Chapter 2

Generation of nonclassical states by time-dependent perturbations on oscillator

In this chapter\(^1\), the generation and dynamics of nonclassical states in a quantum harmonic oscillator whose frequency is time dependent is studied. The interest in this problem arises from the fact that many physical systems, especially electromagnetic radiation, can be successfully modelled by harmonic oscillators and the results for this generic system can be Carried over to various specific systems. For example, as will be shown in the next chapter, the model can be applied to the motion of a trapped ion in a Paul trap.

The plan of this chapter is as follows. In section 2.1, some general results, which are independent of the explicit time dependence of the frequency are given. In section 2.2, a specific time dependence, namely a linear sweep of \(\omega^2\) is considered, and the Heisenberg equations of motion for the position and momentum of the quantum oscillator are solved \textit{exactly}. In section 2.3 it is then shown that the fluctuations in the quadratures \(\hat{X}\) and \(X\) show squeezing and the photon number statistics is shown to display sub-Poissoaian nature. Thus, the quantum harmonic oscillator with a time dependent frequency is shown to generate nonclassical states. Two limits, sudden and adiabatic, also manifest. It is shown that the nonclassical nature is maximal for sudden changes in the frequency, whereas for adiabatic changes it is minimal.

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2.1 Exact analytical approach

Consider a harmonic oscillator Hamiltonian [1]
\[ \hat{H} = \frac{\omega^2}{2} + \frac{1}{2}(1 + \beta(t))\hat{x}^2 \] (2.1)
where we have set \( m = 1 \). The operators \( x \) and \( p \) satisfy the commutation relation
\[ [\hat{x}, \hat{p}] = i \quad (\hbar = 1). \]
(2.2)

The explicit time dependence of the frequency \( \omega \) appears through the \( \beta \)-term in (2.1). In what follows, the exact form of the time dependence of the frequency is of no consequence.

For convenience, let us define dimensionless quadrature operators \( \hat{X} \) and \( \hat{P} \), as follows:
\[ \hat{X} = \sqrt{\frac{\omega}{2}}, \quad \hat{P} = \sqrt{\frac{1}{2\omega}} \hat{p} \] (2.3)
and they satisfy the commutation relation
\[ [\hat{X}, \hat{P}] = \frac{i}{2}. \] (2.4)

It is also convenient to work with a dimensionless time \( \tau = \omega t \). Now, for a given form of the time dependence, let us assume that the solutions are known for the Heisenberg equations of motion for the quadratures \( \hat{X} \) and \( \hat{P} \), the equations of motion being:
\[ \frac{d\hat{X}}{d\tau} = -\hat{P}, \quad \frac{d\hat{P}}{d\tau} = -(1 + \beta(\tau))\hat{X}. \] (2.5)

Let \( U(\tau) \) and \( V(\tau) \) be two functions which are the independent solutions of (2.5) or equivalently of the second order differential equation
\[ \frac{d^2\phi}{d\tau^2} = -(1 + \beta(\tau))\phi. \] (2.6)
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Tims $U'$ and $V$ form the fundamental system of solutions. Then, the general solution of
this homogeneous system (2.5) can be written as

\[
\begin{pmatrix}
\dot{X}(\tau) \\
\dot{P}(\tau)
\end{pmatrix} =
\begin{pmatrix}
U(\tau) & V(\tau) \\
\dot{U}(\tau) & \dot{V}(\tau)
\end{pmatrix}
\begin{pmatrix}
X(0) \\
\dot{P}(0)
\end{pmatrix}
\]  

(2.7)

From (2.7) it is natural to choose the initial conditions for the functions $U$ and $V$ as

\[
U(0) = V(0) = 1, \\
U(0) = V(0) = 0.
\]  

(2.8)

Let us now consider a matrix $Y$ defined for the functions $U$, $V$ and their derivatives

\[
Y = \begin{pmatrix}
U & V \\
\dot{U} & \dot{V}
\end{pmatrix}
\]  

(2.9)

The determinant of this matrix is called the Wronskian of the functions $U$ and $V$. A
sufficient condition for the two functions $U$ and $V$ to be linearly independent is that the
Wronskian be non-vanishing. Since $U$ and $V$ are independent, the Wronskian will have
a non-zero value.

