Chapter 4

Nonclassical light generation in a cavity of variable length

In this chapter\(^1\), a simple model consisting of a cavity with a movable mirror in one dimension is considered. This problem has received considerable attention in the recent past in the context of particle creation [1-5]. The radiation that is produced due to the mirror motion is a purely quantum mechanical effect having no classical analogues. Hence we expect nonclassical features to manifest in the fields so produced. This radiation arises due to the interaction of the mirror with the vacuum fluctuations of a quantised field. The mirror need not be a physical one but a sudden change in the refractive index of medium can also produce real photons from an initial vacuum state [6-8]. In this chapter, we study the quantum statistical properties of the field so produced due to accelerated mirror motion and study the nonclassical nature of the field. We restrict to only one dimension. In four dimensions the problem was studied by Candelas et al [9]. The problem of a spherical mirror expanding with uniform acceleration was considered by Frolov et al [10]. By making a conformal transformation, the nonstationary problem is mapped onto that of a stationary one and the field solution inside the cavity is obtained. Then by the application of a canonical quantisation procedure [1,3,11], a quantised version of the solution is derived. It then follows that the "in-out" mirror motion corresponds to a Rogolhibov transformation of the annihilation and creation operators. Then, by a calculation of the variances in the field quadratures, it is shown that initially, the

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state of the field is vacuum or a coherent state, then, the motion of the mirror produces squeezing of fluctuations in the field quadratures. It is also shown that the various modes of the cavity get correlated as a result of the mirror motion.

4.1 The model

Consider a scalar electromagnetic field within a region bounded by two mirrors, one of which executes an "in-out" motion. The requirement that the motion \( q(ct) \) of the mirror starts at some finite past and stops at some finite future, at least asymptotically, is what is classified as an "in-out" motion (see Fig.4.1). We restrict the mirror motion to only one dimension.

Let us consider the wave equation for an electromagnetic field in the region bounded by the two mirrors

\[
\frac{\partial^2 \phi(x, ct)}{\partial x^2} = \frac{\partial^2 \phi(x, ct)}{\partial (ct)^2},
\]

where, \( \phi(x, ct) \) is the scalar electromagnetic field and \( c \) is the velocity of light in vacuum. The usual boundary conditions demand that the field vanishes at the boundaries \( x = 0 \) and \( x = q(ct) \). So we have the boundary condition,

\[ \phi(x = 0, ct) = \phi(x = q(ct), ct) = 0 \quad \forall ct. \]  

The solution of (4.1) with boundary condition (4.2) is simple for the case of stationary mirrors:

\[ u_n \sim \sin\left(\frac{n\pi x}{L}\right) \exp\left(-i\frac{n\pi ct}{L}\right), \]

when the mirrors are located at \( x = 0 \) and \( x = L \). The basic idea in solving the non-stationary problem is to do a conformal transformation [1,3,11], such that it, reduces to that of a stationary problem in the transformed coordinates.

To that end we make the following transformation [1,11]

\[
\begin{align*}
w + s &= R(ct + x) \\
w - s &= R(ct - x)
\end{align*}
\]
where \( R \) is a function that has to be determined. The function \( R \) can be determined by requiring that the transformation (4.4), map the boundary at \( x = 0 \) to \( s = 0 \) and \( x = q(ct) \) to \( s = 1 \). Thus from the transformation (4.4) it immediately follows that, \( R \) must satisfy

\[
R(ct + q(ct)) - R(ct - q(ct)) = 2.
\]  

Furthermore, there is another condition on the function \( R \) arising due to a physical restriction on the mirror motion — that its velocity should be strictly less than that of
light:

$$|\dot{q}(ct)| < 1.$$  \hspace{1cm} (4.6)

Differentiating (4.5) with respect to time, we have,

$$R'(\zeta)(1+q) = R'(\zeta)(1 - q)$$  \hspace{1cm} (4.7)

where $\zeta = ct + q(ct)$ and $\zeta = ct - q(ct)$. Solving for $\dot{q}(ct)$, we have

$$\dot{q}(t) = \frac{R'(ct - q(ct)) - R'(ct + q(ct))}{R'(ct - q(ct)) + R'(ct + q(ct))}.$$  \hspace{1cm} (4.8)

On comparing (4.6) and (4.8), we find that

$$|R'(ct - q(ct)) - R'(ct + q(ct))| < |R'(ct - q(ct)) + R'(ct + q(ct))|.$$  \hspace{1cm} (4.9)

It thus follows that, $R$ should be such that $R'$ should not change its sign if condition (4.6) is to be satisfied.

