Chapter 3

Construction of Three-Associate-Class PBIB Designs

3.1 INTRODUCTION

Designs with more than two-associate classes have also been studied quite extensively in the literature. Vartak (1955) introduced three-associate-class association scheme which is known as a rectangular association scheme. Vartak (1955) also discussed the concept and the construction of three-associate-class PBIB design. Let there be $v=mn$ symbols arranged in a rectangle of m rows and n columns. With respect to each symbol, the first associates are the other (n-1) symbols of the same row, the second associates are the other (m-1) symbols of the same column, and the remaining (m-1)(n-1) symbols are third associates. For further properties of the rectangular association scheme we refer to Shah (1964) and Vartak (1955, 1959) and Raghavarao (1970). Another useful three-associate-class association scheme is the cubic association scheme introduced by Raghavarao and Chandrasekhararao (1964). John (1966) generalized the two-associate-class triangular association scheme to three associate classes. Bose and Laskar (1967) independently gave the construction of three-associate-classes equivalent to that of John (1966). They consider it as a generalization of the triangular strongly regular graph.

Rectangular PBIB designs have been studied by Bhagwandas and Kageyama (1985), Suen (1989), Sinha (1991), Sinha *et.al.* (1993, 1996, 1999), Kageyama and Miao (1995), Sinha *et.al.* (2002b) and so on. The rectangular designs are useful as factorial experiments, having balance as well as orthogonality (Gupta and Mukerjee, 1989). In addition, if λ_3 is bigger then λ_1 and λ_2, the loss of information on the main effects becomes small (Suen, 1989), when these designs are used as an $m \times n$ complete confounded factorial experiments.

Kageyama and Sinha (2003) introduced some new patterned methods of constructions of rectangular designs. Singh *et.al* (2011) also gave some methods to construct rectangular design.

In this chapter, we have discussed five approaches to construct three-associate-class association scheme called as rectangular PBIB designs. The method of construction of rectangular PBIB design is shown in section 3.2.1 and 3.2.2. In section 3.2.1 we have two approaches, in first approach we have used two incidence matrices N_1 and N_2 which are obtained from difference sets of BIB design while in second approach we are using three incidence matrices which are also obtained from difference sets of BIB design to obtain three-associate-class PBIB design. And in section 3.2.2 the incidence matrices are obtained from symmetrical BIB design. Finally at the end of this chapter a table has been provided based on these five approaches and the cases are considered where $r, k \leq 20$.

3.2 METHODS OF CONSTRUCTION

In this section we carried out five approaches based on symmetrical BIB designs and difference sets of BIB designs to construct rectangular PBIB design.

3.2.1 Rectangular PBIB design using difference sets of BIB designs.

In this section we discussed three methods to constructed three-associate-class PBIB design using two incidence matrices which are obtained from difference sets of BIB designs.

Theorem 3.2.1.1: *If (4t-1) is a prime or prime power, where t=1, 2, 3…, then there always exists a three associate class partially balanced incomplete block design having the parameter* $v = 8t$, $b = 2(4t+1)$, $r = 4t+1$, $k = 4t$, $\lambda_1 = 0$, $\lambda_2 = 2t$, $\lambda_3 = 2t+1$, $m = 4t$, $n = 2$, *using the incidence matrix* $N = \begin{bmatrix} N_1^{*\prime} & N_2^{*\prime} \\ N_2^{*\prime} & N_1^{*\prime} \end{bmatrix}$ *where* N_1^* *and* N_2^* *are some incidence matrices of two incomplete block designs.*

Proof: Let $\alpha_0 = 0$, $\alpha_1 = x^0 = 1$, $\alpha_2 = x$, $\alpha_3 = x^2$, $\alpha_4 = x^3 \ldots\ldots$, $\alpha_{4t+1} = x^{4t}$ be the elements of GF(4t-1) where x is the primitive element. By considering the initial blocks $(0, x^0, x^2, x^4, x^6 \ldots, x^{4(t-1)})$ and $(\infty, x, x^3, x^5, x^7 \ldots, x^{4t-3})$ from a difference set for BIB design with parameters $v' = 4t$, $b' = 2(4t-1)$, $r' = 4t-1$, $k' = 2t$ and $\lambda' = 2t-$. *W*ith the help of these two initial blocks two incidence matrices (say) N_1 and N_2 are obtained.

