Chapter 4

Construction of Partially Balanced Residual Treatment Effects Design

4.1 INTRODUCTION

In residual treatment effects designs, each experimental unit receives some or all of the treatments in an appropriate sequence over a number of successive periods and the effect of the treatment continues beyond the period of its application. These designs have been discussed in the literature under various names, viz., cross-over designs, change-over designs, and repeated measurement designs. Residual designs in v treatments on n experimental units in p periods are useful in a broad spectrum of research areas, including agriculture, dairy husbandry, bioassay procedures, clinical trials, psychological experiments and weather modification experiments. The advantages of the residual design are its cost and the elimination of inter unit variability. In the following, we assume that each treatment produces a direct effect in the period of its application and a residual effect in the subsequent period of its application.

Williams (1949, 1950) introduced balanced residual treatment effects designs. In any experiments involving residual effects, if balanced residual effects designs are not available, then the alternative is to use partially balanced design with respect to direct and residual effects. Blaisdell and Raghavarao (1980) have developed and defined the concept of partially balanced changeover designs based on m-associate class PBIB designs. Biswas and Raghavarao (1998) constructed partially balanced residual treatment effects designs, when the treatments have rectangular and group divisible association schemes. Aggarwal and Jha (2006) gave some new series to construct partially balanced residual treatment effects designs. Here we have provided an alternate method of construction of series 3 and 4, of Aggarwal and Jha (2006).

Patterson (1952) gave combinatorial conditions for balance and also gave a number of methods for construction of such designs when $p \leq v$ and when n is as small as possible. Since p and n are small, these designs are very attractive to practitioners. All these designs had the property that no treatment immediately succeeds itself on the same subject.

Hedayat and Afsarinijad (1978) showed that when $p = v$, a balanced design is universally optimal (UO) (as defined in Kiefer, J. (1975)) for estimation of the direct (residual) effects when the designs in the competing class are uniform on periods as well as subjects.

Cheng and Wu (1980) showed that these designs are UO for the estimation of residual effects when the competing designs may not be uniform over subjects or periods, but again no treatment succeeds itself on the same subject. Kunert (1984) showed that when $n = vt$, a balanced uniform design is UO for direct effects if $v \geq 3$ and $t = 1$ or if $v \geq 6$ and $t = 2$. Hedayat and Yang (2003) generalized this to the case where $v \geq 3$ and $t \leq (v - 1)/2$. The results of Kunert (1984) and of Hedayat and Yang (2003) were proved without any condition on the competing designs. However, there do not appear to be any available results on the optimality of balanced crossover designs when $p < v$.

Cheng and Wu (1980) also introduced what are called strongly balanced designs where each of the v^2 pairs of treatments occurs in consecutive periods for the same subject an equal number of times. They established some strong optimality properties for these designs. However, these designs require $p = vt$ or $vt + 1$ and also require n to be large.

Kushner (1997) gave a novel approximate design theory approach to obtain UO designs for arbitrary values of p and v. Further, Kushner (1998) gave exact designs which are UO for direct effects for every pair (v, p) for some n. Kushner's results are very attractive because they do not put any conditions on the competing designs. Their main limitation is that the values of p or of n are large. Further, in almost all cases these optimal designs are non binary (on the subjects). An attractive property of the binary balanced designs is that they are optimal when the residual effects are negligible (1975). Some authors have obtained optimal designs under different models. Kunert and Martin (2000) gave optimal designs under an interference model. Kunert and Martin (2000) considered models with correlated errors. Kunert and Stufken (2002) introduced a model where the residual effect of a treatment on itself is different from the residual effect when the treatment is followed by another treatment. An excellent review of the literature in this broad area up to 1996 is given by Stufken (1996). The balanced designs given by Williams (1949) and by Patterson (1952) are very attractive because they have a small number of periods and often involve a small to moderate number of subjects. These designs have been around for a long time and are generally believed to be efficient. However, precise optimality results are rather limited in nature.

Shah, Bose and Raghavrao (2005) showed that the balanced crossover designs given by Patterson [Biometrika 39 (1952) 32–48] are (a) universally optimal (UO) for the joint estimation of direct and residual effects when the competing class is the class of connected binary designs and (b) UO for the estimation of direct (residual) effects when the competing class of designs is the class of connected designs (which includes the connected binary designs) in which no treatment is given to the same subject in consecutive periods. In both results, the formulation of UO is as given by Shah and Sinha [Unpublished manuscript (2002)]. Further, they introduced a functional of practical interest, involving both direct and residual effects, and establish (c) optimality of Patterson's designs with respect to this functional when the class of competing designs is as in (b) above.