As a result of the transformation (4.4), the wave equation gets transplanted to

$$R'(R^{-1}(w + s))R'(R^{-1}(w - s)) \left( \frac{\partial^2 \phi(s, w)}{\partial w^2} - \frac{\partial^2 \phi(s, w)}{\partial s^2} \right) = 0,$$  \hspace{1cm} (4.9)

and the boundary condition becomes

$$\phi(s = 0, w) = \phi(s = 1, w) = 0 \quad \forall w.$$  \hspace{1cm} (4.11)

The solution for (4.10) with initial conditions (4.11) is, in the transformed $(s, w)$ coordinate system

$$u_n \sim Ne^{-in\pi w} \sin(n\pi s)$$

$$\sim N\frac{e^{-in\pi(w-s)} - e^{-in\pi(w+s)}}{2i},$$  \hspace{1cm} (4.12)

where $N$ is a normalisation constant.

The basis of solutions $\{u_n, u_n^*\}$ is orthogonal in the inner product defined as

$$\langle u_n, u_k \rangle \equiv i \int_{s=0}^{1} [u_n \frac{\partial u_n^*}{\partial w} - u_n^* \frac{\partial u_k}{\partial w}] ds.$$  \hspace{1cm} (4.13)
Since \((un, un) = -n\pi\) the normalisation constant is \(\sqrt{n\pi}\). The inner product has the following properties:

\[
\langle u_n, u_k \rangle = -\delta_{n,k} \\
\langle u_n, u_k^* \rangle = 0 \\
\langle u_n^*, u_k^* \rangle = \delta_{n,k}.
\] (4.14)

Thus,

\[
u_n(s, w) = \frac{1}{2i\sqrt{n\pi}}[e^{-i\pi(w-s)} - e^{-i\pi(w+s)}].
\] (4.15)

To get the solution in the original \((x, ct)\) co-ordinate system, we use the transformation (4.4) in (4.15):

\[
u_n(x, ct) = \frac{1}{2i\sqrt{n\pi}}[e^{-i\pi R(ct-x)} - e^{-i\pi R(ct+x)}].
\] (4.16)

Thus, for a given trajectory, \(q(ct)\) of the mirror, the solution to the wave function (4.1) with boundary condition (4.2) is obtained when the function \(R\) is determined. Alternatively, one could specify the function \(R\) subject to the condition (4.9), so that \(q(ct)\) satisfies (4.6). Solving (4.5) the trajectory corresponding to the function \(R\) can be obtained.

Let us now quantise the field inside the resonator [1,11]. The field inside the cavity is now an operator. Consider a time in the remote past, when the movable mirror was stationary at \(q(ct) = D\). The field inside the cavity can then be written in terms of the complete set \(\{u_n, u_n^*\}\) of solutions as

\[
\phi(x, ct) = \sum_n u_n(x, ct)\hat{a}_n + u_n^*(x, ct)\hat{a}_n^\dagger.
\] (4.17)

The lowering and raising operators \(a_n\) and \(\hat{a}_n^\dagger\) satisfy the bosonic commutation relation

\[
[\hat{a}_n, \hat{a}_j^\dagger] = \delta_{nj}, [\hat{a}_n, \hat{a}_j] = [\hat{a}_n^\dagger, \hat{a}_j^\dagger] = 0.
\] (4.18)

After the mirror starts moving, the field inside the cavity is

\[
\hat{\phi}(x, ct) = \sum_k v_k(x, ct)\hat{a}_k + v_k^*(x, ct)\hat{a}_k^\dagger.
\] (4.19)
where, \{u_k, u^*_{k}\} are the solutions (4.15) and \(fa\) and \(\hat{b}_k^\dagger\) are the raising and lowering operators for the non-stationary problem. Since the set \{\(v_k, v^*_k\)\} forms an orthonormal basis, we can express \((u_n, v^*_n)\) in terms of \((u_k, u^*_{k})\)

\[
u_n(x, ct) = \sum_k \alpha_{nk} v_k(x, ct) + \beta_{nk} v^*_k(x, ct),
\]