Now, Considering following two new incidence matrices N_1^* and N_2^* of dimension $(v'+1) \times (\frac{b'}{2}+1)$.

$$N_1^* = \begin{bmatrix} 1 & E_{1\frac{b'}{2}} \\ E_{v'1} & N_1 \end{bmatrix} \text{ and } N_2^* = \begin{bmatrix} 0 & 0_{1\frac{b'}{2}} \\ 0_{v'1} & N_2 \end{bmatrix}$$

Where, $E_{1b'/2}$ & $E_{v'1}$ are unitary matrices whose all elements are ones and $0_{1b'/2}$ & $0_{v'1}$ are null matrices. By using N_1^* and N_2^* a new incidence matrix N is obtained. Now

$$NN' = \begin{bmatrix} N_1^*{}' & N_2^*{}' \\ N_2^*{}' & N_1^*{}' \end{bmatrix} \begin{bmatrix} N_1^* & N_2^* \\ N_2^* & N_1^* \end{bmatrix}$$

$$\begin{bmatrix} N_1^*{}'N_1^* + N_2^*{}'N_2^* & N_1^*{}'N_2^* + N_2^*{}'N_1^* \\ N_2^*{}'N_1^* + N_1^*{}'N_2^* & N_2^*{}'N_2^* + N_1^*{}'N_1^* \end{bmatrix} \tag{3.1}$$

Solving for each elements of the matrix

$$N_1^*{}'N_1^* + N_2^*{}'N_2^*$$

$$= \begin{bmatrix} 1 & E_{1v'} \\ E_{\frac{b'}{2}1} & N_1' \end{bmatrix} \begin{bmatrix} 1 & E_{1\frac{b'}{2}} \\ E_{v'1} & N_1 \end{bmatrix} + \begin{bmatrix} 0 & 0_{1v'} \\ 0_{\frac{b'}{2}1} & N_2' \end{bmatrix} \begin{bmatrix} 0 & 0_{1\frac{b'}{2}} \\ 0_{v'1} & N_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+v'E_{11} & E_{1\frac{b'}{2}} + k'E_{1\frac{b'}{2}} \\ E_{\frac{b'}{2}1} + k'E_{\frac{b'}{2}1} & E_{\frac{b'}{2}\frac{b'}{2}} + N_1'N_1 + N_2'N_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+v'E_{11} & (k'+1)E_{1\frac{b'}{2}} \\ (k'+1)E_{\frac{b'}{2}1} & (2t+1)E_{\frac{b'}{2}} + 2tI_{\frac{b'}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1+4tE_{11} & (2t+1)E_{1\frac{b'}{2}} \\ (2t+1)E_{\frac{b'}{2}1} & (2t+1)E_{\frac{b'}{2}} + 2tI_{\frac{b'}{2}} \end{bmatrix}$$

Similarly,

$$N_1^*{}'N_2^* + N_2^*{}'N_1^*$$

$$=\begin{bmatrix} 0 & 0_{1v'} \\ 0_{\frac{b'}{2}1} & N_2' \end{bmatrix}\begin{bmatrix} 1 & E_{1\frac{b'}{2}} \\ E_{v'1} & N_1 \end{bmatrix}+\begin{bmatrix} 1 & E_{1v'} \\ E_{\frac{b'}{2}1} & N_1' \end{bmatrix}\begin{bmatrix} 0 & 0_{1\frac{b'}{2}} \\ 0_{v'1} & N_2 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & E_{1v'}N_2 \\ N_2'E_{v'1} & N_2'N_1+N_1'N_2 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 2tE_{1\frac{b'}{2}} \\ 2tE_{\frac{b'}{2}1} & t(E_{\frac{b'}{2}}-I_{\frac{b'}{2}})+t(E_{\frac{b'}{2}}-I_{\frac{b'}{2}}) \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 2tE_{1\frac{b'}{2}} \\ 2tE_{\frac{b'}{2}1} & 2t(E_{\frac{b'}{2}}-I_{\frac{b'}{2}}) \end{bmatrix}$$