In this chapter, we have constructed partially balanced residual treatment effects designs using group divisible (GD) designs, the mutually orthogonal Latin square designs, group and set of blocks constituting replications based on group divisible designs respectively. We have also estimated the efficiency for the estimated direct effect and estimated elementary contrast.

4.2 Mathematical Model

Let us consider a residual treatment effects design in which v treatments are compared using N experimental units and the experiment lasts for k periods. Let us consider the linear model for the analysis is

$$y_{ij} = \mu + \alpha_i + \beta_j + \tau_{d(i,j)} + \rho_{d(i-1,j)} + e_{ij}$$

where, y_{ij} (i= 1, 2, …, k; j = 1, 2,…, b) is the response observed in the i^{th} period on the jth unit, μ the general mean, α_i the ith period effect, β_j the j^{th} unit effect, $\tau_{d(i,j)}$ the direct effect of the treatment *d(i, j)*, $\rho_{d(i-1,j)}$ the first order residual effect of the treatment *d(i, j)* with $\rho_{d(0,j)} = 0$ for all *j*, e_{ij} are random errors assumed to be normally and independently distributed with mean zero and variance σ^2. All parameters in the model are assumed to be fixed.

Let *M, L, N, L** and *N** denote the respective incidence matrices for the periods-units, direct effects-periods, direct effects-units, residual effects-units and direct effects-residual effects. The following results are due to Raghavarao and Blaisdell (1985).

Raghavarao and Blaisdell (1985) gave the efficiency for the estimated elementary contrasts of direct effects, E_d, which is given by

$$E_d = \frac{v-1}{k^2 t \sum_{i=1}^{v-1} \{(k^2 t - \rho_i) - \{(2k\theta_i - \rho_i - \gamma_i + t)^2 / 4(k(k-1)t - \gamma_i)\}\}^{-1}}$$

And the efficiency for estimated elementary contrasts of residual effects, E_r, is given by

$$E_r = \frac{v-1}{k(k-1)t \sum_{i=1}^{v-1} \{(k(k-1)t - \gamma_i) - \{(2k\theta_i - \rho_i - \gamma_i + t)^2 / 4(k^2 t - \rho_i)\}\}^{-1}}$$

where the ρ_i's are the eigenvalues of $L^* L^{*'}$, γ_i's are the eigenvalues of $N^* N^{*'}$ and the θ_i's are the eigenvalues of *M* for i=1,2,3….,

4.3 METHODS OF CONSTRUCTION

In this section we discuss the methods of construction to obtain partially balanced residual treatment effects designs using group divisible (GD) designs, the mutually orthogonal Latin square designs, group and set of blocks constituting replications based on group divisible designs.

4.3.1 PBRTED is obtained using group (*m*, n) of GD design where *n* is even.

In this section, we considered a group divisible design with (m, n) as group, where m denotes number of groups and n be the number of treatments per group. Next for a given treatment *i,* number of treatments present in that column as second associates and the remaining as third associates. We first construct the $n \times n$ Latin square for each row and perform the following

I. Take the first row of the group (m, n).

II. For the second row take the second element of the first row and the remaining element for the second row will be obtained as $(i + m)$ with reduced mod v.

III. For the third row take the last element that is n of the first row and the remaining are obtained as described in step II.

IV. Now for the fourth row we take the third element of the first row and for fifth row second last element and so on. i.e. 3, n-1, 4, n-2, 5, n-3……

Then repeat each Latin square $(m - 1)$ times and place the entire Latin squares side by side. Add $(n + 1)^{th}$ row in the above Latin squares, consisting of the second associates in the n^{th} row. Then the PBRTED is obtained with parameters

$v = mn,\ t = m\text{-}1,\ k = n\text{+}1,\ N = vt,\ \lambda_1 = m\text{-}1,\ \lambda_2 = 1,\ \lambda_3 = 0,\ \mu_1 = nt,\ \mu_2 = 2 = \mu_3,\ v_1 = nt,$

$v_2 = 0 = v_3.$

Example 4.3.1: Consider a group (3, 4)

