Chapter 4

Semidirect Product and Wreath Product of a Semigroup and a \( \Gamma \)-Semigroup

4 Introduction

The term semidirect product was first used for semigroups by Neumann in 1960 [42] as a stepping stone to constructing wreath product of semigroups. Neumann’s definition was as follows: let \( T \) and \( S \) be two semigroups and \( \theta : S \not\rightarrow \text{End}(T) \) be an antimorphism of \( S \) into the endomorphism semigroup of \( T \) (we shall use an arrow \( \not\rightarrow \) throughout to indicate an antimorphism). If \( s \in S \), denote \( t(s\theta) \) by \( t^\theta \). Then the semidirect product of \( T \) and \( S \), in that order with structure map \( \theta \), consists of the set \( T \times S \) equipped with the product \((t, s)(t_1, s_1) = (tt^\theta, ss_1)\). This product was denoted by \( T_\theta \times S \). In 1983, Nico [43] defined the semidirect product of two monoids. His definition was as follows: let \( S \) and \( T \) be two monoids. \( \text{End}(T) \), the endomorphism monoid of \( T \) and he wrote endomorphism as exponent to the right of arguments. Let \( \alpha : S \rightarrow \text{End}(T) \), \( s \rightarrow \alpha(s) \) be a given homomorphism and \( t^{\alpha(s)} \) is denoted by \( t^\alpha \). Then the semidirect product of \( S \) and \( T \), in that order with structure map \( \alpha \), consists of the set \( S \times T \) equipped with the product \((s, t)(r, u) = (sr, t^\alpha u)\). This product was denoted by \( S \times_\alpha T \). Nico obtained necessary and sufficient conditions for the semidirect product of two monoids to be regular and to be inverse respectively and he applied these results to the wreath product since the semidirect product is a generalization of the wreath product. Again in 1989, Saito [52] determined the necessary and sufficient conditions for the semidirect
product to be orthodox, to be left inverse and to be right inverse respectively and these results were applied to the wreath product. In this chapter our aim is to generalize the semidirect product in $\Gamma$-semigroup theory. In section 1, we define semidirect product of a monoid and a $\Gamma$-semigroup and study it. In section 2, we study that product of a semigroup and a $\Gamma$-semigroup and in section 3, we define wreath product of a semigroup and a $\Gamma$-semigroup which is a generalization of the research paper of R. Zhang [76].

4.1 Semidirect product of a monoid and a $\Gamma$-semigroup

In this section we define semidirect product of a monoid and a $\Gamma$-semigroup and determine the necessary and sufficient conditions for the semidirect product to be right orthodox, to be left inverse and to be right inverse $\Gamma$-semigroup respectively.

Definition 4.1.1 Let $S$ be a monoid and $T$ be a $\Gamma$-semigroup. Let $\text{End}(T)$ denote the set of all endomorphisms on $T$ i.e., the set of all mappings $f : T \rightarrow T$ satisfying $f(ab) = f(a)f(b)$ for all $a, b \in T$, $\alpha \in \Gamma$. Clearly $\text{End}(T)$ is a semigroup. Let $\Phi : S \not\rightarrow \text{End}(T)$ be a given $1$-preserving antimorphism i.e, $\Phi(rs) = \Phi(r)\Phi(s)$ for all $r, s \in S$ and $\Phi(1)$ is the identity mapping from $T$ to $T$. If $s \in S$ and $t \in T$, we write $t^s$ for $\Phi(s)(t)$ and $T^s = \{t^s : t \in T\}$. Let $S \times \Phi T = \{(s, t) : s \in S, t \in T\}$. We define $(s_1, t_1)\alpha(s_2, t_2) = (s_1, s_2, t_1^s \alpha t_2)$ for all $(s_i, t_i) \in S \times \Phi T$, $i = 1, 2$ and $\alpha \in \Gamma$. Then $S \times \Phi T$ is a $\Gamma$-semigroup. This $\Gamma$-semigroup $S \times \Phi T$ is called the semidirect product of the monoid $S$ and the $\Gamma$-semigroup $T$.