Substituting the values in (3.1) we get

$$NN'=\begin{bmatrix} \begin{pmatrix} (1+4t)E_{11} & (2t+1)E_{1\frac{b'}{2}} \\ (2t+1)E_{\frac{b'}{2}1} & (2t)I_{\frac{b'}{2}}+(2t+1)E_{\frac{b'}{2}} \end{pmatrix} & \begin{pmatrix} 0 & (2t)E_{1\frac{b'}{2}} \\ (2t)E_{\frac{b'}{2}1} & (2t)(E_{\frac{b'}{2}}-I_{\frac{b'}{2}}) \end{pmatrix} \\ \begin{pmatrix} 0 & (2t)E_{1\frac{b'}{2}} \\ (2t)E_{\frac{b'}{2}1} & (2t)(E_{\frac{b'}{2}}-I_{\frac{b'}{2}}) \end{pmatrix} & \begin{pmatrix} (1+4t)E_{11} & (2t+1)E_{1\frac{b'}{2}} \\ (2t+1)E_{\frac{b'}{2}1} & (2t)I_{\frac{b'}{2}}+(2t+1)E_{\frac{b'}{2}} \end{pmatrix} \end{bmatrix}$$

Here the diagonal elements are (4t+1) and off diagonal elements are 0, (2t) and (2t+1) and after satisfying all the primary and secondary parameters the design obtained is a rectangular design with parameters *v= 8t, b= 2(4t+1),* *r = 4t+1, k=4t,* $\lambda_1 = 0$, $\lambda_2 = 2t$, $\lambda_3 = 2t+1$, $n_1 = 1$, $n_2 = 4t-1$, $n_3 = 4t-1$ *m = 4t,* *n = 2.*

Example 3.2.1.1 If t=2, then a three associate class partially balanced incomplete block design with parameters *v* = 16, *b*= 18, *r* = 9, *k*= 8, $\lambda_1 = 0$, $\lambda_2 = 5$, $\lambda_3 = 4$, m = 8, n = 2 are obtained by considering initial sets $(0, x^0, x^2, x^4)$ and (∞, x, x^3, x^5) form a difference set of BIB designs.

Since x = 3 is a primitive root of GF (7) hence the initial sets are [0 1 2 4] and [∞ 3 6 5], give the design as follows

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Then using *Theorem 3.2.1.1* The blocks of three associate class PBIB designs are:

1	1	1	1	9	4	5	2	1	1	1	1	3	2	2	3	1	2
2	2	2	3	10	5	6	6	2	3	4	2	7	4	3	4	10	3
3	3	3	4	11	7	8	7	4	5	6	5	8	8	5	6	11	4
4	6	4	5	12	9	9	9	5	6	7	7	9	9	9	9	12	5
5	8	7	8	13	10	10	11	6	7	8	8	10	11	12	10	13	6
6	12	13	10	14	11	11	12	11	10	10	11	12	13	14	13	14	7
7	13	14	14	15	14	12	13	15	12	11	12	13	14	15	15	15	8
8	15	16	15	16	16	15	16	16	16	13	14	14	15	16	16	16	9

Since $v=mn$ where, $m=8$ (rows) and $n=2$ (columns) so the existing design is a PBIB design. Here the diagonal elements are 7 and off diagonal elements are 0, 5 and 4 hence the design is a rectangular three associate PBIB design.

In theorem 3.2.1.2 we obtain three-associate-class PBIB design using difference sets of BIB designs.

Theorem 3.2.1.2: *If (4t-1) is a prime or prime power where t is any positive integer then there always exists a three associate class PBIB design with parameters v= 8t, b=2(4t-1),*

r = 4t-1, k=4t, λ_1 = 0, λ_2 = 2t-1, λ_3=2t, m = 4t, n = 2, using the incidence matrix $N = \begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix}$ *where N_1 and N_2 are some other incidence matrices of two incomplete block designs.*

Proof: Let α_0=0, α_1=x^0=1, α_2=x, α_3=x^2, α_4=x^3…….. , α_{4t+1}=x^{4t} be the elements of GF(4t-1) where x is the primitive root. Considering the initial blocks $(0,x^0,x^2,x^4,x^6\ldots..,x^{4(t-1)})$ and $(\infty,x,x^3,x^5,x^7\ldots..,x^{4t-3})$ from a difference set for BIB design with parameters v' = $4t$, b'= 2(4t-1), r' = 4t-1, k'= 2t and λ' = 2t-1 with the help of two initial blocks we have obtained two incidence matrices (say) N_1 and N_2 . With the help of two incidence matrices we obtained a new incidence matrix N, where

