Dissipation effect on MHD stagnation-point flow of Casson fluid over a stretching sheet through porous media

4.1 Introduction

The exact solution of the problem of the flow in the vicinity of a stagnation-point for both two-dimensional and three-dimensional flows of a viscous fluid may be obtained from the consideration that at large distance from the stagnation-point the flow is essentially the same as that of the corresponding potential flow problem. Thus, the solution of viscous flow may be derived from the solution, of the potential flow problem. The study of stagnation-point flow over a stretching sheet has attracted many researchers ([107], [142], [77], [144], [182]).

The unsteady boundary layer flow and heat transfer of a Casson fluid over a moving flat plate with a parallel free stream were studied by Mustafa et al. [121] and they solved the problem analytically using homotopy analysis method (HAM). Bhattacharyya et al. ([97], [96]) reported the exact solution for boundary layer flow of Casson fluid over a permeable stretching/shrinking sheet with and without external magnetic field. Recently, Bhukta et al. [43] have studied heat and mass transfer on MHD flow of a viscoelastic fluid through porous media over a shrinking sheet.

It is well known that Casson fluid is a shear thinning liquid which is assumed to have an infinite viscosity at zero rate of shear, a yield stress below which no flow occurs, and a zero viscosity at an infinite rate of shear, i.e., if a shear stress less than the yield stress is applied to the fluid, it behaves like a solid, where as if a shear stress greater than yield stress, it starts to move (Dash et al. [156]). The examples of Casson fluid are as follows: jelly, tomato sauce, honey, soup, concentrated fruit juices, etc. Human blood can also be treated as Casson fluid. Due to the presence of several substances like, protein, fibrinogen, and globulin in aqueous base plasma, human red blood cells can form a chainlike structure,
known as aggregates or rouleaux. If the rouleaux behave like a plastic solid, then there exists a yield stress that can be identified with the constant yield stress in Casson’s fluid (Fung [228]).

The objective of the present study is to generalize the work of Bhattacharyya [95] by incorporating the loss due to Julian dissipation in the energy equation. Moreover, the flow is subject to pass through a porous medium which has been effectively accounted with the help of non-Darcy model. The extension is justified as because no system is full proof to trap the thermal energy loss. Secondly, the flow through a porous media is more practical having a numerous applications in the field of oil recovery and saline aquifer.

4.2 Mathematical analysis

Consider the flow of an incompressible Casson fluid past a flat sheet which coincides with the plane \( y = 0 \). The fluid flow is confined to \( y > 0 \). Two equal and opposite forces are applied along \( x \)-axis to initiate the formation of the fluid. The rheological equation of state for an isotropic and incompressible flow of Casson fluid is as follows;

\[
\tau_{ij} = \begin{cases} 
2(\mu + \frac{p_y}{\sqrt{2\pi}})e_y, & \pi > \pi_c, \\
2(\mu + \frac{p_y}{\sqrt{2\pi_c}})e_y, & \pi < \pi_c. 
\end{cases} \tag{4.2.1}
\]

![Flow geometry](image)

The governing continuity, momentum and energy equations of such type of flow is written as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.2.2}
\]

\textbf{Fig.4.1 Flow geometry}
\[ u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = U_s \frac{dU_s}{dx} + \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} - \left( \frac{\sigma B_0^2}{\rho} + \frac{\nu}{k_p} \right) (u - U_s), \]  
(4.2.3) 

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p} \frac{\partial q_x}{\partial y} - \frac{\sigma B_0^2}{\rho} u^2, \]  
(4.2.4) 

The appropriate boundary conditions are

\[
\begin{align*}
\begin{cases}
  u = U_w, \quad v = 0, \quad T = T_w & \text{at } y = 0, \\
  u \to U_s, \quad T \to T_\infty & \text{as } y \to \infty
\end{cases}
\end{align*}
\]  
(4.2.5) 

where \( u \) and \( v \) are the velocity components in \( x \) and \( y \) directions respectively. \( U_s = ax \) is the straining velocity of the stagnation-point with \( a(>0) \) being the straining constant and \( U_w = cx \) is the stretching velocity of the sheet with \( c(>0) \) being the stretching constant.