(4.20)

where the expansion coefficients \(a_{nk}\) and \(\beta_{nk}\) are determined from the definition of the inner product (4.13)

\[
\alpha_{nk} = -\langle u_n, v_k \rangle, \quad \beta_{nk} = \langle u_n, v^*_k \rangle.
\]

(4.21)

Substituting (4.20) in (4.17), we have

\[
\phi(x, ct) = \sum_n \left( \sum_k \alpha_{nk} v_k(x, ct) + \beta_{nk} v^*_k(x, ct) \right) a_n + \\text{free and rjtod} + \beta_{nk}^* v_k(x, ct) \hat{a}_n^\dagger.
\]

(4.22)

Rearranging the order of summation,

\[
\hat{\phi}(x, ct) = \sum_k v_k(x, ct) \hat{b}_k + v^*_k(x, ct) \hat{b}_k^\dagger,
\]

(4.23)

where

\[
\hat{b}_k = \sum_n \alpha_{nk} \hat{a}_n + \beta_{nk}^* \hat{a}_n^\dagger, \quad \hat{b}_k^\dagger = \sum_n \alpha_{nk}^* \hat{a}_n^\dagger + \beta_{nk} \hat{a}_n
\]

(4.24)

and the operators \(b_k\) and \(\hat{b}_k^\dagger\) satisfy the commutation relations

\[
[b_k, \hat{b}_j^\dagger] = \delta_{kj}, [b_k, b_j] = [\hat{b}_k^\dagger, \hat{b}_j^\dagger] = 0.
\]

(4.25)

Thus the mirror motion corresponds to a Bogoliubov transformation of the annihilation and creation operators.
Let us now assume that before the mirror motion starts, the state of the field inside the cavity was the vacuum field. Thus, to begin with there are no photons in any of the modes of the cavity
\[ a_k |0_k; in\rangle = 0, \quad (4.26) \]
where \( O_k \) in \( |0_k; in\rangle \) refers to the vacuum of the \( k \)-th mode and 'in' refers to the remote past. After the mirror starts moving, the operators \( a_k \) and \( \hat{a}_k^\dagger \) get transformed to new operators \( b_k \) and \( efj \). With respect to these new operators, the state \( |0_k; in\rangle \) will no longer be a vacuum state, but will be a squeezed vacuum state. Similarly, if the initial state was a coherent state, with respect to the new operators, it will no longer be coherent state, but will be a squeezed state.

4.2 Specific case: demonstration of squeezing

Let us consider a specific example. There are two approaches: (1) either one can specify a specific trajectory \( q(\xi) \) of the mirror and from that determine the function \( ? \) or (2) one can specify a function \( R \) and then work out a trajectory corresponding to it. Here we choose the second approach. In [11], it was shown that the criterion for particle (photon) creation in the infinite future, due to the mirror motion is that if \( R_{in}(\xi) \) satisfies
\[ R_{in}(\xi) \underset{\xi \rightarrow +\infty}{\rightarrow} \frac{\xi}{d} + \text{constant}, \quad (4.27) \]
then, \( R_{in}(\xi) \) should not satisfy,
\[ R_{in}(\xi) \underset{\xi \rightarrow +\infty}{\rightarrow} \frac{\xi}{d} + \text{constant}'. \quad (4.28) \]
On the contrary, if it satisfies (4.28), then there is no particle creation and the states \( |0_k; in\rangle \) are the same as \( |0_k; out\rangle \). So, based on this criterion, consider the function \( R \) of the following form [11] (see Fig.4.2):
\[ \begin{cases} \frac{\xi}{d} & \text{if } \xi \leq 0 \\ \xi + \frac{A}{e} \sin \left( \frac{\xi}{d} \right) & \text{if } \xi > 0. \end{cases} \quad (4.29) \]
where
\[ A = \frac{d}{D} - 1. \]  
(4.30)