$N = \begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix}$ resultant design N is a PBIB design and $N' = \begin{bmatrix} N_1' & N_2' \\ N_2' & N_1' \end{bmatrix}$ so,

$$NN' = \begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix}\begin{bmatrix} N_1' & N_2' \\ N_2' & N_1' \end{bmatrix}$$

$$NN' = \begin{bmatrix} N_1N_1' + N_2N_2' & N_1N_2' + N_2N_1' \\ N_2N_1' + N_1N_2' & N_2N_2' + N_1N_1' \end{bmatrix} \qquad (3.2)$$

Solving for each elements of the matrix

$N_1N_1' + N_2N_2'$

$$= \begin{pmatrix} t(E_{\frac{b'}{2}} + I_{\frac{b'}{2}}) & 0_{1\frac{b'}{2}} \\ 0_{\frac{b'}{2}1} & 0 \end{pmatrix} + \begin{pmatrix} (t-1)E_{\frac{b'}{2}} + tI_{\frac{b'}{2}} & (2t-1)E_{1\frac{b'}{2}} \\ (2t-1)E_{\frac{b'}{2}1} & (4t-1)E_{11} \end{pmatrix}$$

$$=\begin{pmatrix} (2t-1)E_{\frac{b'}{2}}+2tI_{\frac{b'}{2}} & (2t-1)E_{1\frac{b'}{2}} \\ (2t-1)E_{\frac{b'}{2}1} & (4t-1)E_{11} \end{pmatrix}$$

Similarly,

$$N_1N_2'+N_2N_1'$$

$$=\begin{pmatrix} t(E_{\frac{b'}{2}}-I_{\frac{b'}{2}}) & (2t)E_{1\frac{b'}{2}} \\ 0_{\frac{b'}{2}1} & 0 \end{pmatrix}+\begin{pmatrix} t(E_{\frac{b'}{2}}-I_{\frac{b'}{2}}) & 0_{1\frac{b'}{2}} \\ (2t)E_{\frac{b'}{2}1} & 0 \end{pmatrix}$$

$$=\begin{pmatrix} (2t)(E_{\frac{b'}{2}}-I_{\frac{b'}{2}}) & (2t)E_{1\frac{b'}{2}} \\ (2t)E_{\frac{b'}{2}1} & 0 \end{pmatrix}$$

Substituting the values back to *NN’* we get,

$$NN'=\begin{bmatrix} \begin{pmatrix} (2t-1)E_{\frac{b'}{2}}+2tI_{\frac{b'}{2}} & (2t-1)E_{1\frac{b'}{2}} \\ (2t-1)E_{\frac{b'}{2}1} & (4t-1)E_{11} \end{pmatrix} & \begin{pmatrix} (2t)(E_{\frac{b'}{2}}-I_{\frac{b'}{2}}) & (2t)E_{1\frac{b'}{2}} \\ (2t)E_{\frac{b'}{2}1} & 0 \end{pmatrix} \\ \begin{pmatrix} (2t)(E_{\frac{b'}{2}}-I_{\frac{b'}{2}}) & (2t)E_{1\frac{b'}{2}} \\ (2t)E_{\frac{b'}{2}1} & 0 \end{pmatrix} & \begin{pmatrix} (2t-1)E_{\frac{b'}{2}}+2tI_{\frac{b'}{2}} & (2t-1)E_{1\frac{b'}{2}} \\ (2t-1)E_{\frac{b'}{2}1} & (4t-1)E_{11} \end{pmatrix} \end{bmatrix}_{8t\times 8t}$$

Here the diagonal elements are (4t-1) and off diagonal elements are 0, (2t) and (2t-1) and satisfies all the primary and secondary parameters so, the design is a rectangular design with parameters $v= 8t$, $b=2(4t-1)$, $r = 4t-1$, $k=4t$, $\lambda_1 = 0$, $\lambda_2 = 2t-1$, $\lambda_3=2t$, $m = 4t$, $n = 2$.