The equation of continuity equation (4.2.2) is identically satisfied if we take the stream function \( \psi(x, y) \) such that

\[ u = -\frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \]  
(4.2.6) 

Using (4.2.6) the momentum equation (4.2.3) takes the form

\[
\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = U_s \frac{dU_s}{dx} + \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^3 \psi}{\partial y^3} - \left( \frac{\sigma B_0^2}{\rho} + \frac{\nu}{k_p} \right) \frac{\partial \psi}{\partial y}. \]  
(4.2.7) 

The boundary condition in (4.2.5) reduces to

\[
\begin{align*}
\begin{cases}
  \frac{\partial \psi}{\partial y} = U_w, \quad \frac{\partial \psi}{\partial x} = 0 & \text{at } y = 0, \\
  \frac{\partial \psi}{\partial y} \to U_s, & \text{as } y \to \infty
\end{cases}
\end{align*}
\]  
(4.2.8) 

Using the Rosseland approximation for radiation \( q_r = -(4\sigma^* / 3k_i) \frac{\partial T^4}{\partial y} \) is obtained, where \( \sigma^* \) is the Stefan-Boltzmann constant, \( k_i \) is the absorption coefficient. Using Taylor’s series expanding \( T^4 \) about \( T_\infty \) and neglecting higher-order terms we get \( T^4 = 4T_\infty^3 T - 3T_\infty^4 \).

Now, equation (4.2.4) becomes

\[
\frac{u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}}{\frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} + \frac{16\sigma T_\infty^3}{3k_i \rho c_p} \frac{\partial^3 T}{\partial y^3} - \frac{\sigma B_0^2}{\rho} u^2}, \]  
(4.2.9) 

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The momentum and energy equations (4.2.7) and (4.2.9) can be transformed into the corresponding ordinary differential equations by introducing the following similarity transformations:

\[
\begin{align*}
\psi(x, y) &= \sqrt{\nu xf(\eta)}, \\
\frac{T - T_\infty}{T_w - T_\infty} &= \theta(\eta)
\end{align*}
\]  

(4.2.10)

where \(\eta = \frac{y}{\sqrt{c/D}}\)  

(4.2.11)

The momentum and energy equations (4.2.7) and (4.2.9) are transformed to

\[
\left(1 + \frac{1}{\beta}\right) f'' + ff''' - f''^2 - \left(M + \frac{1}{K_p}\right)(f' - B) + B^2 = 0,
\]

(4.2.12)

\[
(3R + 4) \theta'' + 3RP_f \theta' - 3RP_ME_f f'' = 0,
\]

(4.2.13)

The non-dimensional parameters are

\[
M = \frac{\sigma B_0^2}{\rho c}, \quad B = \frac{a}{c}, \quad P_r = c\mu/k, \quad E_i = \frac{1}{c_p(T_w - T_\infty)}c^2 x^2, \quad R = k^* k_1/4\sigma T_\infty^2
\]

Subject to the boundary conditions

\[
\begin{align*}
&f(\eta) = 0, \quad f'(\eta) = 1, \quad \theta(\eta) = 1 \quad \text{at} \quad \eta = 0, \\
&f'(\eta) \to B, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty
\end{align*}
\]

(4.2.14)

\[\boxed{4.3 \text{ Numerical method}}\]

The set of coupled non-linear governing boundary layer equations (4.2.2) to (4.2.4) together with the boundary conditions (4.2.5) are solved numerically using Runge-Kutta method along with shooting technique. First of all, higher order non-linear differential equations (4.2.2) to (4.2.4) are converted into a set of simultaneous non-linear differential equations of first order and they are further transformed into initial value problem by applying the shooting technique. The transformed initial value problem is solved by employing Runge-Kutta fourth order method. The step-size \(\Delta \eta = 0.05\) is used to obtain the numerical solution with five decimal place of accuracy as the criterion of convergence. In course of numerical computation, the skin-friction coefficient, the Nusselt number which are respectively proportional to \(f''(0), \theta'(0)\) are also calculated and their numerical values are presented in the Table-4.2. The nomenclature is given in Appendix-3.

\[\boxed{4.4 \text{ Results and discussion}}\]

The following discussion is based upon the numerical solution of both the boundary layer equations i.e. velocity boundary layer and thermal boundary layer. At the out-set the
validity check is carried out by comparing the present result with the work of Bhattacharyya [95] (without the porous medium and Julian dissipation). The specialty of the present study is to highlight the effect of porous medium and Julian dissipation on the stagnation-point flow of Casson fluid. From equation (4.2.12) it can be seen that the importance of Julian heating can be indicated by the combined effect of magnetic parameter $M$, and Eckert number $E_c$. The Eckert number is a measure of dissipation effect in the flow. Since this grows in proportion to the square of the velocity it can be neglected for small velocities.