1	4	7	10
2	5	8	11
3	6	9	12

We first obtain the 4×4 Latin squares for each row. Then repeat each Latin square *m*-1 times i.e., twice, now putting all Latin squares side by side we get,

1	4	7	10	1	4	7	10	2	5	8	11	2	5	8	11	3	6	9	12	3	6	9	12
4	7	10	1	4	7	10	1	5	8	11	2	5	8	11	2	6	9	12	3	6	9	12	3
10	1	4	7	10	1	4	7	11	2	5	8	11	2	5	8	12	3	6	9	12	3	6	9
7	10	1	4	7	10	1	4	8	11	2	5	8	11	2	5	9	12	3	6	9	12	3	6

Now, add 5th row in the above Latin squares, consisting of the second associates of the treatments in the 4th row. Hence we get, the PBRTED

1	4	7	10	1	4	7	10	2	5	8	11	2	5	8	11	3	6	9	12	3	6	9	12
4	7	10	1	4	7	10	1	5	8	11	2	5	8	11	2	6	9	12	3	6	9	12	3
10	1	4	7	10	1	4	7	11	2	5	8	11	2	5	8	12	3	6	9	12	3	6	9
7	10	1	4	7	10	1	4	8	11	2	5	8	11	2	5	9	12	3	6	9	12	3	6
8	11	2	5	9	12	3	6	9	12	3	6	7	10	1	4	7	10	1	4	8	11	2	5

with parameters

$v = 12, t = 2, k = 5, N = 24, \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0, \mu_1 = 8, \mu_2 = 2 = \mu_3, v_1 = 8, v_2 = 0 = v_3.$

$$E_d = 0.828,\ E_r = 0.811$$

4.3.2 PBRTED using group (*m, n*) of GD design where *n* is odd.

In this section we have considered the group (*m, n*) where *n* is odd. The treatments in the same row are termed as first associates, treatments in the corresponding column as second associates and the remaining as third associates. Here we have constructed PBRTED using mutually orthogonal Latin square (MOLS) design. We first obtain the $n \times n$ Latin square for each row. Repeat each LSD up to (n - 1) times because the (n - 1) LSD are MOLS. Further add (n + 1)th row in the above Latin squares, consisting of the second associates in the n^{th} row. The resulting design so obtained is PBRTED. Here we discuss the following cases.

4.3.2.1 Group (*m*,*n*) with $m \le n$

This section is further divided into two sub section based on MOLS. In 4.3.2.1.1 we construct PBRTED if number of MOLS is divisible by number of second associates.

4.3.2.1.1 If number of MOLS is divisible by number of second associates, then PBRTED will be obtain with parameters

$v = mn,\ N = vt,\ t= n-1,\ k= n+1,\ \lambda_1 = n-1,\ \lambda_2 = (n-1)/(m-1),\ \lambda_3 = 0,$

$\mu_1 = nt,\ \mu_2 = 2\lambda_2 = \mu_3,\ v_1 = nt,\ v_2 = 0 = v_3.$

Example 4.3.2.1.1 Consider a group (3, 3) as

1	4	7
2	5	8
3	6	9

We first obtain (n-1)=2, 3×3 Latin squares for each row using MOLS.

1	4	7	1	4	7	2	5	8	2	5	8	3	6	9	3	6	9
4	7	1	7	1	4	5	8	2	8	2	5	6	9	3	9	3	6
7	1	4	4	7	1	8	2	5	5	8	2	9	3	6	6	9	3

Now, add 4th row in the above Latin squares, consisting of the second associates of the treatments in the 3rd row. Hence we get, the PBRTED

1	4	7	1	4	7	2	5	8	2	5	8	3	6	9	3	6	9
4	7	1	7	1	4	5	8	2	8	2	5	6	9	3	9	3	6
7	1	4	4	7	1	8	2	5	5	8	2	9	3	6	6	9	3
8	2	5	6	9	3	9	3	6	4	7	1	7	1	4	5	8	2

with parameters

$v = 9, t = 2, k = 4, N = 18, \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0, \mu_1 = 6, \mu_2 = 2 = \mu_3, \nu_1 = 6, \nu_2 = 0 = \nu_3.$

$$E_d = 0.771,\ E_r = 0.742$$

4.3.2.1.2 When m is even, then PBRTED will be obtained with parameters

$v = mn, N = vt, t = (m-1)(n-1), k = n+1, \lambda_1 = (m-1)(n-1), \lambda_2 = n-1,$

$\lambda_3 = 0, \mu_1 = nt, \mu_2 = 2\lambda_2 = \mu_3, \nu_1 = nt, \nu_2 = 0 = \nu_3.$

Example 4.3.2.1.2 Consider a group (2, 3) as

1	3	5
2	4	6

We first obtain (n-1)=2, 3×3 Latin squares for each row using MOLS.