Example 3.2.1.2 If t=2, then a three associate rectangular partially balanced incomplete block design with parameters $v = 16$, $b= 14$, $r = 7$, $k= 8$, $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 =4$, m = 8, n = 2 is obtained by considering 2 initial sets ($0, x^0, x^2, x^4$) and (∞, x, x^3, x^5) from a difference set of BIB designs.

Since x = 3 is a primitive root of GF (7) hence the initial sets are [0 1 2 4] and [∞ 3 6 5], hence design is as follows:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Then using *Theorem 3.2.1.2* the blocks of three associate class PBIB designs are:

1	1	2	3	1	2	1	3	4	1	1	2	1	2
2	2	3	4	4	5	3	5	6	5	2	3	3	4
4	3	4	5	5	6	6	6	7	7	6	7	4	5
7	5	6	7	6	7	7	8	8	8	8	8	8	8
11	12	9	9	10	9	10	9	9	9	11	9	10	9
13	14	13	10	11	11	12	10	10	10	12	12	13	11
14	15	15	14	15	12	13	12	11	11	13	13	14	14
16	16	16	16	16	16	16	15	13	13	15	14	15	15

Since v=mn where, m= 8 (rows) and n=2 (columns) so the existing design is a PBIB design. Here the diagonal elements are 7 and off diagonal elements are 0, 3 and 4 hence the design is a rectangular three associate PBIB design.

Next in theorem 3.2.1.3 we obtain three-associate-class PBIB design using difference sets of BIB designs.

Theorem 3.2.1.3: *If (4t+1) is prime or prime power where t is any positive integer then there always exist a three associate class partially balanced incomplete block design having parameters v= 2(4t+1) = b, r = 4t =k,* $\lambda_1 = 0$, $\lambda_2 = 2t-1$, $\lambda_3=2t$, *m = 4t+1, n = 2, using incidence matrix* $N = \begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix}$ *where* N_1 *and* N_2 *are some other incidence matrices of two incomplete block designs.*

Proof: Let $\alpha_0=0$, $\alpha_1=x^0=1$, $\alpha_2=x$, $\alpha_3=x^2$, $\alpha_4=x^3$…….. , $\alpha_{4t+1}=x^{4t}$ be the elements of GF(4t+1) where x is the primitive element. Considering the two initial blocks $(x^0,x^2,x^4,x^6\ldots\ldots,x^{4t})$ and $(x,x^3,x^5,x^7\ldots\ldots,x^{4t-1})$ form a difference set for BIB design with parameters $v'=4t+1$, $b'=2(4t+1)$, $r'=4t$, $k'=2t$, $\lambda'=2t-1$ with the help of two initial blocks we have obtained two incidence matrices (say) N_1 and N_2. Considering this two incidence matrix we obtain a new incidence matrix N.

Now,

$$NN' = \begin{bmatrix} N_1N_1'+N_2N_2' & N_1N_2'+N_2N_1' \\ N_2N_1'+N_1N_2' & N_2N_2'+N_1N_1' \end{bmatrix}$$

$$NN' = \begin{bmatrix} (2t-1)E_{v'}+(2t+1)I_{v'} & 2t(E_{v'}-I_{v'}) \\ 2t(E_{v'}-I_{v'}) & (2t-1)E_{v'}+(2t+1)I_{v'} \end{bmatrix}_{2(4t+1)\times 2(4t+1)}$$

Here it is obvious that the diagonal elements are 2(2t+1) and off diagonal elements are 0, (2t-1) and (2t). Further verifying all the primary and secondary parameter of PBIB design we found the design to be a rectangular design with parameters *v= 2(4t+1)- b, r =4t=k,* $\lambda_1 = 0, \lambda_2 = 2t-1, \lambda_3 = 2t, m = 4t+1, n = 2.$

Example 3.2.1.3 If t=1, then a three associate class partially balanced incomplete block design with parameters $v = 10 = b$, $r = 4 = k$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$, m = 5, n = 2 is obtained by considering two initial sets (x^0, x^2, x^4) and (x, x^3, x^5) .