The requirement demanded by the condition (4.6) or (4.9) is guaranteed by
\[ |\lambda| < 1. \]  
(4.31)

For this functional form of \( R \),
\[ \alpha_{nk} = \sqrt{\frac{n}{n-k}} J_{n-k}(-n\lambda) \]
\[ f_{nk} = -\sqrt{\frac{k}{n+k}} J_{n+k}(-n\lambda), \]  
(4.32)

where \( J_n \) is a Bessel function of order \( n \). The steps involved in the evaluation of \( \alpha_{nk} \) and \( f_{nk} \) are provided in the Appendix 4A.

Now, the number of photons created in the mode 'k' which had no photons to begin with (i.e., the mode is in the state \( |0_k; i\rangle \)), as a result of the mirror motion is [5]
\[ \langle 0; i| b_k d_k^\dagger |0; i\rangle = \sum_{n=1}^{\infty} |\beta_{nk}|^2. \]  
(4.33)

where \( \beta_{nk} \) is given by (4.21) and (4.32) corresponding to \( R \) given in (4.29). In Fig.4.3 we plot the number of photons created in each mode of the cavity. It shows that only low frequency modes get any excitation as \( q(ct) \) has only low frequency components in it. Since the frequency \( \omega \) is given by
\[ \omega = \frac{n\pi c}{L}, \]  
(4.31)

where \( L \) is the cavity dimension, the frequency of light corresponding to the low-order modes \( (n) \) would then correspond to optical frequencies \( (\omega \sim 10^{15}) \) if \( L \) is of the order of a micrometer. Thus, in a micron-sized cavity, one can hope to detect photons in the optical domain.

We can now calculate the variance \( S(\theta) \) in the field quadrature defined as
\[ S(\theta) = \langle \tilde{X}_n^2(\theta) \rangle - \langle \tilde{X}_n(\theta) \rangle^2, \]  
(4.35)
where

\[ X_n(\theta) = e^{-i\theta}b_n + e^{i\theta}\hat{b}^\dagger_n. \] (4.36)

To determine if there is squeezing, we differentiate \( S(q) \) with respect to \( \theta \) to obtain \( S_{\min} \). If \( S_{\min} < 1 \) for a particular mode, then it implies that the state of the radiation in that mode is squeezed. In Fig. 4.4 we plot \( S_{\min} \) versus \( A \) for various modes of the cavity. Curves 4.4(a) through (d) correspond to the modes \( k = 1,5,10 \) and 20 respectively. We see that there is considerable amount of squeezing of fluctuations in all the modes of the cavity.

From (4.24), if we calculate the expectation value of \( \hat{b}^\dagger_m\hat{b}_n \) taken with respect to the initial vacuum state, we have

\[ \langle 0; in|\hat{b}^\dagger_m\hat{b}_n|0; in \rangle = \sum_k \beta_{km}\beta^*_{kn}. \] (4.37)

We thus see that the correlation between the \( n \)th and the \( m \)th mode is nonzero

\[ \langle \hat{b}^\dagger_m\hat{b}_n \rangle - \langle \hat{b}^\dagger_m \rangle \langle \hat{b}_n \rangle \neq 0 \] (4.38)
a.s \( \langle b_n \rangle = 0 \) for initial vacuum state. Thus, in addition to squeezing, the mirror motion also introduces correlations between the various modes of the cavity.

In this chapter, we have shown that an "in-out" motion of a mirror satisfying conditions for particle creation, i.e., satisfying (4.27) and not (4.28), in a one dimensional cavity creates particles (photons) and that the state of this field is nonclassical as it manifests squeezing of fluctuations in one of its quadratures. We have also shown that the mirror motion also introduces correlations between the various modes of the cavity.
Figure 4.2: Trajectory of the mirror corresponding to $R_{\text{in}}(\xi)$ given by (4.29).
Figure 4.3: Spectrum of photons created due to the mirror motion.
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Figure 4.4: Demonstration of squeezing in the cavity.
Appendix 4A