1	3	5	1	3	5	2	4	6	2	4	6
3	5	1	5	1	3	4	6	2	6	2	4
5	1	3	3	5	1	6	2	4	4	6	2

Now, add 4th row in the above Latin squares, consisting of the second associates of the treatments in the 3rd row. Hence we get, the PBRTED

1	3	5	1	3	5	2	4	6	2	4	6
3	5	1	5	1	3	4	6	2	6	2	4
5	1	3	3	5	1	6	2	4	4	6	2
6	2	4	4	6	2	5	1	3	3	5	1

with parameters

$v = 6, t = 2, k = 4, N = 12, \lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 0, \mu_1 = 6, \mu_2 = 4 = \mu_3, \nu_1 = 6, \nu_2 = 0 = \nu_3.$

$$E_d = 0.821,\ E_r = 0.775$$

4.3.2.2 Group (*m*,*n*) with *m* > *n*

In this section we consider a group where $m > n$. Then with this group we construct a PBRTED with parameters.

$v = mn,\ N = vt,\ t = (m-1)(n-1) / (m-n),\ k = n+1,\ \lambda_1 = t,\ \lambda_2 = (n-1) / (m-n),\ \lambda_3 = 0,$ $\mu_1 = nt,\ \mu_2 = 2\lambda_2 = \mu_3,\ \nu_1 = nt,\ \nu_2 = 0 = \nu_3.$

Example 4.3.2.2 Consider a group (4, 3)

1	5	9
2	6	10
3	7	11
4	8	12

We first obtain (n-1) = 2, 3×3 Latin squares for each row using MOLS. Since the number of second associates are more then the number of MOLS so, repeat the MOLS to accumulate all the second associates equal number of times. In this case each MOLS is repeated thrice.

1	5	9	1	5	9	1	5	9	1	5	9	1	5	9	1	5	9	2	6	10	2	6	10
5	9	1	9	1	5	5	9	1	9	1	5	5	9	1	9	1	5	6	10	2	10	2	6
9	1	5	5	9	1	9	1	5	5	9	1	9	1	5	5	9	1	10	2	6	6	10	2

2	6	10	2	6	10	2	6	10	2	6	10	3	7	11	3	7	11	3	7	11	3	7	11
6	10	2	10	2	6	6	10	2	10	2	2	7	11	3	11	3	7	7	11	3	11	3	7
10	2	6	6	10	2	10	2	6	6	10	6	11	3	7	7	11	3	11	3	7	7	11	3

3	7	11	3	7	11	4	8	12	4	8	12	4	8	12	4	8	12	4	8	12	4	8	12
7	11	3	11	3	7	8	12	4	12	4	8	8	12	4	12	4	8	8	12	4	12	4	8
11	3	7	7	11	3	12	4	8	8	12	4	12	4	8	8	12	4	12	4	8	8	12	4

Now, add 4th row in the above Latin squares, consisting of the second associates of the treatments in the 3rd row. Hence we get, the PBRTED

1	5	9	1	5	9	1	5	9	1	5	9	1	5	9	1	5	9	2	6	10	2	6	10
5	9	1	9	1	5	5	9	1	9	1	5	5	9	1	9	1	5	6	10	2	10	2	6
9	1	5	5	9	1	9	1	5	5	9	1	9	1	5	5	9	1	10	2	6	6	10	2
10	2	6	7	11	3	12	4	8	6	10	2	11	3	7	8	12	4	11	3	7	8	12	4

2	6	10	2	6	10	2	6	10	2	6	10	3	7	11	3	7	11	3	7	11	3	7	11
6	10	2	10	2	6	6	10	2	10	2	6	7	11	3	11	3	7	7	11	3	11	3	7
10	2	6	6	10	2	10	2	6	6	10	2	11	3	7	7	11	3	11	3	7	7	11	3
9	1	5	7	11	3	12	4	8	5	9	1	12	4	8	5	9	1	10	2	6	8	12	4

