Chapter 2

Construction of Group Divisible Designs

2.1 INTRODUCTION

The concept of incomplete block designs was introduced by Yates (1936). A special class of incomplete block designs called a balanced incomplete block design (BIBD) was constructed by Bose (1939) and Hinani (1961). The condition of balancing was relaxed due to non availability of BIBD for all parametric combination. Bose and Nair (1939) introduced a class of binary, equireplicate and proper designs, which are called Partially Balanced Incomplete Block (PBIB) designs. In these designs the variance of every estimated elementary contrast among treatment effect is not the same. To make a mathematically tractable statistical analysis for these designs and study them systematically, we need the concept of association scheme on v symbols. Nair and Rao (1942) modified the original definition of PBIB designs. The analysis of these designs was developed by Bose and Shimamato (1952). The two- associate class PBIB design were classified by Bose and Shimamato (1952) into the following types depending on the association scheme:

(1) Group Divisible (GD);

(2) Simple (S.I);

(3) Triangular (T);

(4) Latin Square Type (L_i); and

(5) Cyclic Designs.

A group divisible design is an arrangement of v=mn treatments into b blocks such that each block contains k(<v) distinct treatments which are partitioned into m($\geq$ 2) groups of n($\geq$ 2) treatments each, further any two distinct treatments occurring together in λ_1 blocks if they belong to the same group, and in λ_2 blocks if they belong to different groups. Simple PBIB designs have either $\lambda_1 = 0$, and $\lambda_2 \neq 0$, or $\lambda_1 \neq 0$, and $\lambda_2 = 0$. The known simple designs are partial geometric designs as defined by Bose (1963) or replications of partial geometric designs.

The two-associate-class association schemes that are not covered by Bose and Shimamato classification are of two categories. The first category consists of pseudo-triangular, pseudo-Latin-square type and pseudo-cyclic with any combinatorial structure. The other category are neither simple, nor their parameters satisfy any previously structure. The NL_j family of designs with parameters given for L_i association scheme, where s and i are replaced by $-t$ and $-j$ was studied by Mesner (1967). Other examples for v = 15 was given by Clathworthy (1973) and for v = 50 was given by Hoffman and Singleton (1960). Bose and Connor (1952) subdivided the GD designs into

(i) Singular GD designs with r - λ_1 = 0, rk-vλ_2 > 0.

(ii) Semi-regular GD designs with r - λ_1 > 0, rk-vλ_2 = 0.

(iii) Regular GD designs with r - λ_1 > 0, rk-vλ_2 > 0.

These three classes of GD design posses' different combinatorial properties.

Later Bose (1963) introduced another class of PBIB designs with two associate classes and named it partial geometry. Some of PBIB designs with two associate classes which are not covered in above mentioned designs are called miscellaneous PBIB designs.

The construction of PBIB designs were also given by Bose and Connor (1952), Zoellner and Kempthorne (1954), Pearce (1970), Kegeyama (1972, 73), freeman (1976 a, b), John (1976), Patterson and Williams (1976), Williams, Patterson and John (1976, 77), Jarrett and Hall (1978) and Patterson, Williams and Hunter (1978). The updated tables of two-associate-class PBIB designs was prepared by Clatwothy (1973) and lists the parameters and plans for 124 singular GD designs, 110 semi-regular GD designs, 200 regular GD designs, 100 triangular designs, 146 Latin-square-type designs, 29 cyclic-type designs, 15 partial geometric designs, and 42 miscellaneous designs which do not fit into any of the previous categories.

However the partially balanced incomplete block designs do not have the property of balancing. It generally require less number of experimental units than the balanced incomplete block designs. Detailed literature on block designs are available in Fisher (1926), Kempthorne (1952), Federer (1955), Cochran and Cox (1957), John (1971), Raghavaro (1971), Das and Giri (1979), Dey (1986) and Nigam, Puri and Gupta (1984).

Seberry (1982) constructed some families of partially balanced incomplete block designs. Kageyama and Mohan (1984) had developed three methods of construction of two or three associate PBIB designs. Mohan and Kageyama (1987) constructed some series of pairwise balanced and partially balanced designs. Ghosh and Das (1993) developed the two way group divisible designs with partial balance for group comparisons. For this study examples were also carried out. Ghosh and Divecha (1995) constructed four new semi-regular GD designs which are obtained by using orthogonal main effect plan of size 50 in 11 factors.

Sinha, Dhar, Saha and Kageyama (2002) constructed balanced arrays of strength two using rectangular designs, group divisible designs, and nested balanced incomplete block designs. Some series of such arrays as well as orthogonal arrays were also presented with illustrations.

Satpati and Prasad (2004) proposed the construction and cataloguing of nested partially balanced incomplete block designs and provided some new methods of construction of two and three associate class nested partially balanced incomplete block designs. The methods were based on Latin-square association scheme and triangular association scheme. One method of constructing nested partially balanced incomplete block designs has also been given by incorporating a set of new treatments in place of each treatment in a nested partially balanced incomplete block design. Exhaustive catalogues of nested partially balanced incomplete block designs based on two and three class association schemes with $v \leq 30$ and $r \leq 15$ have also been prepared.

Here, in this chapter we have proposed the alternative methods for constructing the singular, semi regular and regular group divisible designs. In this chapter we gave three approaches to obtain GD design. Here in section *2.2.1* the group divisible designs discussed by Dey (1986) had chosen N_1 and N_2 by trail method but in this chapter a specified method is considered to construct the GD designs. In the section *2.2.2* we obtain a GD design from a resolvable BIB design and in the section *2.2.3* we make use of juxtaposition for obtaining GD design. Here, average efficiency factor can also calculated using

$$E = \frac{v(v-1)\lambda_2\{\lambda_1+(m-1)\}\lambda_2}{rk\{(m-1)\lambda_1+(mv-2m+1)\lambda_2\}}$$

2.2 METHODS OF CONSTRUCTION

Here we consider three methods to obtain the Group divisible designs.

2.2.1 Regular and Semi-regular GD design using SBIB design and difference set of BIB design.

Here in the Theorem 2.2.1.1 we obtained the regular group divisible design by using symmetrical balanced incomplete block designs.