We now prove the following lemmas.

Lemma 4.1.2 Let $S \times \Phi T$ be a semidirect product of a monoid $S$ and a $\Gamma$-semigroup $T$. Then
\[(i) \, (tau)^s = t^s a u^s \text{ for all } s \in S, \, t, u \in T \text{ and } \alpha \in \Gamma.\]

\[(ii) \, (t^s)^r = (t)^{sr} \text{ for all } s, r \in S \text{ and } t \in T.\]

**Proof:** Let \(s, r \in S, \alpha \in \Gamma \text{ and } t, u \in T.\) Now

\[
(tau)^s = (\Phi(s))(tau) \\
= (\Phi(s))(t)\alpha(\Phi(s))(u) \\
= t^s a u^s.
\]

Hence \(i\) follows. Again

\[
(t^s)^r = (\Phi(r))(t^s) \\
= (\Phi(r))(\Phi(s))(t) \\
= (\Phi(r)\Phi(s))(t) \\
= (\Phi(sr))(t) \\
= (t)^{sr}.
\]

Thus \(ii\) follows.

**Lemma 4.1.3** Let \(S\) be a monoid and \(T\) be a \(\Gamma\)-semigroup, \(\Phi : S \not\rightarrow \text{End}(T)\) be a given \(1\)-preserving antimorphism. Then if the semidirect product is

\[(i) \text{ right (resp. left) orthodox } \Gamma\text{-semigroup then } S \text{ is an orthodox semigroup and } T \text{ is a right (resp. left) orthodox } \Gamma\text{-semigroup;}\]

\[(ii) \text{ right (resp. left) inverse } \Gamma\text{-semigroup then } S \text{ is a right (resp. left) inverse semigroup and } T \text{ is a right (resp. left) inverse } \Gamma\text{-semigroup.}\]

**Proof:** \(i\) Suppose that \(S \times_\Phi T\) is a right orthodox \(\Gamma\)-semigroup. Since it is regular, for \((s, t) \in S \times_\Phi T\), there exists \((s', t') \in S \times_\Phi T\) and \(\alpha, \beta \in \Gamma\) such that

\[(s, t) = (s, t)\alpha(s', t')\beta(s, t) = (ss's, ts's\alpha(t')^s\beta t)\]
and

$$(s', t') = (s', t')\beta(s, t)\alpha(s', t') = (s's', (t')^{s'}\beta t'\alpha t').$$

This implies $s' \in V(s)$. Again if we take $s = 1$ then $s' = 1$ and we get $t' \in V_0^\beta(t)$. Thus $S$ is a regular semigroup and $T$ is a regular $\Gamma$-semigroup.

Let $t_1$ be an $\alpha$-idempotent and $t_2$ be a $\beta$-idempotent in $T$ and $e, g \in E(S)$. Then $(1, t_1)\alpha(1, t_1) = (1, t_1, \alpha t_1) = (1, t_1)$. Hence $(1, t_1)$ is an $\alpha$-idempotent in $S \times_\phi T$. Similarly $(1, t_2)$ is a $\beta$-idempotent in $S \times_\phi T$. Now $(1, (t_1, \alpha t_1)\beta(t_1, \alpha t_1)) = (1, t_1, \alpha t_1)\beta(1, t_1, \alpha t_1) = ((1, t_1)\alpha(1, t_2))\beta(1, t_1)\alpha(1, t_2)) = (1, t_1)\alpha(1, t_2) = (1, t_1, \alpha t_2)$. Thus $t_1, \alpha t_2$ is a $\beta$-idempotent. So, $T$ is a right orthodox $\Gamma$-semigroup.