3	7	11	3	7	11	4	8	12	4	8	12	4	8	12	4	8	12	4	8	12	4	8	12
7	11	3	11	3	7	8	12	4	12	4	8	8	12	4	12	4	8	8	12	4	12	4	8
11	3	7	7	11	3	12	4	8	8	12	4	12	4	8	8	12	4	12	4	8	8	12	4
9	1	5	6	10	2	9	1	5	6	10	2	11	3	7	5	9	1	10	2	6	7	11	3

with parameters

$v = 12, t = 6, k = 4, N = 72, \lambda_1 = 6, \lambda_2 = 2, \lambda_3 = 0, \mu_1 = 18, \mu_2 = 4 = \mu_3, v_1 = 18, v_2 = 0 = v_3.$

$E_d = 0.748, E_r = 0.727$

4.3.3 PBRTED using odd set of blocks

In this section we consider a set of blocks constituting replications based on group divisible designs. Let s be the size of the set, where s is odd. Set of block will have s rows and s columns. The treatments in same row are termed as first associates, treatments in the corresponding column as second associates and remaining as third associates. Now consider one row of the set and called that a block and through this develop $s \times s$ Latin square design using MOLS. If the size of the set is s then there will be $(s\text{-}1)$ MOLS. Hence we will have $(s\text{-}1)$ Latin squares of $s \times s$. Since we have s rows so there will be $s(s\text{-}1)$ Latin squares of size s. Now put the entire Latin square side by side. Add $(s+1)$th row in the above Latin square, consisting of the second associates in the s^{th} row. Then the PBRTED will be obtained with parameters

$$v = s^2, N= vt, t= s\text{-}1, k= s+1, \lambda_1= s\text{-}1, \lambda_2= 1, \lambda_3= 0, \mu_1= st, \mu_2= 2 = \mu_3, v_1= st, v_2 = 0 = v_3$$

Example 4.3.3: Consider a set 3.1 [Clatworthy (1971)] of blocks constituting replications based on group divisible designs for s = 3

1	2	3
4	5	6
7	8	9

Considering the treatments in same row as first associates, treatment in corresponding column as second associates and remaining as third associates. We first obtain Latin squares for each row using MOLS.

1	2	3	1	2	3	4	5	6	4	5	6	7	8	9	7	8	9
2	3	1	3	1	2	5	6	4	6	4	5	8	9	7	9	7	8
3	1	2	2	3	1	6	4	5	5	6	4	9	7	8	8	9	7

Now, add 4th row in the above Latin squares, consisting of the second associates of the treatments in the 3rd row. Hence we get, the PBRTED

1	2	3	1	2	3	4	5	6	4	5	6	7	8	9	7	8	9
2	3	1	3	1	2	5	6	4	6	4	5	8	9	7	9	7	8
3	1	2	2	3	1	6	4	5	5	6	4	9	7	8	8	9	7
6	4	5	8	7	9	9	7	8	8	9	7	3	1	2	5	6	4

with parameters

$v = 9, N = 18, t = 2, k = 4, \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0, \mu_1 = 6, \mu_2 = 2 = \mu_3, \nu_1 = 6, \nu_2 = 0 = \nu_3$

$$E_d = 0.771, E_r = 0.742$$

4.3.4 PBRTED using even set of blocks

In this section we consider a set of blocks constituting replications based on group divisible designs with the even size of set. A set constitutes of s rows and s columns. The treatments in same row are termed as first associates, treatments in the corresponding column as second associates and remaining as third associates. Now consider one row of the set and called that to be a block and with this develop $s \times s$ Latin squares. Since the size of the set is s therefore we will obtain $s(s-1)$ Latin squares by repeating each Latin squares $(s-1)$ times. Now put the entire Latin squares side by side. Add $(s+1)^{th}$ row in the above Latin squares, consisting of the second associates in the s^{th} row. Then the PBRTED will be obtained with parameters

$v = s^2$, $N = vt$, $t = s-1$, $k = s+1$, $\lambda_1 = s-1$, $\lambda_2 = 1$, $\lambda_3 = 0$, $\mu_1 = st$, $\mu_2 = 2 = \mu_3$, $\nu_1 = st$, $\nu_2 = 0 = \nu_3$