Theorem 2.2.1.1: *If (4t+3) is prime or prime power, where t=1, 2, 3…., then there always exists a group divisible design with the parameters v= 8(t+1) = b, r = 4t+3 = k, λ_1 = 0, λ_2 = 2t+1, m = 4(t+1), n = 2, which is obtained from the incidence matrix N define as, N=* $\begin{bmatrix} N_1^* & N_2^* \\ N_2^* & N_1^* \end{bmatrix}$ *, where N_1^* and N_2^* are incidence matrices of another two incomplete block designs.*

Proof: Let $\alpha_0=0$, $\alpha_1=x^0=1$, $\alpha_2=x$, $\alpha_3=x^2$, $\alpha_4=x^3$…….. , $\alpha_{4t+2}=x^{4t+1}$ be the elements of GF(4t+3), where x is the primitive element of GF(4t+3). Consider the initial blocks $(x^0,x^2,x^4,x^6\ldots\ldots,x^{4t})$ and $(x,x^3,x^5,x^7\ldots\ldots,x^{4t+1})$. Now we obtain two SBIB designs with parameters $v'=4t+3=b'$, $r'=2t+1=k'$, $\lambda'=t$, by developing two initial blocks with reduced mod(4t+3) whose incidence matrices are N_1 and N_2 respectively. The dimension of both the matrices N_1 and N_2 are $v'\times b'$. Next we consider two new incidence matrices N_1^* and N_2^* of $(v'+1)\times(b'+1)$, which are defined as

$$N_1^* = \begin{bmatrix} 0 & E_{1b'} \\ E_{v'1} & N_1 \end{bmatrix} \text{ and } N_2^* = \begin{bmatrix} 0 & 0_{1b'} \\ 0_{v'1} & N_2 \end{bmatrix}$$

Throughout E_{ij} and 0_{ij} are the matrices with element 1 and 0 having *i*th row and *j*th column. By using N_1^* and N_2^* we can obtain a new incidence matrix *N*, where

$$N = \begin{bmatrix} N_1^* & N_2^* \\ N_2^* & N_1^* \end{bmatrix}$$

Now, *NN'* can be expressed as

$$NN' = \begin{bmatrix} N_1^* & N_2^* \\ N_2^* & N_1^* \end{bmatrix} \begin{bmatrix} N_1^{*'} & N_2^{*'} \\ N_2^{*'} & N_1^{*'} \end{bmatrix}$$

$$NN' = \begin{bmatrix} N_1^*N_1^{*'}+N_2^*N_2^{*'} & N_1^*N_2^{*'}+N_2^*N_1^{*'} \\ N_2^*N_1^{*'}+N_1^*N_2^{*'} & N_2^*N_2^{*'}+N_1^*N_1^{*'} \end{bmatrix} \quad (2.1)$$

Solving for each elements of the matrix

$N_1^* N_1^* ' + N_2^* N_2^* '$

$$= \begin{bmatrix} 0 & E_{1b'} \\ E_{v'1} & N_1 \end{bmatrix} \begin{bmatrix} 0 & E_{1v'} \\ E_{1b'} & N'_1 \end{bmatrix} + \begin{bmatrix} 0 & 0_{1b'} \\ 0_{v'1} & N_2 \end{bmatrix} \begin{bmatrix} 0 & 0_{1v'} \\ 0_{b'1} & N'_2 \end{bmatrix}$$

$$= \begin{bmatrix} E_{1b'} E_{b'1} & E_{1b'} N_1' \\ N_1 E_{b'1} & E_{v'1} E_{1v'} + N_1 N_1' \end{bmatrix} + \begin{bmatrix} 0_{1b'} 0_{b'1} & 0_{1b'} N_2' \\ N_2 0_{b'1} & 0_{v'1} 0_{1v'} + N_2 N_2' \end{bmatrix}$$

$$= \begin{bmatrix} b' E_{11} & k' E_{1b'} \\ r' E_{b'1} & E_{v'v'} + (r' - \lambda') I_{v'} + \lambda' E_{v'v'} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & (r' - \lambda') I_{v'} + \lambda' E_{v'v'} \end{bmatrix}$$

$$= \begin{bmatrix} b' E_{11} & k' E_{1b'} \\ r' E_{b'1} & 2(r' - \lambda') I_{v'} + (2\lambda' + 1) E_{v'v'} \end{bmatrix}$$

$$= \begin{bmatrix} (4t+3) E_{11} & (2t+1) E_{1b'} \\ (2t+1) E_{b'1} & 2(t+1) I_{v'} + (2t+1) E_{v'v'} \end{bmatrix}$$

$N_1^* N_2^* ' + N_2^* N_1^* '$

$$= \begin{bmatrix} 0 & 0_{1b'} \\ 0_{v'1} & N_2 \end{bmatrix} \begin{bmatrix} 0 & E_{1v'} \\ E_{1b'} & N'_1 \end{bmatrix} + \begin{bmatrix} 0 & E_{1b'} \\ E_{v'1} & N_1 \end{bmatrix} \begin{bmatrix} 0 & 0_{1v'} \\ 0_{b'1} & N'_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & E_{1b'} N_2' \\ N_2 E_{b'1} & N_2 N_1' + N_1 N_2' \end{bmatrix}$$

$$=\begin{bmatrix} 0 & k'E_{1b'} \\ r'E_{b'1} & r'E_{v'v'}-r'I_{v'} \end{bmatrix}$$

$$=\begin{bmatrix} 0 & (2t+1)E_{1b'} \\ (2t+1)E_{b'1} & (2t+1)(E_{v'v'}-I_{v'}) \end{bmatrix}$$

After solving the $N_1^*N_1^*{}'+N_2^*N_2^*{}'$ and $N_1^*N_2^*{}'+N_2^*N_1^*{}'$ we substitute their values in the (2.1) and Hence we get,

NN'=

$$\begin{bmatrix} \begin{pmatrix} (4t+3)E_{11} & (2t+1)E_{1b'} \\ (2t+1)E_{b'1} & 2(t+1)I_{v'}+(2t+1)E_{v'v'} \end{pmatrix} & \begin{pmatrix} 0 & (2t+1)E_{1b'} \\ (2t+1)E_{b'1} & (2t+1)(E_{v'v'}-I_{v'}) \end{pmatrix} \\ \begin{pmatrix} 0 & (2t+1)E_{1b'} \\ (2t+1)E_{b'1} & (2t+1)(E_{v'v'}-I_{v'}) \end{pmatrix} & \begin{pmatrix} (4t+3)E_{11} & (2t+1)E_{1b'} \\ (2t+1)E_{b'1} & 2(t+1)I_{v'}+(2t+1)E_{v'v'} \end{pmatrix} \end{bmatrix}_{8(t+1)\times 8(t+1)}$$

Here it is obvious that diagonal elements are (4t+3) and off diagonal elements are 0 and (2t+1). Further we verified that all the primary and secondary parameters of PBIB design along with v =mn and n = n_1+1 is satisfied this shows that the existing PBIB design is Group Divisible design with parameters $v = 8(t+1) = b$, $r = 4t+3 = k$, $\lambda_1 = 0$, $\lambda_2 = 2t+1$, $n_1=1$, $n_2=2(4t+3)$, $m = 4(t+1)$, n = 2. Further we found that for $t\geq 0$, $r-\lambda_1>0$ and $rk-v\lambda_2>0$ holds true for the resulting GD designs and hence we claim that the resulting GD design is Regular Group Divisible design.