To show that the value of the Wronskian is invariant, we consider the following matrix
differential equation

\[
\frac{dY}{d\tau} = A(\tau)Y
\]  

(2.10)

where, the co-efficient matrix is given by (2.5)

\[
A(\tau) = \begin{pmatrix}
0 & 1 \\
-(1 + \beta(\tau)) & 0
\end{pmatrix}.
\]  

(2.11)

Using Abel's identity for the Wronskian from the theory of differential equations [2,3],
we have

\[
\det Y(\tau) = \det Y(0) \exp \int_{\tau_0}^{\tau} \text{Trace} A(\tau_1) d\tau_1.
\]  

(2.12)

Since $\text{Trace} A(\tau) = 0$,

\[
\det Y(\tau) = \det Y(0).
\]  

(2.13)
Thus, the Wronskian remains invariant. This implies that

\[ UV - UV = I \]  \hspace{1cm} (2.14)

holds true for all times.

We now define lowering and raising operators as

\[ a = X + iP \]
\[ a^+ = X - iP \] \hspace{1cm} (2.15)

which at \( t = 0 \) satisfy the Bosonic commutation relation

\[ [a, a^+] = 1. \] \hspace{1cm} (2.16)

Using the definition (2.15) and the solution (2.7), \( a(T) \) and \( a^+(T) \) become:

\[ a(T) = a(T)\alpha(T) + \beta(T) \]
\[ a^+(T) = \alpha^*(T)a^+(0) + \beta^*(T)a^+(0) \] \hspace{1cm} (2.17)

where, \( u(T) \) and \( v(T) \) are given in terms of the solution (2.7) as

\[ u(T) = \frac{1}{2} \left[ (U + V) + i(U - V) \right] \]
\[ v(T) = \frac{1}{2} \left[ (U - V) + i(U + V) \right]. \] \hspace{1cm} (2.18)

It then follows from the invariance of the Wronskian (condition (2.14)), that

\[ |u|^2 - |v|^2 = 1 \] \hspace{1cm} (2.19)

for all times. Condition (2.19) also follows from requiring that the Bosonic commutation relation (2.16) holds true for the transformed operators \( \hat{a}(T) \) and \( \hat{a}^+(T) \), i.e.,

\[ [\hat{a}(T), \hat{a}^+(T)] = 1. \] \hspace{1cm} (2.20)

This implies that the transformation (2.17) is a canonical transformation. Thus, we have a Bogoliubov transformation of the lowering and raising operators as a result of the time dependent frequency and the co-efficients of the transformation depend explicitly on time.
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Every canonical transformation can be represented as an unitary transformation [4], and hence we have

$$a(r) = S^{-1}(r)a(0)S(t). \quad (2.21)$$

The explicit form of the time evolution operator $S(r)$ can be obtained as follows. One can rewrite (2.17) as

$$a(r) = c''\{ |\tau(r)|a + |\tau(r)|e^{i(\theta_\tau - \theta_\tau)}a^\dagger \}. \quad (2.22)$$

The term in the square brackets can be obtained by the action of a "squeeze" operator [5-9]. Thus if $T = \exp\left(\frac{i}{2}(z \hat{a} \hat{a}^\dagger - h.c.)\right)$, where $z = |z|e^{i\theta}$, then

$$\hat{T}a \hat{T}^{-1} = \cosh |z|a + e^{i\theta} \sinh |z|a^\dagger \quad (2.23)$$

where, the operator identity ([10], p136)

$$e^{\xi a} B e^{-\xi a} = B + \frac{\xi}{\tau_1} [\hat{A}, B] + \frac{\xi^2}{2!} [\hat{A}, [A, B]] + \cdots \quad (2.24)$$

has been used. If one identifies

$$|z| = \cosh^{-1} |u| \text{ and } \theta = \theta_u - \theta_u, \quad (2.25)$$

then (2.23) yields,

$$\hat{T}^{-1}a \hat{T} = |u|a + |v|e^{i(\theta_v - \theta_v)}a^\dagger \quad (2.26)$$

Again using the identity (2.24), we can show that

$$\exp(-i\theta_v \hat{a}^\dagger \hat{a}) \hat{a} \exp(i\theta_v \hat{a}^\dagger \hat{a}) = \hat{a} e^{i\theta_v}. \quad (2.27)$$

Hence we have,

$$\hat{a}(r) = \hat{u}(\tau) \hat{a} + \hat{v}(\tau) \hat{a}^\dagger$$

$$= e^{i\theta_v} \left[ |u(\tau)| \hat{a} + |v(\tau)| e^{i(\theta_v - \theta_v)} \hat{a}^\dagger \right]$$

$$= \hat{S}^{-1} \hat{a} \hat{S} \quad (2.28)$$
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where $S(\tau)$ is

$$
S(\tau) = \exp(i\theta_u \dot{\hat{a}}\hat{a}) \exp(\frac{1}{2} \left[ e^{i\theta_u - \frac{v}{u} \dot{\hat{a}}} \cosh^{-1} |u| \hat{a}^{\dagger 2} + \text{h.c.} \right]).
$$

