation expansion method of Rowlands and Infeld is applied. It is found that a kinetic Alfvén soliton is unstable if the direction along which the plane-wave perturbation is given lies inside a core. The growth rate of instability attains maximum for perturbations given along a direction lying in the plane containing the external magnetic field and the direction of propagation of the solitary wave. Figures are drawn showing the square of the dimensionless growth rate of instability as a function of $\theta$ for some different values of $\alpha$, where $\alpha$ and $\theta$ are the angles made by the direction of external magnetic field with the direction of propagation of the solitary wave and the direction of the plane-wave perturbation respectively. For ion-acoustic solitons it is found that there is instability if the perturbation is given in a direction lying in the region bounded by two planes intersecting along the direction of propagation of the solitary wave. The growth rate of instability attains maximum for perturbations given in a plane perpendicular to the direction of propagation of the solitary wave. Graphs are drawn showing the variation of the square of the dimensionless growth rate of instability against $\theta'$ for some different values of $\alpha$, where $\theta'$ is the angle made by the direction of perturbation with the plane containing the external magnetic field and the direction of propagation of the soliton. Higher order nonlinear and dispersive effects have been obtained for both kinetic Alfvén and ion-acoustic solitary waves by the method developed by Das and Majumder (1991). Assuming a solitary wave propagation a single equation is obtained for the perturbed electron or ion number density. Perturbation solution of this equation is obtained by the Bogoliubov-Mitropolsky method. The secularity removing condition at each order gives corrections to the Mach number. It is found that both the modes have the same solitary wave profile. Each of the quantities, width of the common profile, Mach numbers and amplitudes, have been expressed as functions of a parameter. Eliminating this parameter the width of the common profile and Mach numbers can be expressed as functions of solitary wave amplitude. Graphs are plotted for these two quantities showing their variations with the amplitude. It is found that solitary kinetic Alfvén waves move with a faster speed and solitary ion-acoustic wave with a
slower speed than the speeds predicted by the KdV theory, and the width of the higher order soliton is greater than that of a KdV soliton.

APPENDIX - 2A

\[ \beta = \lambda_2 \cos \alpha, \quad \gamma = \lambda_1 \sin^2 \alpha \cos \alpha, \]
\[ a_1 = \lambda_1 \sin \alpha \left( \sin^2 \alpha - 2 \cos^2 \alpha \right), \quad a_2 = \lambda_1 \cos \alpha \left( 2 \sin^2 \alpha - \cos^2 \alpha \right) \]
\[ a_3 = \lambda_1 \cos^2 \alpha \sin \alpha, \quad a_4 = -\lambda_1 \cos \alpha, \quad a_5 = \lambda_1 \sin \alpha \]
\[ b_1 = \lambda_2 \sin \alpha, \quad b_2 = \lambda_3 \sin \alpha \left( \sin^2 \alpha - \cos^2 \alpha \right) \]
\[ b_3 = \lambda_3 \sin^2 \alpha \cos \alpha \]
\[ b_4 = -\lambda_3 \sin \alpha \cos^2 \alpha \]
\[ b_5 = \lambda_3 \cos \alpha \left( \sin^2 \alpha - \cos^2 \alpha \right), \quad b_6 = \lambda_3 \sin \alpha \cos^2 \alpha \]
\[ b_7 = \lambda_3 \sin \alpha, \quad b_8 = -\lambda_3 \cos \alpha \]
\[ \lambda = \lambda_3 \sin^2 \alpha, \]
\[ c_2 = \lambda_3 \cos^2 \alpha \]

APPENDIX - 2B

\[ \beta' = \lambda_4 \cos \alpha, \]
\[ \gamma' = \lambda_4 \sin^2 \alpha \cos \alpha, \]
\[ a_1' = \lambda_4 \sin \alpha \left( \sin^2 \alpha - 2 \cos^2 \alpha \right), \quad a_2' = \lambda_4 \cos \alpha \left( 2 \sin^2 \alpha - \cos^2 \alpha \right) \]
\[ a_3' = \lambda_4 \cos^2 \alpha \sin \alpha, \quad a_4' = -\lambda_4 \cos \alpha \]
\[ a_5' = \lambda_4 \sin \alpha \]

APPENDIX - 2C

\[ \delta_1^{(0)} = \frac{1}{k_1^2 k_3^2 M^{(0)}^2} \left( M^{(0)^2} - k_1^2 \right) \left( M^{(0)^2} - k_2^2 \right) \]
\[ \delta_1^{(1)} = -\frac{2M M^{(1)}}{k_1^2 k_3^2 M^{(0)^2}} \left( M^{(0)^4} - k_1^2 k_2^2 \right) \]
\[ \delta_1^{(2)} = \frac{1}{k_1^2 k_3^2 M^{(6)}} \left[ -2M^{(6)} M^{(2)} (M^{(6)} - k_1^2 k_2^2) \right. \\
- M^{(1)2} (M^{(6)} + 3k_1^2 k_2^2)] \\

\delta_1^{(3)} = \frac{1}{k_1^2 k_3^2 M^{(6)}} \left[ -2M^{(6)} M^{(6)} (M^{(6)} - k_1^2 k_2^2) \right. \\
- 2M^{(2)} M^{(1)} M^{(6)} (M^{(6)} + 3k_1^2 k_2^2) + 4M^{(1)3} k_1^2 k_2^2 \right]

\delta_2^{(0)} = -\frac{1}{k_1^2 k_3^2 M^{(6)}} \left[ 2M^{(6)} - 2M^{(6)} (k_1^2 + 3k_2^2) + 2k_1^2 k_2^2 \right]

\delta_2^{(1)} = -\frac{2M^{(1)}}{k_1^2 k_3^2 M^{(6)}} \left[ 2M^{(6)} M^{(2)} (M^{(6)} - k_1^2 k_2^2) \right. \\
+ M^{(1)} (M^{(6)} + 3k_1^2 k_2^2)] \\

\delta_3^{(0)} = -\frac{1}{3k_1^2 k_3^2 M^{(6)}} \left[ M^{(6)} + (k_1^2 - 6k_2^2) M^{(6)} + k_1^2 k_2^2 \right]

\delta_3^{(1)} = -\frac{2M^{(1)}}{6k_1^2 k_3^2 M^{(6)}} \left( M^{(6)} - k_1^2 k_2^2 \right)

\delta_4^{(0)} = -\frac{1}{6k_1^2 k_2^2 M^{(6)}} \left[ M^{(6)} + k_1^2 M^{(6)} + k_1^2 k_2^2 \right]