Example 2.2.1.1 If t=1, then a regular group divisible design with parameters $v = 16 = b$, $r = 7 = k$, $\lambda_1 = 0$, $\lambda_2 = 3$, $n_1=1$, $n_2=14$, m = 8, n = 2 is obtained. Let the two initial sets of SBIB design are (x^0, x^2, x^4) and (x, x^3, x^5). Consider these two initial sets as two initial blocks of a SBIBD.

Since 3(=x) is a primitive element of GF (7) hence the two initial blocks are [1 2 4] and [3 6 5]. After developing these two initial blocks with reduced mod 11, we get SBIB design with parameters v = 7 =b, r = 3= k, λ=1 and whose blocks are following

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Considering two incidence matrices N_1^* and N_2^* of order $(v'+1)$ x $(b'+1)$ where,

$$N_1^* = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad N_2^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

By using N_1^* and N_2^* we obtain a new incidence matrix N ,

$$N = \begin{bmatrix} N_1^* & N_2^* \\ N_2^* & N_1^* \end{bmatrix} \text{ and } N' = \begin{bmatrix} N_1^{*}{}' & N_2^{*}{}' \\ N_2^{*}{}' & N_1^{*}{}' \end{bmatrix}$$

Next using *Theorem 2.2.1.1*, we obtain the blocks of RGD design, which are following

2	1	1	1	1	1	1	1	10	4	5	2	2	3	2	3
3	2	3	4	5	2	3	2	11	6	7	6	3	4	4	5
4	3	4	5	6	6	7	4	12	7	8	8	7	8	5	6
5	5	6	7	8	7	8	8	13	9	9	9	9	9	9	9
6	12	13	10	10	11	10	11	14	10	11	12	13	10	11	10
7	14	15	14	11	12	12	13	15	11	12	13	14	14	15	12
8	15	16	16	15	16	13	14	16	13	14	15	16	15	16	16

This RGD design is reported as R117a by Dey(1977). However we have developed this design using systematic method while Dey(1977) constructed it by using trial and error method. Its dual is truly self dual, Non-Resolvable and its complement exist with parameters v=16=b, r=9=k, λ_1=2, λ_2=5, n_1=1, n_2=14 , m=8, n=2. The blocks of the complement design are the following:

1	4	2	2	2	3	2	3	1	1	1	1	1	1	1	1
9	6	5	3	3	4	4	5	2	2	2	3	4	2	3	3
10	7	7	6	4	5	5	6	3	3	3	4	5	5	6	4
11	8	8	8	7	8	6	7	4	5	4	5	6	6	7	7
12	9	9	9	9	9	9	9	5	8	6	7	8	7	8	8
13	10	10	11	12	10	11	10	6	12	10	10	10	11	10	11
14	11	11	12	13	13	14	12	7	14	13	11	11	12	12	13
15	13	12	13	14	14	15	15	8	15	15	14	12	13	13	14
16	16	14	15	16	15	16	16	9	16	16	16	15	16	14	15

This design is also happened to be a RGD design which is also reported by Dey(1977). Dey(1977) has constructed it with trial and error method while we developed it as the completement of R117a.

Next in Theorem 2.2.1.2 we again obtained regular group divisible designs by using the difference set of balanced incomplete block design.

Theorem 2.2.1.2: *If (4t+1) is prime or prime power, where t=1, 2, 3…., then there always exists a group divisible design with the parameters v= 4(2t+1) = b, r= (4t+1) = k,* $\lambda_1=0$*,* $\lambda_2=2t$*, m= 2(2t+1), n=2, which is obtained from the incidence matrix N defined as* $N=\begin{bmatrix} N_1^* & N_2^* \\ N_2^* & N_1^* \end{bmatrix}$ *, where* N_1^* *and* N_2^* *are the incidence matrices of another two incomplete block designs.*

Proof: Let $\alpha_0=0$, $\alpha_1=x^0=1$, $\alpha_2=x$, $\alpha_3=x^2$, $\alpha_4=x^3,\ldots,\alpha_{4t+1}=x^{4t}$ be the elements of GF(4t+1), where x is the primitive element of GF(4t+1). Consider the initial blocks (x^0, x^2, x^4, $x^6,\ldots,x^{4t-2}$) and (x, x^3, x^5, $x^7,\ldots,x^{4t-1}$) from a difference set for BIB design with parameters $v'=(4t+1)$, $b'=2(4t+1)$, $r'=4t$, $k'=2t$ and $\lambda'=2t-1$. With the help of two initial blocks we obtained two incomplete block designs whose incidence matrices are N_1 and N_2 .

Next consider another two new incidence matrices N_1^* and N_2^* of dimension $(v'+1) \times (\frac{b'}{2}+1)$ as,

$$N_1^* = \begin{bmatrix} 0 & E_{1\frac{b'}{2}} \\ E_{v'1} & N_1 \end{bmatrix} \text{ and } N_2^* = \begin{bmatrix} 0 & 0_{1\frac{b'}{2}} \\ 0_{v'1} & N_2 \end{bmatrix}$$

Where, $E_{1b'/2}$ & $E_{v'1}$ is a unitary matrix whose all elements are one and $0_{1b'/2}$ & $0_{v'1}$ is a null matrix. By using N_1^* and N_2^* we can obtain a new incidence matrix N where,