Derivation of $\alpha_{nk}$ and $\beta_{nk}$

In this appendix the quantities $a_{nk}$ and $b_{nk}$ corresponding to the function $R_m$ given by (4.29) are derived. From (4.21), $\alpha_{nk}$ and $\beta_{nk}$ are defined as

$$\alpha_{nk} = -\langle u_n, v_k \rangle,$$

$$\beta_{nk} = \langle u_n, v_k^* \rangle. \tag{4A.1}$$

Let us consider the evaluation of $\beta_{nk}$. We need to evaluate

$$\beta_{nk} = \langle u_n, v_k^* \rangle,$$

$$= i \int_0^{\theta} dx [v_k^* \frac{\partial u_n^*}{\partial t} - u_n^* \frac{\partial v_k^*}{\partial t}]. \tag{4A.2}$$

Since the inner product (4.13) is independent of time [11], let us choose $t = 2d$. Then,

$$f_t = -\frac{1}{2\pi \sqrt{nk}} \int_0^d dx \sin \left( \frac{k\pi x}{d} \right) \left[ \frac{\sin \left( \frac{n\pi}{c} \right)}{\cos \left( \frac{\pi x}{d} \right)} + \frac{n\pi}{d} \right] \sin \left( \frac{n\pi x}{d} \right) \sin \left( \frac{n\pi}{d} \right). \tag{4A.4}$$

On substituting the functional form of $R_m(z)$, we have

$$\beta_{nk} = \frac{-1}{2\pi \sqrt{nk}} \int_0^d dx \sin \left( \frac{k\pi x}{d} \right) \left[ \frac{\sin \left( \frac{n\pi}{c} \right)}{\cos \left( \frac{\pi x}{d} \right)} + \frac{n\pi}{d} \right] \sin \left( \frac{n\pi x}{d} \right) \sin \left( \frac{n\pi}{d} \right). \tag{4A.5}$$

Upon redefining $\frac{n\pi}{d}$ as $x$, we have

$$\beta_{nk} = \frac{-1}{\pi \sqrt{nk}} \int_0^x dx \sin \left( \frac{k\pi x}{d} \right) \left[ (n - k) + n\pi \cos(x) \right] \sin \left( \frac{x}{d} + n\pi \sin(x) \right). \tag{4A.6}$$

Using the properties of trigonometric identities, we then obtain

$$\beta_{nk} = \frac{-1}{\pi \sqrt{nk}} \int_0^\pi dx \frac{1}{2} \left[ (n - k) + n\pi \cos(x) \right] \left[ \cos \left( \frac{x}{d} + n\pi \sin(x) \right) \right].$$
Appendix 4A. Derivation of $\alpha_{nk}$ and $\beta_{nk}$

\begin{equation}
\frac{1}{2\pi\sqrt{n}k}\int_0^\pi dx\left\{ (n - k + n\lambda\cos(x))\cos((n - k)x) + n\lambda\sin(x) \right\}
\end{equation}

\begin{equation}
-2k\cos[(n + k)x + n\lambda\sin(x)].
\end{equation}

Defining $\delta = (n - k)x + n\lambda\sin(x)$ and $\theta = (n - k)x + n\lambda\sin(x)$

\begin{equation}
\beta_{nk} = \frac{1}{2\pi\sqrt{n}k}\int_0^\pi dx\cos[(n + k)x + n\lambda\sin(x)]
\end{equation}

\begin{equation}
\int_0^\pi d\delta\cos(\delta) - \int_0^\pi d\delta\cos(\delta)
\end{equation}

\begin{equation}
= -\frac{1}{\pi} \int_0^\pi dx\left[ \cos((n + k)x) + n\lambda\sin(x) \right].
\end{equation}

Using the integral representation of Bessel functions (\cite{13}, §8.411, Eqn (1), p 952), we thus have

\begin{equation}
\beta_{nk} = -\sqrt{\frac{k}{\pi}} J_{n+k}(-n\lambda),
\end{equation}

where $J_v$ is Bessel function of the first kind of order $v$. Along similar lines, we obtain for $\alpha_{nk}$

\begin{equation}
\alpha_{nk} = \sqrt{\frac{k}{\pi}} J_{n-k}(-n\lambda).
\end{equation}
References