Example 4.3.4 Consider a set 4.1 [Clatworthy (1971)] of blocks constituting replications based on group divisible designs for $s = 4$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

We first obtain the 4×4 Latin squares for each row. Then repeat each Latin square (s-1) times, i.e., thrice, and then on putting all Latin squares side by side we get,

1	2	3	4	1	2	3	4	1	2	3	4	5	6	7	8	5	6	7	8
2	3	4	1	2	3	4	1	2	3	4	1	6	7	8	5	6	7	8	5
4	1	2	3	4	1	2	3	4	1	2	3	8	5	6	7	8	5	6	7
3	4	1	2	3	4	1	2	3	4	1	2	7	8	5	6	7	8	5	6

5	6	7	8
6	7	8	5
8	5	6	7
7	8	5	6

9	10	11	12	9	10	11	12	9	10	11	12	13	14	15	16
10	11	12	9	10	11	12	9	10	11	12	9	14	15	16	13
12	9	10	11	12	9	10	11	12	9	10	11	16	13	14	15
11	12	9	10	11	12	9	10	11	12	9	10	15	16	13	14

13	14	15	16
14	15	16	13
16	13	14	15
15	16	13	14

13	14	15	16
14	15	16	13
16	13	14	15
15	16	13	14

Now, add 5th row in the above Latin squares, consisting of the second associates of the treatments in the 4th row. Hence we get, the PBRTED

1	2	3	4
2	3	4	1
4	1	2	3
3	4	1	2
7	8	5	6

1	2	3	4
2	3	4	1
4	1	2	3
3	4	1	2
11	12	9	10

1	2	3	4
2	3	4	1
4	1	2	3
3	4	1	2
15	16	13	14

5	6	7	8
6	7	8	5
8	5	6	7
7	8	5	6
11	12	9	10

5	6	7	8
6	7	8	5
8	5	6	7
7	8	5	6
11	12	9	10

5	6	7	8
6	7	8	5
8	5	6	7
7	8	5	6
15	16	13	14

5	6	7	8
6	7	8	5
8	5	6	7
7	8	5	6
3	4	1	2

9	10	11	12
10	11	12	9
12	9	10	11
11	12	9	10
15	16	13	14

9	10	11	12
10	11	12	9
12	9	10	11
11	12	9	10
3	4	1	2

9	10	11	12
10	11	12	9
12	9	10	11
11	12	9	10
7	8	5	6

13	14	15	16
14	15	16	13
16	13	14	15
15	16	13	14
3	4	1	6

13	14	15	16
14	15	16	13
16	13	14	15
15	16	13	14
7	8	5	2

13	14	15	16
14	15	16	13
16	13	14	15
15	16	13	14
11	12	9	10

with parameters

$v = 16, t = 3, k = 5, N = 48, \lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0, \mu_1 = 12, \mu_2 = 2 = \mu_3, \nu_1 = 12, \nu_2 = 0 = \nu_3.$

$$E_d = 0.81, E_r = 0.797$$

4.3.5 PBRTED using GD design where k^* is even

Aggarwal and Jha (2006) has constructed PBRTED using all the blocks of GD design with parameter $v^*, b^*, k^*, r^*, \lambda_1^*, \lambda_2^*, m, n$ (k^* is even). Where the resulting parameters of PBRTED $v = v^*$, $k= k^*$, $N = vt$, $t = r^*$, $\lambda_1 = \lambda_1^*$, $\lambda_2 = \lambda_2^*$,$\mu_1 = k \lambda_1^*$, $\mu_2 = k \lambda_2^*$, $\nu_1 = (k-2) \lambda_1^*$, $\nu_2 = (k-2) \lambda_2^*$. They have developed PBRTED considering k^* even and odd where they have not added (n+1)th row as well as they asked to repeat each Latin square (m-1) times but in series 3 they are using the block design. However in our investigation we have constructed PBRTED for both the cases

(1) With v treatments in k plots

(2) With v treatments in $(k+1)^{th}$ plots.

Also we have not repeated the Latin square design (m-1) times which reveals that in our investigation number of blocks will be smaller than Aggarwal and Jha (2006).

Remark: Sometime it is not possible to construct PBRTED by adding $(k^*+1)^{th}$ row because in the respective column some treatment is being repeated, i.e., the design becomes non-binary.