Again $(e, t_1^e)$ is an $\alpha$-idempotent since $(t_1^e)^g = t_1^e$ and $(g, t_2^g)$ is a $\beta$-idempotent. Since $S \times_\phi T$ is a right orthodox $\Gamma$-semigroup, $(e, t_1^e)\alpha(g, t_2^g) = (e, t_1^e)\alpha(g, t_2^g) = ((e, t_1^e)\alpha(g, t_2^g))\beta((e, t_1^e)\alpha(g, t_2^g)) = (e, t_1^e)\alpha(g, t_2^g)\beta((e, t_1^e)\alpha(g, t_2^g)) = ((e, t_1^e)\alpha(g, t_2^g))\beta((e, t_1^e)\alpha(g, t_2^g))$. Hence $(e, g)^2 = eg$. So, $S$ is an orthodox semigroup. The case of left orthodox $\Gamma$-semigroup is similar to that of right orthodox $\Gamma$-semigroup.

(ii) Let $S \times_\phi T$ be a right inverse $\Gamma$-semigroup. Then by (i) $S$ is a regular semigroup and $T$ is a regular $\Gamma$-semigroup. Let $t_1$ be an $\alpha$-idempotent and $t_2$ is a $\beta$-idempotent in $T$. Let $e, g \in E(S)$. Since $(1, t_1)$ is an $\alpha$-idempotent, $(1, t_2) is a $\beta$-idempotent and $S \times_\phi T$ is a right inverse $\Gamma$-semigroup, we have $(1, t_1, \alpha t_1, \beta t_1) = (1, t_1, \alpha(1, t_2))\beta(1, t_1) = (1, t_2)\beta(1, t_1) = (1, t_2, \beta t_1)$ which shows that $t_1, \alpha t_2, \beta t_1 = t_2, \beta t_1$. So $T$ is a right inverse $\Gamma$-semigroup. Again since $(e, t_1^e)$ is an $\alpha$-idempotent and $(g, t_2^g)$ is a $\beta$-idempotent, $(ge, t_2^g)\beta t_1^e) = (g, t_2^g)\beta e, t_1^e) = (e, t_1^e)\alpha(g, t_2^g)\beta(e, t_1^e) = (e, t_1^e)\alpha g, t_2^g)\beta t_1^e)$. Which shows $ge = ege$ for $e, g \in E(S)$. Thus $S$ is a right inverse semigroup. The case of left inverse $\Gamma$-semigroup is similar to that of right inverse $\Gamma$-semigroup.

**Theorem 4.1.4** Let $S \times_\phi T$ be the semidirect product of a monoid $S$ and a $\Gamma$-semigroup
corresponding to a given 1-preserving antimorphism $\Phi : S \rightarrow \text{End}(T)$ and let $(s, t) \in S \times T$, then

(i) if $(s', t') \in V^B_\alpha((s, t))$ then $(s', t') \in V^B_\alpha((s, t^\alpha t'))$. In particular if $s \in \mathcal{E}(S)$, then $(s, (t')^s \beta t^\alpha s \alpha t') \in V^B_\alpha((s, t^\alpha t')).

(ii) if $t^s$ is an $\alpha$-idempotent and $s' \in V(s)$, then $(s', t^s t') \in V^B_\alpha((s, t^s t'))$.

Proof: (i) Since $(s', t') \in V^B_\alpha((s, t))$ we have,

$$(s', t') = (s', t') \beta(s, t) \alpha(s', t') = (ss's', (t')^s \beta t^\alpha s \alpha t')$$

and

$$(s, t) = (s, t) \alpha(s', t') \beta(s, t) = (ss's, t^s \alpha(t')^s \beta t).$$

This shows that $s' \in V(s)$ and

$$t^s \alpha(t')^s \beta t = t,$$  \hspace{1cm} \ldots (4.1.1)

$$(t')^s \beta t^\alpha s \alpha t' = t'.$$  \hspace{1cm} \ldots (4.1.2)