Since x = 2 is a primitive root of GF (5) hence the initial sets are [1 4] and [2 3], hence design is as follows:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \end{bmatrix}$$

Then using *Theorem 3.2.1.3* the blocks of three associate class PBIB designs are:

1	2	1	2	3	2	3	4	1	1
4	5	3	4	5	3	4	5	5	2
7	8	9	6	6	6	7	6	7	8
8	9	10	10	7	9	10	8	9	10

Since v=mn where, m= 5 (rows) and n=2 (columns) so the existing design is a PBIB design. Here the diagonal elements are 4 and off diagonal elements are 0, 1 and 2 and after satisfying all the primary and secondary parameters of PBIB designs we obtain a rectangular design with parameters $v = 10 = b$, $r = 4 = k$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$, $m = 5$, $n = 2$.

In Theorem 3.2.1.4 we have discussed a three-associate-class PBIB design using three incomplete block design to obtain the desired result.

Theorem 3.2.1.4: *If (3t+1) is prime or prime power where t is any positive integer then we construct a three associate class partially balanced incomplete block design having parameters* $v = 3(3t+1) = b$, $r = 3t = k$, $\lambda_1 = 0$, $\lambda_2 = t-1$, $\lambda_3 = t$, $m = 3t+1$, $n = 3$, *using incidence matrix* $N = \begin{bmatrix} N_1 & N_3 & N_2 \\ N_3 & N_2 & N_1 \\ N_2 & N_1 & N_3 \end{bmatrix}$ *where* N_1, N_2 *and* N_3 *are some incidence matrices of incomplete block designs.*

Proof: Let $\alpha_0=0$, $\alpha_1=x^0=1$, $\alpha_2=x$, $\alpha_3=x^2$, $\alpha_4=x^3$……. , $\alpha_{3t+1}=x^{3t-1}$ be the elements of GF(3t+1) where x is the primitive element. Considering the initial blocks $(x^0,x^3,x^6,x^9\ldots..,x^{3t-3})$, $(x,x^4,x^7\ldots..,x^{3t-2})$ and $(x^2,x^5,x^8\ldots..,x^{3t-1})$ form a difference set for BIB design with parameters $v'=3t+1$, $b'= 3(3t+1)$, $r'=3t$, $k' = t$, $\lambda' = t-1$ with the help of three initial blocks we have obtained three incidence matrices (say) N_1, N_2 and of dimension $(v'+1)\times(\frac{b'}{3}+1)$

Now,

$$NN' = \begin{bmatrix} N_1 & N_3 & N_2 \\ N_3 & N_2 & N_1 \\ N_2 & N_1 & N_3 \end{bmatrix}\begin{bmatrix} N_1' & N_3' & N_2' \\ N_3' & N_2' & N_1' \\ N_2' & N_1' & N_3' \end{bmatrix}$$

$$
= \begin{bmatrix} N_1N_1^{'} + N_2N_2^{'} + N_3N_3^{'} & N_1N_3^{'} + N_2N_2^{'} + N_3N_1^{'} & N_3N_1^{'} + N_2N_3^{'} + N_1N_2^{'} \\ N_3N_1^{'} + N_2N_3^{'} + N_1N_2^{'} & N_1N_1^{'} + N_2N_2^{'} + N_3N_3^{'} & N_1N_3^{'} + N_2N_2^{'} + N_3N_1^{'} \\ N_1N_3^{'} + N_2N_2^{'} + N_3N_1^{'} & N_3N_1^{'} + N_2N_3^{'} + N_1N_2^{'} & N_1N_1^{'} + N_2N_2^{'} + N_3N_3^{'} \end{bmatrix}
$$

Solving for each elements of the matrix

$$
N_1N_1^{'} + N_2N_2^{'} + N_3N_3^{'} = \lambda^{'} E_{v'v'} + (2t+1)I_{v'} = (t-1)E_{v'v'} + (2t+1)I_{v'}
$$

next,

$$
N_1N_3^{'} + N_2N_2^{'} + N_3N_1^{'} = t(E_{v'v'} - I_{v'})
$$

And finally

$$
N_3N_1^{'} + N_2N_3^{'} + N_1N_2^{'} = t(E_{v'v'} - I_{v'})
$$

Hence,

$$
NN' = \begin{bmatrix} (t-1)E_{v'v'} + (2t+1)I_{v'} & t(E_{v'v'} - I_{v'}) & t(E_{v'v'} - I_{v'}) \\ t(E_{v'v'} - I_{v'}) & (t-1)E_{v'v'} + (2t+1)I_{v'} & t(E_{v'v'} - I_{v'}) \\ t(E_{v'v'} - I_{v'}) & t(E_{v'v'} - I_{v'}) & (t-1)E_{v'v'} + (2t+1)I_{v'\,v'} \end{bmatrix}_{3(3t+1)\times 3(3t+1)}
$$