$N = \begin{bmatrix} N_1^* & N_2^* \\ N_2^* & N_1^* \end{bmatrix}$ resultant design N is a PBIB design and $N' = \begin{bmatrix} N_1^*{}' & N_2^*{}' \\ N_2^*{}' & N_1^*{}' \end{bmatrix}$ so,

$$NN' = \begin{bmatrix} N_1^* & N_2^* \\ N_2^* & N_1^* \end{bmatrix} \begin{bmatrix} N_1^*{}' & N_2^*{}' \\ N_2^*{}' & N_1^*{}' \end{bmatrix}$$

$$NN' = \begin{bmatrix} N_1^* N_1^*{}' + N_2^* N_2^*{}' & N_1^* N_2^*{}' + N_2^* N_1^*{}' \\ N_2^* N_1^*{}' + N_1^* N_2^*{}' & N_2^* N_2^*{}' + N_1^* N_1^*{}' \end{bmatrix} \tag{2.2}$$

Solving for each elements of the matrix

$N_1^* N_1^*{}' + N_2^* N_2^*{}'$

$$= \begin{bmatrix} 0 & E_{1\frac{b'}{2}} \\ E_{v'1} & N_1 \end{bmatrix} \begin{bmatrix} 0 & E_{1v'} \\ E_{1\frac{b'}{2}} & N'_1 \end{bmatrix} + \begin{bmatrix} 0 & 0_{1\frac{b'}{2}} \\ 0_{v'1} & N_2 \end{bmatrix} \begin{bmatrix} 0 & 0_{1v'} \\ 0_{\frac{b'}{2}1} & N'_2 \end{bmatrix}$$

$$= \begin{bmatrix} E_{1\frac{b'}{2}}E_{\frac{b'}{2}1} & E_{1\frac{b'}{2}}N_1' \\ N_1E_{\frac{b'}{2}1} & E_{v'1}E_{1v'} + N_1N_1' \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & N_2N_2' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{b'}{2}E_{11} & k'E_{1\frac{b'}{2}} \\ k'E_{\frac{b'}{2}1} & E_{v'v'} + N_1N_1^{'} + N_2N_2^{'} \end{bmatrix}$$

$$= \begin{bmatrix} (4t+1)E_{11} & (2t)E_{1\frac{b'}{2}} \\ (2t)E_{\frac{b'}{2}1} & (2t+1)I_{v'} + (2t)E_{v'v'} \end{bmatrix}$$

$N_1^*N_2^*{}' + N_2^*N_1^*{}'$

$$= \begin{bmatrix} 0 & 0_{1\frac{b'}{2}} \\ 0_{v'1} & N_2 \end{bmatrix} \begin{bmatrix} 0 & E_{1v'} \\ E_{1\frac{b'}{2}} & N'_1 \end{bmatrix} + \begin{bmatrix} 0 & E_{1\frac{b'}{2}} \\ E_{v'1} & N_1 \end{bmatrix} \begin{bmatrix} 0 & 0_{1v'} \\ 0_{\frac{b'}{2}1} & N'_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ N_2E_{\frac{b'}{2}1} & N_2N_1' \end{bmatrix} + \begin{bmatrix} 0 & E_{1\frac{b'}{2}}N_2' \\ 0 & N_1N_2' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & E_{1\frac{b'}{2}}N_2' \\ N_2E_{\frac{b'}{2}1} & N_2N_1' + N_1N_2' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & k'E_{1\frac{b'}{2}} \\ k'E_{\frac{b'}{2}1} & k'E_{v'v'} - k'I_{v'} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (2t)E_{1\frac{b'}{2}} \\ (2t)E_{\frac{b'}{2}1} & (2t)(E_{v'v'} - I_{v'}) \end{bmatrix}$$

we substitute their values in the (2.2) and Hence we get,

$$NN' = \begin{bmatrix} \begin{pmatrix} (4t+1)E_{11} & 2t\,E_{1\frac{b'}{2}} \\ 2t\,E_{\frac{b'}{2}1} & (2t+1)I_{v'} + 2t\,E_{v'v'} \end{pmatrix} & \begin{pmatrix} 0 & 2t\,E_{1\frac{b'}{2}} \\ 2t\,E_{\frac{b'}{2}1} & 2t(E_{v'v'} - I_{v'}) \end{pmatrix} \\ \begin{pmatrix} 0 & 2t\,E_{1\frac{b'}{2}} \\ 2t\,E_{\frac{b'}{2}1} & 2t(E_{v'v'} - I_{v'}) \end{pmatrix} & \begin{pmatrix} (4t+1)E_{11} & 2t\,E_{1\frac{b'}{2}} \\ 2t\,E_{\frac{b'}{2}1} & (2t+1)I_{v'} + 2t\,E_{v'v'} \end{pmatrix} \end{bmatrix}_{4(2t+1)\times 4(2t+1)}$$

Here the diagonal elements are (4t+1) and off diagonal elements are 0 and 2t, so the design is PBIB with two associate classes. Further it is verified that this series of PBIB designs satisfied the parameters of RGD designs and hence the resulting designs are RGD designs with parameters $v= 4(2t+1) = b$, $r= (4t+1) = k$, $\lambda_1=0$, $\lambda_2=2t$, $n_1=1$, $n_2=2(4t+1)$, $m= 2(2t+1)$, $n=2$.

Example 2.2.1.2 If t=1, then a regular group divisible design with parameters $v = 12 = b$, $r = 5 = k$, $\lambda_1 = 0$, $\lambda_2 = 2$, m = 6, n = 2 is obtained by considering 2 initial sets (x^0,x^2) and (x,x^3) from a difference set of BIB designs.

Since 2(=x) is a primitive element of GF (5) hence the initial sets are [1 4] and [2 3]. Consider these two initial sets as the two initial blocks. Next developing these two blocks with reduced mod 5 we get two incomplete block design, say, d_1 and d_2 whose blocks are following

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \end{bmatrix}$$

Considering two incidence matrices N_1^* and N_2^* of order *(v'+1)* x *(b'+1)* where,

$$N_1^* = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \text{ and } N_2^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

By using $N_1{}^*$ and $N_2{}^*$ we obtain a new incidence matrix N,

$$N = \begin{bmatrix} N_1^* & N_2^* \\ N_2^* & N_1^* \end{bmatrix} \text{ and } N' = \begin{bmatrix} N_1^{*\prime} & N_2^{*\prime} \\ N_2^{*\prime} & N_1^{*\prime} \end{bmatrix}$$

Next using *Theorem 2.2.1.2*, we get a RGD designs, the blocks of this designs are as follows

2	1	1	1	1	2	1	8	3	4	5	2
3	2	3	2	3	3	4	9	4	5	6	6
4	5	6	4	5	7	6	10	7	7	7	7
5	9	10	11	8	10	8	11	8	9	8	9
6	10	11	12	2	12	9	12	11	12	10	11