(2.29)

As a result of the time dependent frequency, one expects transition to occur between the levels of the oscillator. The probability of finding the oscillator in the state $|n\rangle$ at time $r$ given that at $r = 0$ it was in the state $|m\rangle$ is

$$
p_{nm}(\tau) = |\langle n | \hat{S}(\tau) | m \rangle|^2
$$

(2.30)

The evaluation of these matrix elements can be done if one expresses $S$ in a normally ordered form. This is done using the disentangling theorem for the SU(1,1) group [11-13]. The transition probability then becomes

$$
p_{nm} = \left( \frac{1}{|u|^{2m-3}} \right) \frac{1}{n!m!} |\langle 0 | (\hat{a} + \frac{v}{u} \hat{a}^{\dagger})^n (\hat{a}^{\dagger} - \frac{v^{*}}{u^{*}} \hat{a})^m |0 \rangle|^2.
$$

(2.31)

The actual steps involved obtaining (2.31) are provided in the Appendix 2A. As an example, let $m = 0$ and $n = 2$. Then

$$
|\langle 2 | \hat{S}(\tau) | 0 \rangle|^2 = \frac{|u|^{3}}{\alpha} |\langle 0 | (\hat{a} + \frac{v}{u} \hat{a}^{\dagger})^2 |0 \rangle|^2
$$

\[ - \frac{|v||v^{*}|}{2}. \quad (2.32)

Quasiprobability distributions (a resume is given in chapter 6), like the Glauber-Sudarshan $P$-function, $Q$-function and Wigner function are very useful in gaining insight into the quantum statistical aspects of a problem. Many states which are highly nonclassical have simple and rather well behaved representations in terms of the Wigner function. The Wigner function is defined as the Fourier transform of the quantum characteristic function $\chi_w$: [10] and [14] (section 4.4)

$$
\Phi(\alpha) = \frac{1}{\pi^2} \int d^2 \xi \exp(\xi^* \alpha - \xi \alpha^*) \chi_w(\xi, \xi^*),
$$

(2.33)

where

$$
\chi_w(\xi, \xi^*) = \text{Tr}\{\hat{\rho} \exp[\xi \hat{a}^{\dagger} - \xi^* \hat{a}]\}
$$

(2.34)
and \( \Phi(\alpha) \) satisfies the normalisation condition \( \int d^{2}\alpha \Phi(\alpha) = 1 \).

What is the effect of the transformation that arises due to the time dependent frequency on the Wigner function? To see that, we substitute the transformation (2.17) into (2.33) and we have,

\[
\Phi(\alpha, \alpha^*, \tau) = \frac{1}{\pi^{2}} \int d^{2}\xi \exp(\xi\alpha - \xi\alpha^*) \text{Tr}\{\hat{\rho} \exp[\xi(\alpha^\dagger a + a^\dagger) - \text{h.c.}]\}. \tag{2.35}
\]

On changing the integration variable \( \xi \) to \( \lambda = (\xi u^* - \xi v) \), and noting that the Jacobian of the transformation is unity, we have

\[
\Phi(\alpha, \alpha^*, \tau) = \frac{1}{\pi^{2}} \int d\lambda \exp\{\lambda^*(\lambda^* u - \eta^* v) - \lambda(\lambda u^* - v^* \alpha)\} \text{Tr}\{\hat{\rho} \exp(\lambda \hat{\alpha}^\dagger - \lambda^* \hat{\alpha})\}. \tag{2.36}
\]

Comparing the equation (2.36) and (2.33) we can express the Wigner function at time \( \tau \), in terms of the function at \( t = 0 \) [1]:

\[
\Phi(\alpha, \alpha^*, \tau) = \Phi \left([u^*(\tau) \alpha - v^*(\tau) \alpha^*], [u(\tau) \alpha^* - v^*(\tau) \alpha], 0\right). \tag{2.37}
\]