Here the diagonal elements are $3t$ and off diagonal elements are 0, (t-1) and (t). Further verifying all the primary and secondary parameter of PBIB design we found the design obtain is a rectangular PBIB design with parameters $v = 3(3t+1) = b,\ r = 3t = k,\ \lambda_1 = 0,\ \lambda_2 = t-1,\ \lambda_3 = t,\ ,\ n_1 = 2,\ n_2 = 3t,\ n_3 = 6t,\ m = 3t+1,\ n = 3.$

Example 3.2.1.4 If t=2, then a three associate rectangular partially balanced incomplete block design with parameters v = 21= b, r = 6 =k, λ_1 = 0, λ_2 =1, λ_3 = 2, m = 7, n = 3 is obtained by considering three initial sets (x^0,x^3), (x,x^{4}) and (x^2,x^5)

Since x = 3 is a primitive root of GF (7) hence the initial sets are [1 6], [3 4] and [2 5], hence design is as follows:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{bmatrix} \text{and} \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Then using *Theorem 3.2.1.4* the blocks of three associate class PBIB designs are:

1	2	1	2	2	3	4	1	3	4	5	6	3	4	5	2	3	1	1	1	2
6	7	3	4	5	6	7	5	4	5	6	7	5	6	7	6	7	4	7	2	3
9	10	11	8	10	11	12	13	8	9	8	9	9	10	8	8	8	9	10	11	12
12	13	14	12	11	12	13	14	13	14	10	11	13	14	11	14	9	10	12	13	14
17	18	19	20	15	16	15	16	16	17	18	15	15	15	16	17	18	19	16	17	15
18	19	20	21	20	21	17	18	19	20	21	19	21	16	17	19	20	21	20	21	18

Since v=mn where, m= 7 (rows) and n=3 (columns) so the existing design is a PBIB design. Here the diagonal elements are 4 and off diagonal elements are 0, 1 and 2 hence the design is a rectangular three associate PBIB design. Further verifying all the primary and secondary parameter of PBIB design we found the design to be a rectangular design with parameters v = 21= b, r = 6 =k, λ_1 = 0, λ_2 =1, λ_3 = 2, m = 7, n = 3.

3.2.2 Rectangular PBIB design using SBIB design.

In this section a three-associate-class PBIB designs is constructed by using two incidence matrices which are obtained from SBIB designs.

Theorem 3.2.2.1: *If (4t+3) is prime or prime power where t is any positive integer then we construct a three associate class PBIB design with parameters* $v = 2(4t+3) = b$, $r = 2(2t+1) = k$, $\lambda_1 = 0$, $\lambda_2 = 2t$, $\lambda_3 = 2t+1$, $m = 4t+3$, $n = 2$, *using incidence matrix* $N = \begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix}$ *where* N_1 *and* N_2 *are some other incidence matrices of two incomplete block designs.*

Proof: Let $\alpha_0=0$, $\alpha_1=x^0=1$, $\alpha_2=x$, $\alpha_3=x^2$, $\alpha_4=x^3$…….. , $\alpha_{4t+2}=x^{4t+1}$ be the elements of GF(4t+3) by considering the initial blocks $(x^0,x^2,x^4,x^6\ldots ,x^{4t})$ and $(x,x^3,x^5,x^7\ldots ,x^{4t+1})$ we obtain two SBIB designs with parameters $v' = 4t+3 = b'$, $r' = 2t+1 = k'$, $\lambda' = t$. By the help of two SBIB we have obtained two incidence matrices (say) N_1 and N_2 of order $v' \times b'$. Now, considering a new incidence matrix