This is a RGD design with parameters $v = 12 = b$, $r = 7 = k$, $\lambda_1 = 2$, $\lambda_2 = 4$, $n_1=1$, $n_2=10$, m = 6, n = 2 and is being reported in Clatworthy (1973) as R144, and hence it is a known RGD design. Moreover its dual is truly self dual, non-resolvable and its complement exist with parameters $v = 12 = b$, $r = 7 = k$, $\lambda_1 = 2$, $\lambda_2 = 4$, n_1=1, n_2=10, m = 6, n = 2. This happened to be RGD design. The blocks of the complementary designs are following

1	3	2	3	2	2	1	1	1	1	1	1
7	4	4	5	4	3	2	2	2	2	3	4
8	6	5	6	6	5	3	5	3	3	4	5
9	7	7	7	7	7	4	6	6	4	5	6
10	8	8	8	9	10	5	9	8	9	8	8
11	9	9	9	10	11	6	10	10	11	10	9
12	12	12	10	11	12	7	12	11	12	12	11

This RGD is also reported in the Clatworthy (1973) as R175 but the two designs differ in triplets so we claim the two designs are non- isomorphic to each other.

Next in theorem 2.2.1.3 we discussed the construction of semi regular group divisible designs by using the difference set of balanced incomplete block design.

Theorem 2.2.1.3: *If (4t-1) is prime or prime power, where t=1, 2, 3…., then there always exists a group divisible design with the parameters v= 2(4t-1), b = 8t, r = 4t, k = 4t-1,* $\lambda_1 = 0$, $\lambda_2 = 2t$, *m = 4t-1, n = 2, which is obtained from the incidence matrix N defined as* $N=\begin{bmatrix} N_1' & N_2' \\ N_2' & N_1' \end{bmatrix}$, *where* N_1' *and* N_2' *are the transpose of the incidence matrices, say,* N_1 *and* N_2 *of two different incomplete block designs.*

Proof: Let $\alpha_0=0$, $\alpha_1=x^0=1$, $\alpha_2=x$, $\alpha_3=x^2$, $\alpha_4=x^3$,…, $\alpha_{4t+1}=x^{4t}$ be the elements of GF(4t-1), where x is the primitive element of GF(4t-1). Consider the two initial blocks $(0, x^0, x^2, x^4, x^6,\ldots,x^{4(t-1)})$ and $(\infty, x, x^3, x^5, x^7,\ldots,x^{4t-3})$ from a difference set for a BIB design with parameters $v' = 4t$, $b'=2(4t-1)$, $r' = 4t-1$, $k'= 2t$ and $\lambda' = 2t-1$. Now with the help of two initial blocks we have obtained two incidence matrices say, N_1 and N_2. Let N_1' and N_2' be the transpose of N_1 and N_2 respectively. By using N_1' and N_2' we obtain the new incidence matrix *N*.

$$N = \begin{bmatrix} N_1^{'} & N_2^{'} \\ N_2^{'} & N_1^{'} \end{bmatrix}$$

Now, we can express *NN'* as

$$NN^{'} = \begin{bmatrix} N_1^{'} & N_2^{'} \\ N_2^{'} & N_1^{'} \end{bmatrix} \begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix}$$

$$NN^{'} = \begin{bmatrix} N_1^{'}N_1 + N_2^{'}N_2 & N_1^{'}N_2 + N_2^{'}N_1 \\ N_1^{'}N_2 + N_2^{'}N_1 & N_1^{'}N_1 + N_2^{'}N_2 \end{bmatrix} \tag{2.3}$$

Solving each element of the matrix separately

$$N_1^{'}N_1 + N_2^{'}N_2 = t(E_{\frac{b'}{2}} + I_{\frac{b'}{2}}) + t(E_{\frac{b'}{2}} + I_{\frac{b'}{2}})$$

$$= (2t)(E_{\frac{b'}{2}} + I_{\frac{b'}{2}})$$

Next,

$$N_1^{'}N_2 + N_2^{'}N = t(E_{\frac{b'}{2}} - I_{\frac{b'}{2}}) + t(E_{\frac{b'}{2}} - I_{\frac{b'}{2}})$$

$$= (2t)(E_{\frac{b'}{2}} - I_{\frac{b'}{2}})$$

Substituting their values in the (2.3) and hence we get

$$NN' = \begin{pmatrix} 2t(E_{\frac{b'}{2}} + I_{\frac{b'}{2}}) & 2t(E_{\frac{b'}{2}} - I_{\frac{b'}{2}}) \\ 2t(E_{\frac{b'}{2}} - I_{\frac{b'}{2}}) & 2t(E_{\frac{b'}{2}} + I_{\frac{b'}{2}}) \end{pmatrix}_{2(4t-1)\times 2(4t-1)}$$

Here the diagonal elements are $4t$ and off diagonal elements are 0 and $2t$. Further all the primary and secondary parameters for a GD design are satisfied which shows the existing PBIB design is a GD design with parameters $v= 2(4t-1)$, $b=8t$, $r= 4t$, $k= 4t-1$, $\lambda_1=0$, $\lambda_2= 2t$, $n_1= 1$, $n_2= 4(2t-1)$, $m=4t-1$, $n=2$. Further we found that for $t>0$, $r-\lambda_1>0$ and $rk-v\lambda_2 =0$ holds true for the resulting GD design and hence we claim that the resulting GD design is a Semi Regular Group Divisible design.

Example 2.2.1.3 If t=2, then a group divisible design with parameters $v = 14$, $b= 16$, $r = 8$, $k= 7$, $\lambda_1 = 0$, $\lambda_2 = 4$, m = 7, n = 2 is obtained by considering two initial sets (0, x^0, x^2, x^4) and (∞, x, x^3, x^5) from a difference set of BIB designs.