Fig.4.2 exhibits the effect of velocity ratio in porous and non-porous medium. For the purpose of comparison with Bhattacharyya, the dotted curve for $K_p = 10, B = 2.0, M = 0.5$ and $\beta = 2$ has been drawn. It is seen that the curve coincides with the curve presented in Fig. 6.1 of Bhattacharyya [95]. It is interesting to note that the velocity profile clearly display three distinct characteristics for ratios of straining and stretching i.e. $B < 1, B = 1$ and $B > 1$. $B = 1$ implies the equality of straining velocity of the stagnation point flow and stretching velocity of the sheet. This amounts to no motion. Therefore, the velocity remains constant throughout. On the other hand, $B > 1$ represents the dominance of straining over stretching which leads to increase the velocity in the layer close to bounding surface i.e. stretching sheet, there after the velocity remains constant. Moreover, it is seen that the effect of porous matrix is to increase the velocity profile. Further, it is seen that the profiles for $B < 1$ and $B > 1$ are almost symmetrical about $B = 1$. This means a three layer character exists for $B < 1, B = 1$ and $B > 1$.

Figs.4.3 and 4.4 exhibit the velocity variation for various values of the parameter, $\beta$ showing the characteristics of Casson fluid. It is to note that an increase in $\beta$, leads to increase the velocity in both the cases i.e. in the presence of porous medium and without it. Thus, it is concluded that the non-Newtonian property of Casson fluid model is responsible to diffuse the momentum through more number of layers of fluid contributing to thickening of boundary layer when the effect of straining dominates over stretching.
Fig. 4.2 Variation of $B$ on velocity profile

Fig. 4.3 Variation of $\beta$ on velocity profile
For $B < 1$, the profiles are mirror image of $B > 1$ about the profile $B = 1$. Further it is to note that the effect of Lorentz force is to reduce the velocity at all points. This is due to resistive property of the pondermotive force generated due to interaction of magnetic field with conducting fluid (Fig.4.4). On careful observation it is further revealed that slight change in magnetic field does not affect the velocity profile in both porous and nonporous medium.

Figs.4.5, 4.6 and 4.7 are drawn when $P_t = 1$. $P_t$ is the salient characteristic number. This imposes a pre-condition of equality and kinematic viscosity and thermal diffusivity. Figs.4.5 and 4.6 depicts the temperature variation for various values of characterizing parameters.

On careful observation it is seen that decrease in $M$ leads to decrease the temperature at all layers (curves II and III) but the reverse effect is observed in case of $E_c$ (curves II and IV) as well as for $B$, ($B < 1$) (curves II and VI). It is concluded that thinning of thermal boundary layer occurs due to higher thermal dissipation and stretching rate. It is also to note that presence of porous matrix increases the temperature. Moreover, (curves V and VII) of fig.4.6 show that non-Newtonian property of the Casson fluid enhance the temperature in the presence of porous matrix.

Fig.4.7 (curve I) intends to present a very special case when the effect of magnetic field, dissipation and porosity are absent. This leads to lower down the temperature which is otherwise established, in their presence, in earlier discussions.

Fig.4.8 aims at showing the relative importance of kinematic viscosity and thermal diffusivity. It is evident that higher kinematic viscosity, in case of liquid, restricts the heat transfer to a fewer layers resulting a thinner boundary layer where as for gas $P_t = 0.71$, the heat energy diffuses to a larger layers.

Fig.4.9 presents the effect of $R$, the radiation parameter. This shows that thermal radiation associated with magnetic effect lowers down the temperature (curves I and IV) when kinematic viscosity and thermal diffusivity enjoy the same order of priority.
Fig. 4.4 Variation of $M$ on velocity profile

Fig. 4.5 Variation of $B, E_c, M$ and $K_p$ on temperature profile
Fig. 4.6 Variation of $\beta, E_c, M$ and $K_p$ on temperature profile

<table>
<thead>
<tr>
<th>Curve</th>
<th>$\beta$</th>
<th>$E_c$</th>
<th>$M$</th>
<th>$K_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.5</td>
<td>0</td>
<td>0.5</td>
<td>10</td>
</tr>
<tr>
<td>II</td>
<td>0.5</td>
<td>0.2</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>III</td>
<td>0.5</td>
<td>0.2</td>
<td>0.5</td>
<td>10</td>
</tr>
<tr>
<td>IV</td>
<td>0.5</td>
<td>0.5</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>V</td>
<td>0.5</td>
<td>0.2</td>
<td>0.5</td>
<td>0.1</td>
</tr>
<tr>
<td>VI</td>
<td>1.0</td>
<td>0.2</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>VII</td>
<td>1.0</td>
<td>0.2</td>
<td>0.5</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Fig. 4.7 Variation of $M, E_c$ and $K_p$ on temperature profile