The Wigner function thus evolves along the classical trajectories. This is expected as the Hamiltonian is quadratic in \( \dot{\alpha} \) and \( \dot{P} \). The time evolution is especially simple if the Wigner function associated with the initial state is a Gaussian [15]. According to the moment theorem for Gaussian processes [16], all moments of order higher than two are expressible in terms of those of order one and two. Thus, the Gaussian nature of the Wigner function simplifies the calculation of higher order moments. The Wigner function for a very large class of harmonic oscillator states has the following general Gaussian structure, [15]

\[
\Phi(\alpha) = \frac{1}{\pi \sqrt{(\gamma^2 - 4|\mu|^2)}} \exp \left(-\frac{\mu(\alpha - \alpha_0)^2 + \mu^*(\alpha^* - \alpha_0^*)^2 + \gamma|\alpha - \alpha_0|^2}{\gamma^2 - 4|\mu|^2}\right) \tag{2.38}
\]

where the parameters in (2.38) have the following meaning in terms of the mean values and variances

\[
\langle \hat{a} \rangle = \alpha_0
\]
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\[ \langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2 = -2 \mu^2 \]  
(2.40)

\[ \frac{1}{\pi} \langle (\hat{a}^\dagger \hat{a}) \rangle = 7. \]  
(2.41)

The function (2.38) corresponds to the Wigner function for (a) coherent states when \( \mu = 0 \) and \( \gamma = 5 \); (b) squeezed states when \( \mu \neq 0, \alpha_0 \neq 0 \) and \( \gamma^2 - 4|\mu|^2 = \bar{\gamma} \); (c) thermal states when \( \mu = 0, \alpha_0 = 0 \) and \( \gamma > 1/2 \) and (d) also to mixtures of thermal and coherent states.

As an example of the use of the Gaussian nature of Wigner function, let us evaluate the Mandel's Q-parameter, which involves the evaluation of second moments. The Mandel's Q-parameter as defined in chapter 1 gives a quantitative measure of how much the photon number distribution deviates from a Poissonian distribution. In order to determine Q, we need to evaluate \( \langle n^2 \rangle = \langle (\hat{a}^\dagger \hat{a})^2 \rangle \) and \( \langle n \rangle = \langle \hat{a}^\dagger \hat{a} \rangle \). The mean value of \( (\hat{a}^\dagger \hat{a})^2 \) in terms of the Wigner function is

\[ \langle \hat{a}^\dagger \hat{a} \rangle = \int\Phi(\alpha)(|\alpha|^4 - |\alpha|^2) d^2 \alpha. \]  
(2.42)

According to the moment theorem for Gaussian processes [16], all higher order moments (> 2) can be expressed in terms of the second and first moments. In particular, if the first moment is zero, then all odd order moments are zero and the even order moments are given by

\[ \langle X_i X_j X_k \ldots \rangle = \frac{(2N)!}{N!2^N} \{\sigma_{ij} \sigma_{kl} \sigma_{mn} \ldots \}_{\text{sym}}, \]  
(2.43)

where the subscript 'sym' means symmetric form of the product of the variance matrices \( \sigma \) and \( 2N \) is the order of the moment. Since, in (2.42), \( \langle \hat{a} \rangle = \alpha_0 \neq 0 \), in general, we can define a such that \( a = a + \alpha_0 \), so that we can apply the result (2.43) for the barred variables. Then (2.42) becomes,

\[ \langle \hat{a}^\dagger \hat{a} \rangle = \int\Phi(\alpha)(|\bar{\alpha} + \alpha_0|^4 - |\bar{\alpha} + \alpha_0|^2) d^2 \alpha, \]  
(2.44)

\[ = \langle |\bar{\alpha} + \alpha_0|^4 \rangle - \langle |\bar{\alpha} + \alpha_0|^2 \rangle. \]  
(2.45)

Expanding the quantities \( |\bar{\alpha} + \alpha_0|^4 \) and \( |\bar{\alpha} + \alpha_0|^2 \) and noting that for the barred variables

\[ \langle |\bar{\alpha}|^4 \rangle = 2\langle |\bar{\alpha}|^2 \rangle^2 + \langle \bar{\alpha} \rangle^2 \langle \bar{\alpha}^* \rangle^2, \]  
(2.46)
\[ \langle \hat{a}^3 \rangle = \langle \hat{a} \rangle = 0, \]  

we have

\[ \langle (\hat{a}^\dagger \hat{a})^2 \rangle = 2|\alpha|^2 + |\alpha_0|^4 + 4|\alpha_0|^2|\alpha|^2 - |\alpha|^2 - \langle |\alpha|^2 \rangle - \langle \hat{a}^2 \rangle \alpha_0^2 + \langle \hat{a}^2 \rangle + \alpha_0^2 + \langle \hat{a}^2 \rangle \langle \alpha^* \alpha \rangle. \]  

(2.47)

Using the definitions (2.39) to (2.41), we have

\[ \langle (\hat{a}^\dagger \hat{a})^2 \rangle = 2\gamma^2 + |\alpha_0|^4 - |\alpha_0|^2 - 7 - 2\mu^* \alpha_0^2 + 2\mu \alpha_0^2 + 4|\mu|^2 + 4|\alpha_0|^2 \gamma \]  