Now,

$$NN' = \begin{bmatrix} N_1N_1' + N_2N_2' & N_1N_2' + N_2N_1' \\ N_2N_1' + N_1N_2' & N_2N_2' + N_1N_1' \end{bmatrix}$$

$$NN' = \begin{bmatrix} 2(2tE_{v'} + I_{v'}) & (2t+1)(E_{v'} - I_{v'}) \\ (2t+1)(E_{v'} - I_{v'}) & 2(2tE_{v'} + I_{v'}) \end{bmatrix}_{2(4t+3)\times 2(2t+1)}$$

Here the diagonal elements are 2(2t+1) and off diagonal elements are 0, (2t) and (2t+1). Further verifying all the primary and secondary parameter of PBIB design we found the design obtain is a rectangular PBIB design with parameters $v = 2(4t+3) = b$, $r = 2(2t+1) = k$, $\lambda_1 = 0$, $\lambda_2 = 2t$, $\lambda_3 = 2t+1$, $m = 4t+3$, $n = 2$.

Example 3.2.2.1 If t=1, then a three associate class partially balanced incomplete block design with parameters $v = 14 = b$, $r = 6 = k$, $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 3$, m = 7, n = 2 is obtained by considering 2 initial sets (x^0, x^2, x^4) and (x, x^3, x^5) .

Since x = 3 is a primitive root of GF (7) hence the initial sets are [1 2 4] and [3 6 5], hence design is as follows:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Then using *Theorem 3.2.2.1* The blocks of three associate class PBIB designs are:

1	2	3	4	3	4	1	1	1	2	1	2	1	2
2	3	4	5	5	6	5	2	5	6	3	3	3	4
4	5	6	7	6	7	7	6	6	7	7	7	4	5
10	11	8	8	8	9	10	11	9	8	9	8	9	8
12	13	12	9	9	10	11	12	10	10	11	12	13	10
13	14	14	13	11	12	13	14	14	11	12	13	14	14

Since v=mn where, m= 7 (rows) and n=2 (columns) so the existing design is a PBIB design. Here the diagonal elements are 6 and off diagonal elements are 0, 2 and 3 hence the design is a three associate class PBIB design. Further verifying all the primary and secondary parameter of PBIB design we found the design to be a rectangular design with parameters $v = 14 = b$, $r = 6 = k$, $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 3$, m = 7, n = 2.

Table 3.1: Rectangular Designs with r, k ≤ 20

S. No.	v	b	r	k	λ_1	λ_2	λ_3	n_1	n_2	n_3	m	n	Remark
1	8	10	5	4	0	2	3	1	3	3	4	2	Theorem 3.2.1.1
2	8	6	3	4	0	1	2	1	3	3	4	2	Theorem 3.2.1.2
3	10	10	4	4	0	1	2	1	4	4	5	2	Theorem 3.2.1.3
4	14	14	6	6	0	2	3	1	7	7	7	2	Theorem 3.2.2.1
5	16	14	7	8	0	3	4	1	7	7	8	2	Theorem 3.2.1.2
6	16	18	9	8	0	4	5	1	7	7	8	2	Theorem 3.2.1.1
7	18	18	8	8	0	3	4	1	8	8	9	2	Theorem 3.2.1.3
8	22	22	10	10	0	4	5	1	10	10	11	2	Theorem 3.2.2.1
9	21	21	6	6	0	1	2	2	6	12	7	3	Theorem 3.2.1.4
10	24	22	11	12	0	5	6	1	11	11	12	2	Theorem 3.2.1.2
11	24	26	13	12	0	6	7	1	11	11	12	2	Theorem 3.2.1.1
12	26	26	12	12	0	5	6	1	12	12	13	2	Theorem 3.2.1.3
13	34	34	16	16	0	7	8	1	16	16	17	2	Theorem 3.2.1.3
14	38	38	18	18	0	8	9	1	18	18	19	2	Theorem 3.2.2.1
15	39	39	12	12	0	3	4	2	12	24	13	3	Theorem 3.2.1.4
16	40	38	19	20	0	9	10	1	19	19	20	2	Theorem 3.2.1.2
17	48	48	15	15	0	4	5	2	15	30	16	3	Theorem 3.2.1.4
18	57	57	18	18	0	5	6	2	18	36	19	3	Theorem 3.2.1.4