4.3.5.1 PBRTED with v treatments in k^* plots

Consider a blocks of group divisible design with parameters $v^*, b^*, k^*, r^*, \lambda_1^*, \lambda_2^*, m, n$. Obtain the $k^* \times k^*$ Latin square for each blocks as per *4.3.1* but without repetition and then put all the Latin squares side by side. Next we have two possibility of having

PBRTED design (1) by adding (k*+1)th row in the resulting Latin square design of size k*×k* and (2) deleting the last row of the resulting Latin square of size k*×k*.

Add $(k^{*}+1)^{th}$ row in the above Latin squares, consisting of the second associates of the group (*m, n*). the resulting design is PBRTED with parameters

$v = v^{*},\ N = vt,\ t = r^{*},\ k = k^{*}+1,\ \lambda_1 = \lambda_1^{*},\ \lambda_2 = 2\lambda_2^{*},\ \lambda_3 = \lambda_2^{*},\ \mu_1 = k^{*}\lambda_1,\ \mu_2 = 2k^{*}\lambda_2^{*} = \mu_3,$,

$\nu_1 = k^{*}\lambda_1^{*},\ \ \nu_2 = k^{*}\lambda_2^{*} = \nu_3.$

Example 4.3.5.1 Consider the blocks of group divisible design $v^{*} = 6$, $r^{*} = 4$, $k^{*} = 4$, $b^{*} = 6$, $\lambda_1^{*} = 4$, $\lambda_2^{*} = 2$, $m = 3$, $n = 2$,

1	4	2	5
2	5	3	6
3	6	1	4
4	1	5	2
5	2	6	3
6	3	4	1

We first obtain the 4×4 Latin squares for each blocks.

1	4	2	5	2	5	3	6	3	6	1	4	4	1	5	2	5	2	6	3	6	3	4	1
5	1	4	2	6	2	5	3	4	3	6	1	2	4	1	5	3	5	2	6	1	6	3	4
4	2	5	1	5	3	6	2	6	1	4	3	1	5	2	4	2	6	3	5	3	4	1	6
2	5	1	4	3	6	2	5	1	4	3	6	5	2	4	1	6	3	5	2	4	1	6	3

Now, add 5^{th} row in the above Latin squares, consisting of the second associates of the group (3, 2). Hence we get, the PBRTED

1	4	2	5	2	5	3	6	3	6	1	4	4	1	5	2	5	2	6	3	6	3	4	1
5	1	4	2	6	2	5	3	4	3	6	1	2	4	1	5	3	5	2	6	1	6	3	4
4	2	5	1	5	3	6	2	6	1	4	3	1	5	2	4	2	6	3	5	3	4	1	6
2	5	1	4	3	6	2	5	1	4	3	6	5	2	4	1	6	3	5	2	4	1	6	3
3	6	3	6	1	4	1	4	2	5	2	5	6	3	6	3	4	1	4	1	5	2	5	2

with parameters

$v = 6$, $t = 4$, $k = 5$, N= 24, $\lambda_1 = 4$, $\lambda_2 = 4$, $\lambda_3 = 2$, $\mu_1 = 16$, $\mu_2 = 16 = \mu_3$, $\nu_1 = 16$, $\nu_2 = 8 = \nu_3$.

$$E_d = 0.91,\ E_r = 0.872$$

4.3.5.2 PBRTED using *v* treatments in (*k + 1) plots**

In this sub sections we obtain PBRTED after deleting the last row of the above Latin square. Hence the parameters of PBRTED are

$v = v^*$, $N = vt$, $t = r^*$, $k = k^*$, $\lambda_1 = \lambda_1^*$, $\lambda_2 = \lambda_2^* = \lambda_3$, $\mu_1 = k\lambda_1^*$, $\mu_2 = k^*\lambda_2^* = \mu_3$, $\nu_1 = (k-2)\lambda_1^*$, $\nu_2 = (k-2)\lambda_2^* = \nu_3$

Example 4.3.5.2 Consider the blocks of group divisible design $v^* = 6$, $r^* = 4$, $k^* = 4$, $b^* = 6$, $\lambda_1^* = 4$, $\lambda_2^* = 2$, $m = 2$, $n = 3$,

1	2	4	6
2	3	5	1
3	4	6	2
4	5	1	3
5	6	2	4
6	1	3	5

We first obtain the 4×4 Latin squares for each blocks.