From (4.1.1) we have, $(t^s \alpha(t')^s \beta t)^{t^s} = (t)^{t^s}$ i.e., $t^s \alpha(t')^s \beta t^s = t^s$ and from (4.1.2), $(t')^s \beta t^s \alpha(t')^s = (t')^s$ i.e., $(t')^s \beta t^s \alpha(t')^s = (t')^s$. Now $(s', t') \beta(s, t^s t') \alpha(s', t') = (ss's', (t')^s \beta t^s \alpha t') = (s', t')$ (by (4.1.2)) and

$$(s, t^s) \alpha(s', t') \beta(s, t^s') = (ss's, t^s \alpha(t')^s \beta t^s s)$$

$$(s, t^s) \alpha(t')^s \beta t^s s$$

$$(s, t^s'.)$$

Thus we have $(s', t') \in V^B_\alpha((s, t^s))$. Again if $s \in \mathcal{E}(S)$, $(t')^s \beta t^s \alpha(t')^s = (t')^s \beta t^s \alpha(t')^s = (t')^s$. Moreover

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\[(s, t')\alpha(s, (t')^s \beta t'^s \alpha t') \beta(s, t') = \left( sss, t'^s \alpha(t')^s \beta t'^s \alpha t' \right)\]
\[= (s, t'^s \alpha(t')^s \beta t'^s)\]
\[= (s, t'^s).
\]

and

\[(s, (t')^s \beta t'^s \alpha t') \beta (s, t') \alpha(s, (t')^s \beta t'^s \alpha t') = \left( s, (t')^s \beta t'^s \alpha t' \right) \beta t'^s \alpha(t')^s \beta t'^s \alpha t'\]
\[= (s, (t')^s \beta t'^s \alpha t') \beta t'^s \alpha t'\]
\[= (s, (t')^s \beta t'^s \alpha t').
\]

Hence \((s, (t')^s \beta t'^s \alpha t') \in V^\alpha_\alpha(s, t')\).

(ii) Since \(t^s\) is an \(\alpha\)-idempotent, \((s, t^s) \alpha(s', t'^s) \alpha(s, t^s) = (ss's, t'^ss' \alpha t'^ss' \alpha t^s) = (s, t^s)\)

and

\[(s', t'^s) \alpha(s, t^s) \alpha(s', t'^s) = (s' ss', t'^ss' \alpha t'^ss' \alpha t^s)\]
\[= (s', t'^ss' \alpha t'^ss' \alpha t^s)\]
\[= (s', (t^s \alpha t^s t^s))\]
\[= (s', t'^s).
\]

i.e., \((s', t'^s) \in V^\alpha_\alpha(s, t^s)\).

The following theorem shows the relation between the idempotent elements of a monoid, \(\Gamma\)-semigroup and their semidirect product.

**Theorem 4.1.5** Let \(S\) be a monoid and \(T\) be a \(\Gamma\)-semigroup and \(S \times \Phi T\) be the semidirect product corresponding to a given \(1\)-preserving antimorphism \(\Phi : S \not\rightarrow \text{End}(T)\). Moreover, let \(t \in t^e T T\) for every \(e \in E(S)\) and every \(t \in T\). Then

(i) \((e, t)\) is an \(\alpha\)-idempotent if and only if \(e \in E(S)\) and \(t^e\) is an \(\alpha\)-idempotent;

(ii) if \((e, t)\) is an \(\alpha\)-idempotent, then \((e, t^e) \in V^\alpha_\alpha((e, t))\).
Proof: (i) If \((e, t)\) is an \(\alpha\)-idempotent then \((e, t) = (e, t)\alpha(e, t) = (e^2, t^\alpha t)\) i.e., \(e = e^2\) and \(t^\alpha t = t\) \[\ldots(4.1.3)\)
So, \(t^\alpha = (t^\alpha t)^{\alpha} = t^\alpha t^\alpha\) which implies that \(t^\alpha\) is an \(\alpha\)-idempotent. Conversely, let \(e \in E(S)\) and \(t^\alpha\) be an \(\alpha\)-idempotent. Since \(t \in t^\alpha T\), \(t = t^\alpha \beta t_1\) for some \(\beta \in \Gamma, t_1 \in T\) and hence \(t^\alpha t = t^\alpha t^\alpha \beta t_1 = t\). Thus \((e, t)\alpha(e, t) = (e, t^\alpha t) = (e, t)\) i.e., \((e, t)\) is an \(\alpha\)-idempotent.