Since 3(=x) is a primitive element of GF (7) hence the initial sets are [0 1 2 4] and [∞ 3 6 5]. Considering these two sets as two initial blocks and then developing these two initial blocks with reduced mod 7 we get two sets of BIB design whose blocks are following

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
4	5	6	0	1	2	3

∞	∞	∞	∞	∞	∞	∞
3	4	5	6	0	1	2
6	0	1	2	3	4	5
5	6	0	1	2	3	4

Here we are considering 0 and ∞ as the 7th and 8th treatments of the blocks. Using *Theorem 2.2.1.3* we get the blocks of GD design are as follows

1	1	2	1	2	3	1	4	3	8	1	2	1	1	2	1
2	2	3	3	4	5	4	5	4	9	5	6	3	2	3	2
5	3	4	4	5	6	6	7	6	10	6	7	7	4	5	3
7	6	7	5	6	7	7	8	8	11	9	8	9	10	8	4
10	11	8	9	8	8	9	9	9	12	10	10	11	12	11	5
11	12	12	13	10	9	10	10	12	13	11	11	12	13	13	6
13	14	13	14	14	11	12	13	14	14	12	12	13	14	14	7

Here we found that the resulting GD design is a semi regular group divisible design with parameters $v = 14$, $b= 16$, $r = 8$, $k= 7$, $\lambda_1 = 0$, $\lambda_2 = 4$, m = 7, n = 2. This SRGD is being reported in Clatworthy (1973) as SR82. Its complement is truly self complement, resolvable and its dual gives a rectangular PBIB design with three associate class with parameters $v=16$, $b=14$, $r=7$, $k=8$, $n_1=1$, $n_2=7$, $n_3=7$, $\lambda_1=0$, $\lambda_2=3$, $\lambda_3=4$, $m=8$, $n=2$.

And at last we have discuss the construction a regular group divisible design by the difference set of balanced incomplete block designs. Here we have three different sets which is arranged as given in Theorem 2.2.1.4.

Theorem 2.2.1.4 *If (3t+1) is prime or prime power, where t=1, 2, 3…., then there always exists a group divisible design with the parameters* $v= 3(3t+2) = b$, $r = (3t+1) =$

k, $\lambda_1 = 0$, $\lambda_2 = t$, $m = (3t+2)$, $n = 3$, which is obtained from the incidence matrix N define as, $N=\begin{bmatrix} N_1^* & N_3^* & N_2^* \\ N_3^* & N_2^* & N_1^* \\ N_2^* & N_1^* & N_3^* \end{bmatrix}$*, where N_1^*, $N_2^*$and N_3^* are incidence matrices of three incomplete block designs.*

Proof: Let α_0=0, α_1=x^0=1, α_2=x, α_3=x^2, α_4=x^3…….. , α_{3t+1}=x^{3t-1} be the elements of GF(3t+1), where x is the primitive element of GF(3t+1). Consider the initial blocks (x^0,x^3,x^6….. ,x^{3t-3}), (x,x^4,x^7….. ,x^{3t-2}) and (x^2,x^5,x^8….. ,x^{3t-1}) from difference set for a BIB design with parameters *$v'= 3t+1$, $b'= 3(3t+1)$, $r'= 3t$, $k'= t$, $\lambda'=t-1$*. With the help of three initial blocks we have obtained three incidence matrices say, N_1, N_2 and N_3 respectively. Using these matrices we constructed other incidence matrices of the form

$$N_1^* = \begin{bmatrix} 0 & 0_{1\frac{b'}{3}} \\ 0_{v'1} & N_1 \end{bmatrix}, \quad N_2^* = \begin{bmatrix} 0 & E_{1\frac{b'}{3}} \\ E_{v'1} & N_2 \end{bmatrix} \text{ and } N_3^* = \begin{bmatrix} 0 & 0_{1\frac{b'}{3}} \\ 0_{v'1} & N_3 \end{bmatrix}$$

Throughout E_{ij} and 0_{ij} are the matrices with element 1 and 0 having ith row and jth column. By using N_1^*, N_2^* and N_3^* we can obtain a new incidence matrix *N*, where

$$N=\begin{bmatrix} N_1^* & N_3^* & N_2^* \\ N_3^* & N_2^* & N_1^* \\ N_2^* & N_1^* & N_3^* \end{bmatrix}$$

Now NN' can be expressed as

$$NN' = \begin{bmatrix} N_1^* & N_3^* & N_2^* \\ N_3^* & N_2^* & N_1^* \\ N_2^* & N_1^* & N_3^* \end{bmatrix} \begin{bmatrix} N_1^{*'} & N_3^{*'} & N_2^{*'} \\ N_3^{*'} & N_2^{*'} & N_1^{*'} \\ N_2^{*'} & N_1^{*'} & N_3^{*'} \end{bmatrix}$$

$$= \begin{bmatrix} N_1^*N_1^{*'} + N_2^*N_2^{*'} + N_3^*N_3^{*'} & N_1^*N_3^{*'} + N_3^*N_2^{*'} + N_2^*N_1^{*'} & N_1^*N_2^{*'} + N_3^*N_1^{*'} + N_2^*N_3^{*'} \\ N_1^*N_2^{*'} + N_3^*N_1^{*'} + N_2^*N_3^{*'} & N_1^*N_1^{*'} + N_2^*N_2^{*'} + N_3^*N_3^{*'} & N_1^*N_3^{*'} + N_3^*N_2^{*'} + N_2^*N_1^{*'} \\ N_1^*N_3^{*'} + N_3^*N_2^{*'} + N_2^*N_1^{*'} & N_1^*N_2^{*'} + N_3^*N_1^{*'} + N_2^*N_3^{*'} & N_1^*N_1^{*'} + N_2^*N_2^{*'} + N_3^*N_3^{*'} \end{bmatrix} \quad (2.4)$$

Solving each element of the matrix separately

$$N_1^*N_1^{*'} + N_2^*N_2^{*'} + N_3^*N_3^{*'} =$$

$$= \begin{bmatrix} 0 & 0_{1\frac{b'}{3}} \\ 0_{v'1} & N_1 \end{bmatrix} \begin{bmatrix} 0 & 0_{1v'} \\ 0_{\frac{b'}{3}1} & N_1^{'} \end{bmatrix} + \begin{bmatrix} 0 & E_{1\frac{b'}{3}} \\ E_{v'1} & N_2 \end{bmatrix} \begin{bmatrix} 0 & E_{1v'} \\ E_{\frac{b'}{3}1} & N_2^{'} \end{bmatrix} + \begin{bmatrix} 0 & 0_{1\frac{b'}{3}} \\ 0_{v'1} & N_1 \end{bmatrix} \begin{bmatrix} 0 & 0_{1v'} \\ 0_{\frac{b'}{3}1} & N_3^{'} \end{bmatrix}$$

$$= \begin{bmatrix} E_{1\frac{b'}{3}} E_{\frac{b'}{3}1} & E_{1\frac{b'}{3}} N_2^{'} \\ N_2 E_{\frac{b'}{3}1} & N_1N_1^{'} + N_2N_2^{'} + N_3N_3^{'} + E_{v'1}E_{1v'} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{b'}{3} E_{11} & k'E_{1\frac{b'}{3}} \\ k'E_{\frac{b'}{3}1} & (2k'+1)I_{v'} + \lambda' E_{v'v'} + E_{v'v'} \end{bmatrix}$$

$$= \begin{bmatrix} (3t+1)E_{11} & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & (2t+1)I_{v'} + tE_{v'v'} \end{bmatrix}$$