(2.48)

and \( \langle \hat{a}^\dagger \hat{a} \rangle \) is given by

\[ \langle \hat{a}^\dagger \hat{a} \rangle = \gamma - \frac{1}{\alpha} + |\alpha_0|^2. \]  

(2.49)

Combining (2.49) and (2.50), we have the final expression for Mandel's Q-parameter

\[ Q = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 - \langle \hat{n} \rangle}{\gamma}, \]  

\[ \sim \left( \frac{\gamma^2 + 2|\alpha_0|^2 \gamma - 1 - 2(\alpha_0^*)^2 \mu^* - 2\alpha_0^2 \mu + 4|\mu|^2}{7 + |\alpha_0|^2 - \frac{1}{2}} \right) - 1. \]  

(2.51)

(2.52)

It is clear from (2.37) and (2.38) that the Wigner function will remain Gaussian with time dependent parameters \( \alpha_0, \gamma \) and \( /z \):

\[ \alpha_0(\tau) = u(\tau) \alpha_0 + v(\tau) \alpha_0^* \]  

(2.53)

\[ \mu(\tau) = u^2 + v^* \mu^* - u^* v^* \gamma \]  

(2.54)

\[ \gamma(\tau) = (u^* u + v^* v) \gamma - 2\mu u^* v - 2\mu^* u v^*, \]  

(2.55)

where \( u(\tau) \) and \( v(\tau) \) are the time dependent co-efficients of the Bogoliubov transformation (2.17). Thus, if the initial Wigner function of a state was Gaussian, as a result of the Bogoliubov transformation (2.17) the Gaussian nature remains intact after the transformation, but for a redefinition of the parameters.
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2.2 Linear sweep of \( w^2 \), a specific case

In this section, we consider as an example, a specific form of the time dependent frequency. Let \( \beta(t) \) in (2.1) be of the following form (see Fig.2.1):

\[
\beta(t) = \begin{cases} 
0 & \text{for } -\infty < t < 0 \\
\beta_0 \frac{t}{T} & \text{for } 0 < t < T \\
\beta_0 & \text{for } T < t < \infty.
\end{cases}
\] (2.55)

With this form of the time dependence one has to solve the Heisenberg equations of motion (2.5) for the dimensionless operators \( X \) and \( P \). In region I, we have just an oscillatory solution at the oscillation frequency \( \omega \):

\[
\tilde{X}^{(I)}(\tau) = X(0) \cos(\tau) + P(0) \sin(\tau) \quad (2.56)
\]

\[
\tilde{P}^{(I)}(\tau) = -X(0) \sin(\tau) + P(0) \cos(\tau). \quad (2.57)
\]
To find the solution in the region II, we make the following transformation for the dimensionless time variable, $T \rightarrow \tau'$,

$$\tau' = r + \frac{\omega T}{\beta_0} \quad (2.58)$$

and we define a new variable, $z$

$$z = \frac{2}{3} \left( \frac{\beta_0}{\omega T} \right)^{\frac{1}{3}} \quad (2.59)$$

Then, the Heisenberg equations of motion (2.5) get transformed to

$$\frac{d\dot{X}}{d\tau'} = \dot{\hat{P}}$$

$$\frac{d\dot{P}}{d\tau'} = -\left( \frac{\beta_0}{\omega T} \right) \dot{X} \quad (2.60)$$

or equivalently, $\dot{X}$ and $\dot{P}$ are solutions of the second-order differential equation

$$\frac{d^2 \phi}{d\tau'^2} + \left( \frac{\beta_0}{\omega T} \right) \phi = 0. \quad (2.61)$$

The exact solution for the Heisenberg equation of motion for the operators $X$ and $P$ in the region II can be written in terms of Bessel functions of order $\frac{1}{3}$: [17]

$$\dot{X}^{(II)}(\tau') = C_1 \sqrt{\tau'} J_1(z) + C_2 \sqrt{\tau'} Y_1(z) \quad (2.62)$$

$$\dot{P}^{(II)}(\tau') = C_3 \sqrt{\tau'} J_1(z) + C_4 \sqrt{\tau'} Y_1(z). \quad (2.63)$$