(ii) If \((e, t)\) is an \(\alpha\)-idempotent, from (i) \(e \in E(S)\) and \(t^\alpha\) is an \(\alpha\)-idempotent. Now
\[\begin{align*}
(e, t)\alpha(e, t^\alpha)\alpha(e, t) &= (e, t^\alpha t^\alpha) = (e, t^\alpha) (\text{ from (4.1.3)}) \\
\alpha(e, t^\alpha) &= (e, t^\alpha t^\alpha t^\alpha) = (e, t^\alpha) \\
\alpha(e, t^\alpha) \in V_{\alpha}^\beta((e, t))
\end{align*}\]
Thus \((e, t^\alpha) \in V_{\alpha}^\beta((e, t))\).

We now give the necessary and sufficient conditions for the semidirect product to be a right (resp. left) orthodox \(\Gamma\)-semigroup and right (resp. left) inverse \(\Gamma\)-semigroup.

Theorem 4.1.6 Let \(S\) be a monoid and \(T\) be a \(\Gamma\)-semigroup. Let \(\Phi : S \not\mapsto End(T)\) be a given 1-preserving antimorphism. Then the semidirect product \(S \times_{\Phi} T\) is a right (resp. left) orthodox \(\Gamma\)-semigroup if and only if

(i) \(S\) is an orthodox semigroup and \(T\) is a right (resp. left) orthodox \(\Gamma\)-semigroup,

(ii) for every \(e \in E(S)\) and every \(t \in T, t \in t^\alpha T\) and

(iii) if \(t^\alpha\) is an \(\alpha\)-idempotent, then \(t^\beta\) is an \(\alpha\)-idempotent for every \(g \in E(S)\), where \(e \in E(S), t \in T\).

Proof: Suppose \(S \times_{\Phi} T\) is a right orthodox \(\Gamma\)-semigroup. Then by Lemma 4.1.3 \(S\) is an orthodox semigroup and \(T\) is a right orthodox \(\Gamma\)-semigroup. For (ii), let \((e, t) \in S \times_{\Phi} T\) with \(e \in E(S)\) and let \((e', t') \in V_{\alpha}^\beta((e, t))\) for some \(\alpha, \beta \in \Gamma\). Then by Theorem 4.1.4, \((e', t'), (e, t^\alpha t^\alpha e') \in V_{\alpha}^\beta((e, t^\alpha))\). Thus \(V_{\alpha}^\beta((e, t)) \cap V_{\alpha}^\beta((e, t^\alpha)) \neq \emptyset\) and hence we have \(V_{\alpha}^\beta((e, t)) = V_{\alpha}^\beta((e, t^\alpha))\) by Theorem 3.1.5. So \((e, t^\alpha)^\alpha t^\alpha e' \in V_{\alpha}^\beta((e, t))\).
Thus \((e, t) = (e, t)\alpha(e, (t')^\alpha t^\alpha e')\beta(e, t) = (e, t^\alpha (t')^\alpha t^\alpha e' \beta(t')^\alpha t^\alpha e')\) and hence \(t =\)
For (iii) we shall first show that for an \(\alpha\)-idempotent \(t^e\) of \(T\) if \(e \in E(S)\), \(t^{e'}\) is an \(\alpha\)-idempotent for any \(e' \in V(e)\). If \(e \in E(S)\) and \(t^e\) is an \(\alpha\)-idempotent, then by Theorem 4.1.5, \((e, t)\) is an \(\alpha\)-idempotent in \(S \times_T T\) and \((e, t^e) \in V^\alpha_\alpha((e, t))\). Again since \(t^e\) is an \(\alpha\)-idempotent \((e, t^e)\) is also an \(\alpha\)-idempotent and thus \((e, t^e) \in V^\alpha_\alpha((e, t^e))\) i.e., \(V^\alpha_\alpha((e, t^e)) \cap V^\alpha_\alpha((e, t)) \neq \emptyset\) and so \(V^\alpha_\alpha((e, t^e)) = V^\alpha_\alpha((e, t))\) and by Theorem 4.1.4(ii) \((e', t^{e'}) \in V^\alpha_\alpha((e, t))\) i.e., \((e', t^{e'}) \in V^\alpha_\alpha((e, t))\). Thus \((e, t) = (e, t)\alpha(e', t^{e'})\alpha(e, t) = (ee'e, t^{e'}\alpha t^{e'}\alpha t) = (e, t^{e'}\alpha t^{e'}\alpha t)\) (since \(t = t^\beta u\) for some \(\beta \in \Gamma, u \in T\) and so \(t^{e'}\alpha t^{e'} = t^{e'}\)). Hence \(t^{e'}\) is an \(\alpha\)-idempotent.