$N_1^* N_3^{*'} + N_3^* N_2^{*'} + N_2^* N_1^{*'}$

$$= \begin{bmatrix} 0 & 0_{1\frac{b'}{3}} \\ 0_{v'1} & N_1 \end{bmatrix} \begin{bmatrix} 0 & 0_{1v'} \\ 0_{\frac{b'}{3}1} & N_3^{'} \end{bmatrix} + \begin{bmatrix} 0 & 0_{1\frac{b'}{3}} \\ 0_{v'1} & N_1 \end{bmatrix} \begin{bmatrix} 0 & E_{1v'} \\ E_{\frac{b'}{3}1} & N_2^{'} \end{bmatrix} + \begin{bmatrix} 0 & E_{1\frac{b'}{3}} \\ E_{v'1} & N_2 \end{bmatrix} \begin{bmatrix} 0 & 0_{1v'} \\ 0_{\frac{b'}{3}1} & N_1^{'} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & E_{1\frac{b'}{3}} N_1^{'} \\ k' E_{\frac{b'}{3}1} & N_1 N_3^{'} + N_3 N_2^{'} + N_2 N_1^{'} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & k' E_{1\frac{b'}{3}} \\ k' E_{\frac{b'}{3}1} & k'(E_{v'v'} - I_{v'}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & t(E_{v'v'} - I_{v'}) \end{bmatrix}$$

$N_1^* N_2^{*'} + N_3^* N_1^{*'} + N_2^* N_3^{*'}$

$$= \begin{bmatrix} 0 & 0_{1\frac{b'}{3}} \\ 0_{v'1} & N_1 \end{bmatrix} \begin{bmatrix} 0 & E_{1v'} \\ E_{\frac{b'}{3}1} & N_2^{'} \end{bmatrix} + \begin{bmatrix} 0 & 0_{1\frac{b'}{3}} \\ 0_{v'1} & N_1 \end{bmatrix} \begin{bmatrix} 0 & 0_{1v'} \\ 0_{\frac{b'}{3}1} & N_1^{'} \end{bmatrix} + \begin{bmatrix} 0 & E_{1\frac{b'}{3}} \\ E_{v'1} & N_2 \end{bmatrix} \begin{bmatrix} 0 & 0_{1v'} \\ 0_{\frac{b'}{3}1} & N_3^{'} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & k' E_{1\frac{b'}{3}} \\ k' E_{\frac{b'}{3}1} & k'(E_{v'v'} - I_{v')} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & t(E_{v'v'} - I_{v')} \end{bmatrix}$$

Hence, substituting these values in the (2.4) we get,

$$
NN' = \begin{bmatrix}
\begin{pmatrix} (3t+1)E_{11} & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & (2t+1)I_{v'} + tE_{v'v'} \end{pmatrix} & \begin{pmatrix} 0 & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & t(E_{v'v'} - I_{v'}) \end{pmatrix} & \begin{pmatrix} 0 & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & t(E_{v'v'} - I_{v'}) \end{pmatrix} \\
\begin{pmatrix} 0 & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & t(E_{v'v'} - I_{v'}) \end{pmatrix} & \begin{pmatrix} (3t+1)E_{11} & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & (2t+1)I_{v'} + tE_{v'v'} \end{pmatrix} & \begin{pmatrix} 0 & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & t(E_{v'v'} - I_{v'}) \end{pmatrix} \\
\begin{pmatrix} 0 & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & t(E_{v'v'} - I_{v'}) \end{pmatrix} & \begin{pmatrix} 0 & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & t(E_{v'v'} - I_{v'}) \end{pmatrix} & \begin{pmatrix} (3t+1)E_{11} & tE_{1\frac{b'}{3}} \\ tE_{\frac{b'}{3}1} & (2t+1)I_{v'} + tE_{v'v'} \end{pmatrix}
\end{bmatrix}_{3(3t+2)\times 3(3t+2)}
$$

Here it is obvious that diagonal elements are $(3t+1)$ and off diagonal elements are 0 and t. Further we verified that all the primary and secondary parameters of PBIB design along with $v = mn$ and $n = n_1+1$ is satisfied which shows that the existing PBIB design is Group Divisible design with parameters $v = 3(3t+2) = b$, $r = (3t+1) = k$, $\lambda_1 = 0$, $\lambda_2 = t$, $m = (3t+2)$, $n = 3$. Further we found that for $t > 0$, $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$ holds true for the resulting GD designs and hence we claim that the resulting GD design is Regular Group Divisible design.

Example 2.2.1.4 If t=2, then a group divisible design with parameters $v = 24 = b$, $r = 7 = k$, $\lambda_1 = 0$, $\lambda_2 = 2$, m = 8, n = 2 is obtained by considering two initial sets $(0, x^3)$, (x, x^4) and (x^2, x^5) from a difference set of BIB designs

Since 3(=x) is a primitive element of GF (7) hence the initial sets are [1, 6], [3, 4] and [2, 5]. Considering these sets as initial blocks we develop blocks with reduced mod 7. We get three sets of BIB design whose blocks are the following:

$$
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}
$$

Using Theorem 2.2.1.4 we get a PBIB design whose blocks are as follows:

1	1	1	1	1	1	1	2	2	18	2	3	2	4	5	6	3	10	3	4	5	2	3	4
3	2	2	7	6	5	4	3	5	19	7	8	4	6	7	8	5	11	6	7	8	6	7	8
4	3	8	8	7	6	5	4	9	20	11	12	13	11	12	10	10	12	9	9	9	9	9	9
14	13	12	11	10	11	10	5	11	21	14	15	16	15	16	13	14	13	12	13	14	15	10	10
16	15	14	13	12	16	15	6	12	22	17	17	17	17	17	17	17	14	13	14	15	16	16	11
18	20	19	18	21	20	19	7	22	23	21	21	22	18	18	19	23	15	18	19	20	19	20	21
21	24	23	22	24	23	22	8	24	24	20	22	23	24	19	20	24	16	23	24	19	21	22	23

Here we found that the resulting GD design is a regular group divisible design with parameters $v = 24 = b$, $r = 7 = k$, $\lambda_1 = 0$, $\lambda_2 = 2$, $m = 8$, $n = 2$. This RGD is being not being reported in Clatworthy (1973) Its complement exist, truly a self dual.