The co-efficients $C_1$ to $C_4$ are fixed by requiring $X^{(II)}(\tau') \rightarrow X^{(I)}(\tau= 0) = X(0)$ and $P^{(II)}(\tau') \rightarrow P^{(I)}(\tau = 0) = P(0)$. In region III, the solution is again oscillatory, but with a frequency of oscillation $\sqrt{1 + \beta_0 \omega}$. Thus,

$$\dot{X}^{(III)}(\tau) = C_5 \cos(\sqrt{1 + \beta_0 \tau}) + C_6 \sin(\sqrt{1 + \beta_0 \tau}) \quad (2.64)$$

$$\dot{P}^{(III)}(\tau) = C_7 \cos(\sqrt{1 + \beta_0 \tau}) + C_8 \sin(\sqrt{1 + \beta_0 \tau}). \quad (2.65)$$

As before the constants $C_5$ to $C_8$ are determined by requiring the continuity of $X$ and $P$ at $r = T$. Thus the entire solution is obtained. The variances in the quadratures can then be calculated in a straightforward manner.
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Following our results [1] various authors have extended the results obtained by us\(^2\). Other forms of specific time dependences can also be considered. For example, in the case of quantised motion of an ion in a Paul trap, the time dependence is sinusoidal. This case is discussed in detail in the next chapter.

2.3 Numerical approach

In the previous two sections, the exact analytical solution for the Heisenberg equations of motion (2.5) was considered. The previous section described a specific example where an analytical solution is possible. But there are problems where such an analytical solution is not possible. To handle such situations, in this section a numerical approach to the problem is given.

We rewrite (2.5) in terms of the mean values of the operators \(X\) and \(P\): [21]

\[
\begin{align*}
\varphi_1 &= \varphi_2 \\
\varphi_2 &= -(1 + \text{flr})^\wedge
\end{align*}
\]

where

\[
\begin{align*}
\varphi_1 &= \langle X \rangle \\
\varphi_2 &= \langle P \rangle.
\end{align*}
\]

Also, let

\[
\begin{align*}
\Psi_1 &= \langle \hat{X}^2 \rangle \\
\Psi_2 &\equiv \langle \hat{X}\hat{P} + \hat{P}\hat{X} \rangle \\
\Psi_3 &\equiv \langle \hat{P}^2 \rangle.
\end{align*}
\]

\(^2\)Janszky et al [18,19], based on [1] have shown that a series of well-timed frequency jumps leads to more pronounced squeezing. In a later publication [20], Aliaga et al have obtained our results by a completely equivalent method using a maximum entropy principle procedure.
Then, we have,

\[ \Psi_1 = 2\Psi_2 \]
\[ \Psi_2 = -(1 + \beta(\tau))\Psi_1 + \Psi_3 \]
\[ \Psi_3 = -2(1 + \beta(\tau))\Psi_2. \]  

(2.69)

The initial conditions for (2.66) and (2.69) are determined from (2.67) and (2.68) evaluated at \( \tau = 0 \) for a given initial state. Systems (2.66) and (2.69) are numerically integrated using standard Runge-Kutta algorithm and the fluctuations in the quadratures \( \Delta X \) and \( \Delta P \) at time \( r \) are directly obtained as

\[ \langle (\Delta \bar{X})^2 \rangle = \Psi_1 - \varphi_1^2 \]
\[ \langle (\Delta \bar{P})^2 \rangle = \Psi_3 - \varphi_2^2. \]  

(2.70)

The time dependence of the Mandel's Q-parameter is obtained directly from (2.51) substituting for the time dependent parameters given by (2.52) to (2.54).

2.4 Demonstration of nonclassical properties

In this section, a discussion of the results for the case of \( \beta(\tau) \), given by (2.55) in section 2.2, is given to demonstrate nonclassical effects like, squeezing of fluctuations and sub-Poissonian statistics. One could use either the exact analytical approach of section 2.2 or the numerical approach of section 2.3. We use the exact solutions for the regions II and III (see Fig.2.1) with proper boundary conditions. One can then use this solution to evaluate \( \bar{X}, \bar{P} \) and \( \dot{X}P + \dot{P}X \) and evaluate the expectation values for various initial states. Alternatively, one could determine these quantities by a direct numerical integration of the equations of motion for the expectation values [21].

We next present the numerical results for the non-classical properties like, squeezing and sub-Poissonian statistics of the oscillator. In Fig.2.2 we show the squeezing in the component \( X \) when initially the oscillator is in the ground state. We observe that a linear sweep produces a significant amount of squeezing. The squeezing properties are
much more pronounced for the case of a sudden jump [22]. As expected the adiabatic changes [23] do not produce any noticeable squeezing. From the calculation of the phases $\theta_u$ and $\theta_v$, we also find that the two quadratures $\hat{X}$ and $\hat{P}$ are in general correlated for most of the time (see Fig.2.3). Note that for fast sweeping, the variance exhibits periodic behaviour. For the parameters of the Fig.2.2 this period is found to be $\pi$ which follows from (2.6) and (2.55) as $1 + \beta_0 \rightarrow 2$. In Fig.2.4, we show the squeezing characteristics if initially the oscillator state is squeezed in the quadrature $P$. The quadrature $X$ exhibits quite a significant amount of squeezing which in turn depends on the rate of the frequency sweep. For the initial vacuum state the Wigner function is Gaussian (2.38) with equal noise in the two quadratures ($\mu = 0$ and $\gamma = \frac{1}{2}$). In Fig.2.5, we show the time evolution of the Wigner function (2.38). We show the behaviour at a time when the system shows maximum amount of squeezing in the $X$ quadrature.