Let \(e, g \in E(S)\) and suppose that \(t^{e'}\) is an \(\alpha\)-idempotent for \(t \in T\), then \(t^{e'}\alpha t^{e'} = (t^{e'}\alpha t^{e'})^g = t^{e'}\) i.e., \(t^{e'}\) is an \(\alpha\)-idempotent and we have \(e^g \in E(S)\) and \(g \in V(e^g)\) since \(S\) is right orthodox. Then by the above fact \(t^{e'}\) is an \(\alpha\)-idempotent.

Conversely, suppose that \(S\) and \(T\) satisfy (i), (ii) and (iii). Let \((s, t) \in S \times_T T\) be given. Since \(S\) is regular, there exists \(s' \in S\) such that \(s = ss's\) and \(s' = ss's'.\) We take \(e = ss's\), then \(e \in E(S)\). By (ii) \(t \in t^eT\) which implies \(t = t^e\beta u\) for some \(\beta \in \Gamma, u \in T\).

Since \(T\) is regular, there exist \(v \in T\) and \(\gamma, \delta \in \Gamma\) such that \(v \in V^\delta_\delta(t)\). Let \(t' = v^{s'}\) where \(v \in V^\delta_\delta(t)\) where \(\gamma, \delta \in \Gamma\). Now \(t^{s'}\gamma(t')^s\delta t = t^{s'}\gamma v^{s'}\delta t^{e} \beta u = (t\gamma v^{s'}\delta t\)\beta u = t\) i.e., \((s, t) = (ss's, t^{s'}\gamma(t')^s\delta t = (s, t)\gamma(s', t')\delta(s, t)\). Again \((t')^{s'}\delta t^{s'}\gamma t' = (v^{s'}\delta t^{s'}\gamma v^{s'} = v^{s'} = t'\) i.e., \((s', t') = (ss's, t^{s'}\delta t^{s'}\gamma t') = (s', t')\delta(s, t)\gamma(s', t')\). Thus we have \((s', t') \in V^\delta_\delta(s, t)\) which yields \(S \times_T T\) is a regular \(\Gamma\)-semigroup. The case of left orthodox \(\Gamma\)-semigroup is similar to that of right orthodox \(\Gamma\)-semigroup.

Now let \((e, t)\) be an \(\alpha\)-idempotent and \((g, u)\) be a \(\beta\)-idempotent. Then by Theorem 4.1.5, \(e, g \in E(S), t^e\) is an \(\alpha\)-idempotent and \(u^g\) is a \(\beta\)-idempotent. By (iii) \(t^{e^g}\) is an \(\alpha\)-idempotent, \(u^{e^g}\) is a \(\beta\)-idempotent and \(t^{e^g}\alpha t^{e^g} = (t^{e^g}\alpha t^{e^g})^{e^g} = t^{e^g}\) i.e., \(t^{e^g}\) is an \(\alpha\)-idempotent. By our assumption \(e, g \in E(S)\) and \((t^{e^g}u^g)^{e^g} = t^{e^g}\alpha u^{e^g}\) is a \(\beta\)-idempotent.
Thus by Theorem 4.1.5, \((e,t)\alpha(g,u) = (eg,t^\alpha u)\) is a \(\beta\)-idempotent which shows that \(S \times_\varphi T\) is a right orthodox \(\Gamma\)-semigroup.