2.2.2 Semi-regular GD design using resolvable BIB design

Here in this section we discuss the simple method to construct a Group divisible design with the help of resolvable balanced incomplete block design.

Theorem 2.2.2.1: *Let s is a prime or prime power, deleting one group from a resolvable BIB design belonging to the series* $v= s^2$ *,* $b =s(s+1)$*,* $r= s+1$*,* $k = s$*,* $\lambda = 1$*. Then there exist a group divisible design with parameters* $v= s^2 = b$*,* $r= s = k$*,* $\lambda_1 = 0$*,* $\lambda_2 = 1$*,* $m = s$ *and* $n = s$*.*

Proof: Let us consider a resolvable balanced incomplete block design with parameters $v= s^2$, $b =s(s+1)$, $r= s+1$, $k = s$, $\lambda = 1$. Let us also consider a group of the following type:

1	*2*	…………	*s*	
(s+1)	*(s+2)*	…………	*2s*	
..	..			(2.5)
..	..			
..	..			
s(s-1)+1	*s(s-1)+2*	…………	s^2	

Construct the resolvable BIBD. Since it is a resolvable BIBD and hence it will have r groups such that in each group there will be b/r blocks. Now from this resolvable BIBD we delete one group which contain the blocks shown in (2.5). The resulting design gives another incomplete block designs. This design happens to be a PBIB design. After verifying all the primary and secondary parameters of existing PBIB design is Group Divisible design with parameters $v=s^2 = b$, $r = s = k$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = s$, $n = s$. Further we found that for $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$ holds true for the resulting GD designs and hence we claim that the resulting GD design is semi regular group divisible design.

Example 2.2.2.1 If s = 3, then a group divisible design with parameters $v = 9 = b$, $r = 3 = k$, $\lambda_1 = 0$, $\lambda_2 = 1$, m = 3, n = 3 is obtained from a resolvable BIB design with parameters $v = 9$, $b = 12$, $r = 4$, k = 3, $\lambda = 1$. The blocks of resolvable BIBD are as follows

1	2	3	1	4	7	1	2	3	1	2	3
4	5	6	4	5	8	6	4	5	5	6	4
7	8	9	7	6	9	8	9	7	9	7	8

Using *Theorem 2.2.2.1,* we get a GD design whose blocks as follows.

1	4	7	1	2	3	1	2	3
2	5	8	6	4	5	5	6	4
3	6	9	8	9	7	9	7	8

Here we found that the resulting GD design is a semi regular group divisible design with parameters $v = 9 = b$, $r = 3 = k$, $\lambda_1 = 0$, $\lambda_2 = 1$, m = 3, n = 3. This SRGD is being reported in Clatworthy (1973) as SR23. Its complement design exist which is also a SRGD and is also reported in Clatworthy (1973) as SR65 and it's truly a self dual.

2.2.3 Singular GD design from another singular GD design.

In this case we consider the singular group divisible design and then applying juxtaposition to this design, we get another singular group divisible design. This we show in Theorem 2.2.3.1.

Theorem 2.2.3.1 *Consider a singular group divisible design with parameters* v', b', r', k', λ_1', λ_2', m', n', *then we can construct another singular group divisible design with juxtaposition having the parameters* $v = 2v'$, $b = b'$, $r = r'$, $k = 2k'$, $\lambda_1 = \lambda_1'$, $\lambda_2 = \lambda_2'$, $m = m'$, $n = 2n'$, *using juxtaposition techniques.*

Proof: Lets consider a singular group divisible design with parameters v', b', r', k', λ_1', λ_2', m', n'. Let N' be the incidence matrix of order $v' \times b'$. Now using juxtaposition we obtain a new incidence matrix $N = \begin{bmatrix} N^* \\ \hdashline N^* \end{bmatrix}$ of order $2v' \times b'$

Now, we can express

$$NN' = \begin{bmatrix} N^* \\ \hdashline N^* \end{bmatrix} \begin{bmatrix} N^{*'} & N^{*'} \end{bmatrix}$$

$$= \begin{bmatrix} N^* N^{*'} & N^* N^{*'} \\ N^* N^{*'} & N^* N^{*'} \end{bmatrix}$$

Hence, we find that the design obtain is a PBIB design. Further we verified that all the primary and secondary parameters of PBIB design along with $v = mn$ and $n = n_1 + 1$ is satisfied which shows that the existing PBIB design is Group Divisible design with parameters $v = 2v'$, $b = b'$, $r = r'$, $k = 2k'$, $\lambda_1 = \lambda_1'$, $\lambda_2 = \lambda_2'$, $m = m'$, $n = 2n'$. Further we found that $r - \lambda_1 = 0$ holds true for the resulting GD designs and hence we claim that the resulting GD design is singular group divisible design.

Example 2.2.3.1 Lets consider a S1 from the Clatworthy table (1973) with parameters $v = 6$ $r = 2$ $k = 4$ $b = 3$ $m = 3$ $n = 2$ $\lambda_1 = 2$ $\lambda_2 = 1$. Blocks of S1 are as follows:

[1 4 2 5]

[2 5 3 6]

[3 6 1 4]

Then using *Theorem 2.2.3.1* we obtain a PBIB design with same block size but with twice the number of treatments. The blocks of PBIB design are

[1 4 2 5 7 10 8 11]

[2 5 3 6 8 11 9 12]

[3 6 1 4 9 12 7 10]

After verifying that all the primary and secondary parameters of PBIB design along with $v = mn$ and $n = n_1+1$ is satisfied which shows that the existing PBIB design is Group Divisible design with parameters $v = 12$ $r = 2$ $k = 8$ $b = 3$ $m = 3$ $n = 4$ $\lambda_1 = 2$ $\lambda_2 = 1$. Further we found that $r - \lambda_1 = 0$ holds true for the resulting GD designs and hence we claim that the resulting GD design is singular group divisible design. This SGD is being reported in Clatworthy (1973) as SR53.