Finally, in Fig.2.6, we show the generation of sub-Poissonian statistics when initially the state is a coherent state. The time dependent behaviour of the $Q$ parameter was calculated from (2.51) using (2.52) to (2.54). It is similar to that shown in Fig.2.2. In general this is not expected, except when the mean value of the field is so large, that a linearisation around the mean value can be done. For Fig.2.2, the mean value is zero, but this is not so for Fig.2.6. The linear sweep of the restoring force can produce large amounts of sub-Poissonian statistics. Several possibilities for realising the present model exist; for example, one can use a cavity with a material whose dielectric constant is varied with time.
Figure 2.2: The variance of the quadrature $X$ versus time, $\tau$ for an oscillator initially in the ground state. The parameters are: $\beta_0 = 1$ and $\omega T = (a)10^{-3}$, (b)1, (c)3 and (d)10^3. The cases (a) and (d) correspond, respectively, to sudden and adiabatic limits.
Figure 2.3: Same as Fig.2.2, but with the oscillator initially in a squeezed coherent state $|\alpha, \zeta\rangle$ with $\alpha = 1$ and $\zeta = 0.5e^{-i\tau}$. 
Figure 2.4: The phases of $u$ (curve (a)), $v$ (curve (b)) and $\mu$ (curve (c)) versus time, $\tau$. As the phases are non-zero for most of the time, the quadratures $X$ and $P$ are correlated most of the time.
Figure 2.5: Wigner function $\Phi(\alpha, \alpha^*, \tau)$ with $\alpha = \langle X \rangle + i\langle P \rangle$ for the system initially in vacuum state for $\omega T = 10^{-3}, \beta_0 = 1$ and $\tau = 1.1$ which corresponds to the minimum in Fig.2.2. Squeezing in the quadrature $X$ can be seen in the inset which shows contours of constant values of the Wigner function.
Figure 2.6: Mandel's Q-parameter as a function of \( \tau \) for the case of an oscillator initially in a coherent state \( |\alpha\rangle, \alpha = 1 \). The parameter \( \omega T \) has been chosen as (a) \( 10^{-3} \), (b) 1, (c) 3 and (d) \( 10^3 \).
Appendix 2A

Calculation of the matrix element $\langle n|S|m \rangle$

In this appendix we give the intermediary steps involved in the calculation of the matrix element in (2.30). In calculating such matrix elements the following operator identities are useful [12,13]:

$$e^{\lambda \hat{A}} \hat{B}^m e^{-\lambda \hat{A}} = \left[ \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \{\hat{A}^n, \hat{B} \} \right], \quad (2A.1)$$

$$e^{\lambda \hat{A}} e^{\hat{B}} e^{-\lambda \hat{A}} = \exp \left( \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \{\hat{A}^n, \hat{B} \} \right), \quad (2A.2)$$

$$e^{\lambda \hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{C_2} e^{C_3} \ldots, \quad (2A.3)$$

where

$$\{\hat{A}^0, \hat{B} \} = B \quad (2A.4)$$

$$\{\hat{A}^{n+1}, \hat{B} \} = [\hat{A}, \{\hat{A}^n, \hat{B} \}]. \quad (2A.5)$$

Equation (2A.3) is called Zassenhaus formula and is a dual of the BCH formula (Baker-Campbell-Hausdorff formula). The quantities $C_2$ are

$$C_2 = -\frac{1}{2} [\hat{A}, \hat{B}] \quad (2A.6)$$

$$C_3 = \frac{1}{3} [\hat{B}, [\hat{A}, \hat{B}]] + \frac{1}{2} [\hat{A}, [\hat{A}, \hat{B}]] \quad (2A.7)$$

and so on. If $C_2$ commutes with $A$ and $B$, then all higher terms vanish. Equation (2.29) can be written as:

$$S = e^{-i(A_1)} e^{\lambda \hat{A}} \exp \left( \frac{|k|}{2} e^{i\theta_1} a^\dagger a - \frac{|k|}{2} e^{-i\theta_2} a^\dagger a \right), \quad (2A.8)$$
Appendix 2A. Calculation of the matrix element $\langle n | \hat{S} | m \rangle$  

where

\[ X = i\theta_s \]  \hspace{1cm} (2A.9)
\[ \theta = \theta_v - \theta_s \]  \hspace{1cm} (2A.10)
\[ |\zeta| = \cosh^{-1}|\mu|. \]  \hspace{1cm} (2A.11)