<table>
<thead>
<tr>
<th>Curve</th>
<th>$M$</th>
<th>$E_c$</th>
<th>$K_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>II</td>
<td>0.5</td>
<td>0.1</td>
<td>10</td>
</tr>
<tr>
<td>III</td>
<td>0.5</td>
<td>0.2</td>
<td>10</td>
</tr>
<tr>
<td>IV</td>
<td>1.0</td>
<td>0.1</td>
<td>10</td>
</tr>
<tr>
<td>V</td>
<td>1.0</td>
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</tr>
<tr>
<td>VI</td>
<td>0.5</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

$B=0.1, P_r=1, R=1, \beta=2$
Fig. 4.8 Variation of $M$, $E_c$, and $P_r$ on temperature profile

Fig. 4.9 Variation of $M$, $E_c$, and $R$ on temperature profile
Table-4.1 is prepared for validating and authenticity check by comparing the values of $f''(0)$ with earlier works by Nazar et.al [164] and Bhattacharyya [95]. In case of Newtonian fluid this shows a good agreement. Table-4.2 presents the numerical values of surface criteria such as skin friction and Nusselt number. One most important finding is that skin friction assumes positive values for $B > 1$ i.e. $B = 2$ otherwise negative. This aspect has been clearly shown in the velocity graph. Thus, the predominance of straining rate accounts for the positive values of skin friction. Therefore, straining rate and stretching rate have great influence on flow criteria including stability and growth of boundary layer. It is also noticed that all other parameters reduce the skin friction including Casson fluid parameter, which is desirable, except $E_\gamma$ which has no significant effect.

**Table-4.1 Values of $f''(0)$ for several values of $B$ with $M = 0$ and $\beta = \infty$.**

<table>
<thead>
<tr>
<th>$B$</th>
<th>Nazar et al. [164]</th>
<th>Bhattacharyya et al.[95]</th>
<th>Present study</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.9694</td>
<td>-0.969386</td>
<td>-0.969381</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.9181</td>
<td>-0.918107</td>
<td>-0.918102</td>
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<tr>
<td>0.5</td>
<td>-0.6673</td>
<td>-0.667263</td>
<td>-0.667260</td>
</tr>
<tr>
<td>2</td>
<td>2.0176</td>
<td>2.017503</td>
<td>2.017486</td>
</tr>
</tbody>
</table>

Table-4.2 also focuses light on Nusselt number, presenting rate of heat transfer at the bounding surface. It is evident that stretching rate ratio has no such effect as that of skin friction in reversing the rate of heat transfer at the surface. Further, it is to note that rate of heat transfer is enhanced with an increase of velocity ratio parameter $B$, Eckert number $E_\gamma$, magnetic parameter $M$, Prandtl number $Pr$, and thermal radiation parameter $R$ except Casson parameter $\beta$ and porous matrix $K_p$. Thus it is concluded that non-Newtonian parameter and the presence of porous matrix reduce the rate of heat transfer at the bounding surface on which the flow phenomena occurs.
Table-4.2 Skin friction and Nusselt number at the plate

<table>
<thead>
<tr>
<th>$B$</th>
<th>$E_c$</th>
<th>$M$</th>
<th>$K$</th>
<th>$P_r$</th>
<th>$B$</th>
<th>$R$</th>
<th>$f'(0)$</th>
<th>$-\theta(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>0.71</td>
<td>0.1</td>
<td>1</td>
<td>-0.9061484</td>
<td>0.28598657</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
<td>0</td>
<td>10</td>
<td>0.71</td>
<td>0.1</td>
<td>1</td>
<td>-0.9061484</td>
<td>0.28598657</td>
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<tr>
<td>0.1</td>
<td>0.2</td>
<td>0.5</td>
<td>10</td>
<td>0.71</td>
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<td>1</td>
<td>-0.9262399</td>
<td>0.30087789</td>
</tr>
<tr>
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<td>0.2</td>
<td>0.5</td>
<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>1</td>
<td>-0.360191</td>
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</tr>
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<td>0.37802938</td>
</tr>
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<td>1</td>
<td>0.65106272</td>
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<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>1</td>
<td>-0.360191</td>
<td>0.39828737</td>
</tr>
</tbody>
</table>

4.5 Conclusion

- The non-Newtonian property of Casson fluid is responsible to diffuse the momentum through more number of fluid layers.
- Slight change in magnetic field does not affect the velocity profile in both porous and nonporous medium.
- Absence of magnetic field, Julian dissipation and porosity of the medium lower down the temperature.
- The fluid with higher kinematic viscosity, restricts the heat transfer to a fewer layers of fluid.
- The straining rate and stretching rate have great influence on the flow criteria including stability and growth of boundary layer.
- Casson fluid parameter as well as other parameters reduce the skin friction, which is desirable, except Eckert number $E_c$ which has no significant contribution.

- The non-Newtonian parameter and the presence of porous matrix reduce the rate of heat transfer at the bounding surface.