1	2	4	6	2	3	5	1	3	4	6	2	4	5	1	3	5	6	2	4	6	1	3	5
2	4	6	1	3	5	1	2	4	6	2	3	5	1	3	4	6	2	4	5	1	3	5	6
6	1	2	4	1	2	3	5	2	3	4	6	3	4	5	1	4	5	6	2	5	6	1	3
4	6	1	2	5	1	2	3	6	2	3	4	1	3	4	5	2	4	5	6	3	5	6	1

Deleting the last row of the above design we get PBRTED

1	2	4	6	2	3	5	1	3	4	6	2	4	5	1	3	5	6	2	4	6	1	3	5
2	4	6	1	3	5	1	2	4	6	2	3	5	1	3	4	6	2	4	5	1	3	5	6
6	1	2	4	1	2	3	5	2	3	4	6	3	4	5	1	4	5	6	2	5	6	1	3

with parameters

$v = 6$, $t = 4$, $k = 4$, $N= 24$, $\lambda_1 =3$, $\lambda_2 = 2, = \lambda_3$, $\mu_1 = 12$, $\mu_2 = 8 = \mu_3$, $v_1 = 6$, $v_2 = 4 = v_3$.

$$E_d = 0.821,\ E_r = 0.775$$

4.3.6 PBRTED using GD design where *k** is odd

In this section we consider the blocks of group divisible design with parameters v^*, r^*, k^*, b^*, λ_1^*, λ_2^*, m, n where k^* is even. Obtain the $k^* \times k^*$ Latin square for each blocks using MOLS and put all the Latin squares side by side. Now, delete the last row of the resulting design. Hence we get PBRTED with parameters

$v = v^*$, $t = (k\text{-}1)r^*$, $k = k^*$, $N = vt$, $\lambda_1 =2\ \lambda_1^*$, $\lambda_2 = (k\text{-}1)\lambda_2^* = \lambda_3$, $\mu_1 = 2k\lambda_1^*$, $\mu_2 = k(k\text{-}1)\lambda_2^* = \mu_3$, $v_1 = (k\text{-}1)\ \lambda_1^*$, $v_2 =(k\text{-}1)(k\text{-}2)\lambda_2^* = v_3$

Example 4.3.6 Consider the blocks of group divisible design $v^* = 6$, $r^* = 6$, $k^* = 3$, $b^* = 12$, $\lambda_1^* = 3$, $\lambda_2^* = 2$, $m = 2$, $n = 2$,

1	2	3
1	5	6
2	4	6
3	4	5
1	2	3
1	5	6
2	4	6
3	4	5

We first obtain the 3×3 Latin squares for each blocks using MOLS.

1	2	3	1	2	3	1	5	6	1	5	6	2	4	6	2	4	6	3	4	5	3	4	5
2	3	1	3	1	2	5	6	1	6	1	5	4	6	2	6	2	4	4	5	3	5	3	4
3	1	2	2	3	1	6	1	5	5	6	1	6	2	4	4	6	2	5	3	4	4	5	3

1	2	3	1	2	3	1	5	6	1	5	6	2	4	6	2	4	6	3	4	5	3	4	5
2	3	1	3	1	2	5	6	1	6	1	5	4	6	2	6	2	4	4	5	3	5	3	4
3	1	2	2	3	1	6	1	5	5	6	1	6	2	4	4	6	2	5	3	4	4	5	3

Deleting the last row from the above design

1	2	3	1	2	3	1	5	6	1	5	6	2	4	6	2	4	6	3	4	5	3	4	5
2	3	1	3	1	2	5	6	1	6	1	5	4	6	2	6	2	4	4	5	3	5	3	4

1	2	3	1	2	3	1	5	6	1	5	6	2	4	6	2	4	6	3	4	5	3	4	5
2	3	1	3	1	2	5	6	1	6	1	5	4	6	2	6	2	4	4	5	3	5	3	4

We get, PBRTED with parameters

$v = 6$, $t = 8$, $k = 3$, $N= 48$, $\lambda_1 =0$, $\lambda_2 = 4, = \lambda_3$, $\mu_1 = 0$, $\mu_2 = 12 = \mu_3$, $\nu_1 = 0$, $\nu_2 = 4 = \nu_3$.

$$E_d = 0.655,\ E_r = 0.6$$