Recalling that the state $|n\rangle$ can be obtained from the vacuum by the action of at (see 1.10), we have

\[ \langle n | \hat{S} | m \rangle = \frac{1}{n!m!} (0|\hat{a}^n e^{i\theta_1 \hat{a}} e^{i\theta_2 \hat{a}^+} e^{-i\theta_1 \hat{a}^+} e^{-i\theta_2 \hat{a}}) \hat{a}^m |0\rangle. \]  \hspace{1cm} (2A.12)

The second exponential operator in (2A.12) can then be disentangled using the disentangling theorem for the $SU(1,1)$ group. The $SU(1,1)$ group generators, $A', A'^\dagger$ and $K_3$ satisfy the following commutation relations

\[ [K_-, K_+] = 2K_3 \]  \hspace{1cm} (2A.13)
\[ [K_3, K_\pm] = \pm K_\pm, \]  \hspace{1cm} (2A.14)

where the generators are

\[ K_+ = \frac{\hat{a}^+ \hat{a}}{\lambda^2} \]  \hspace{1cm} (2A.15)
\[ A'_- = \frac{1}{2} \]  \hspace{1cm} (2A.16)
\[ K_3 = \frac{1}{2} \left( \hat{a}^+ \hat{a} + \frac{1}{2} \right). \]  \hspace{1cm} (2A.14)

According to the disentangling theorem [11]

\[ \exp (\gamma_+ K_+ + \gamma_- A'_- + \gamma_3 K_3) = \exp (\Gamma_+ K_+ ) \exp (\ln(K_3) K_3) \exp (\Gamma_- K_-), \]  \hspace{1cm} (2A.18)

where

\[ \phi^2 = \frac{1}{4} \gamma_3 - \gamma_+ \gamma_- \]  \hspace{1cm} (2A.19)
\[ \Gamma_\pm = \frac{2\gamma_\pm \sinh(\phi)}{2\phi \cosh(\phi) - \gamma_3 \sinh(\phi)} \]  \hspace{1cm} (2A.20)
\[ \Gamma_3 = \left( \cosh(\phi) - \frac{\gamma_3}{2\phi} \sinh(\phi) \right)^2. \]  \hspace{1cm} (2A.21)
Appendix 2A. Calculation of the matrix element \((n|\hat{S}|m)\)

Tims we get,

\[
(n|\hat{S}|m) = \frac{1}{\sqrt{n!m!}}(0|\hat{a}^n e^{(\hat{a}^\dagger \hat{a})} e^{\xi \hat{a}^\dagger} e^{\xi^2 \hat{a}^\dagger \hat{a}^\dagger} e^{-\xi^2 \hat{a}^\dagger \hat{a}^\dagger} |0), \tag{2A.22}
\]

where

\[
\xi = \frac{v}{2u} \tag{2A.23}
\]
\[
A = -\ln(|u|). \tag{2A.24}
\]

Using the operator identities (2A.1) and (2A.2) we can reorder these terms. After reordering, we get

\[
(n|\hat{S}|m) = \frac{e^{n \lambda + (m-2) A}}{\sqrt{|u|u!m!}} (0|(\hat{a} + 2\xi \hat{a}^\dagger)^n (\hat{a}^\dagger - 2\xi^2 \hat{a}^\dagger \hat{a})^m |0). \tag{2A.25}
\]

Substituting the definitions (2A.9) to (2A.11) and (2A.23) and (2A.24), we have

\[
(n|\hat{S}|m) = \left( \frac{e^{inh_u}}{|u|^{m-2}} \right) \frac{1}{\sqrt{n!m!}} (0| (\hat{a} + \frac{v}{u})^n (\hat{a}^\dagger - \frac{v^*}{u^*} \hat{a})^m |0), \tag{2A.26}
\]

and hence

\[
p_{nm} = \left( \frac{1}{|u|^{2m-3}} \right) \frac{1}{n!m!} |(0|(\hat{a} + \frac{v}{u})^n (\hat{a}^\dagger - \frac{v^*}{u^*} \hat{a})^m |0)|^2. \tag{2A.27}
\]
References