Theorem 4.1.7 Let \(S\) be a monoid, \(T\) be a \(\Gamma\)-semigroup and \(\Phi : S \not\rightarrow End(T)\) be a given \(1\)-preserving antimorphism. Then the semidirect product \(S \times_\varphi T\) is a right inverse \(\Gamma\)-semigroup if and only if

(i) \(S\) is a right inverse semigroup and \(T\) is a right inverse \(\Gamma\)-semigroup and

(ii) for every \(e \in E(S)\) and every \(t \in T\), \(te_{LT}\).

Proof: Let \(S \times_\varphi T\) be a right inverse \(\Gamma\)-semigroup. Then by Lemma 4.1.3, \(S\) is a right inverse semigroup and \(T\) is a right inverse \(\Gamma\)-semigroup. Again since every right inverse \(\Gamma\)-semigroup is a right orthodox \(\Gamma\)-semigroup from the previous theorem, condition (ii) holds.

Conversely, suppose that \(S\) and \(T\) satisfy (i) and (ii). Regularity of \(S \times_\varphi T\) can be proved by similar way of that of Theorem 4.1.6. Let \((e,t)\) be an \(\alpha\)-idempotent and \((g,u)\) be a \(\beta\)-idempotent in \(S \times_\varphi T\). Then by Theorem 4.1.5, \(e,g \in E(S)\), \(t^\alpha\) is an \(\alpha\)-idempotent, \(u^\beta\) is a \(\beta\)-idempotent. From (ii) \(t = t^\gamma v\) for some \(\gamma \in \Gamma\), \(v \in T\) and thus \(t^\alpha t^\beta = t\) and similarly \(u^\beta u = u\). So \(u^\alpha = (u^\beta u)^{\beta^\gamma} = u^{\beta^\gamma} \beta u^\alpha\) and \(t^\alpha = (t^\alpha t^\beta)^{\alpha^\gamma} = t^\alpha t^\beta t^\gamma\) since \(S\) is a right inverse semigroup. Now by (ii) we have \(u^\alpha \beta = (u^\alpha \beta)^{\gamma \delta v}\) for some \(\delta \in \Gamma\), \(v_i \in T\) and hence \(u^\alpha \beta = u^{\gamma \delta \beta} \beta^{\gamma \delta \delta} v_i = u^{\gamma \delta} \beta^{\gamma \delta} v_i\). Thus we have \((e,t)\alpha(g,u)\beta(e,t) = (ege,t^{\gamma \delta \alpha} \alpha^\beta t) = (ge,t^{\gamma \delta} \alpha u^{\beta^\gamma} \beta t^{\gamma \delta} \delta v_i) = (ge,u^{\gamma \delta} \beta^{\gamma \delta} \delta v_i) = (ge,u^\alpha \beta) = (g,u)\beta(e,t)\) which implies \(S \times_\varphi T\) is a right inverse \(\Gamma\)-semigroup.

Theorem 4.1.8 Let \(S\) be a monoid, \(T\) be a \(\Gamma\)-semigroup and \(\Phi : S \not\rightarrow End(T)\) be a given \(1\)-preserving antimorphism. Then the semidirect product \(S \times_\varphi T\) is a left inverse \(\Gamma\)-semigroup if and only if
(i) \( S \) is a left inverse semigroup and \( T \) is a left inverse \( \Gamma \)-semigroup and
(ii) for every \( e \in E(S) \) and every \( t \in T, \ t = t^* \).

**Proof:** Let \( S \times_\Phi T \) be a left inverse \( \Gamma \)-semigroup. Then by Lemma 4.1.3, \( S \) is a left inverse semigroup and \( T \) is a left inverse \( \Gamma \)-semigroup. For (ii) let \((e, u)\) be an \( \alpha \)-idempotent in \( S \times_\Phi T \). Then \( (e, u) = (e, u)\alpha(e, u) = (e, u^\alpha u) \) i.e., \( u^\alpha u = u \). Again \((e, u^\alpha)\alpha(e, u^\alpha) = (e, u^\alpha\alpha^\alpha u^\alpha) = (e, u^\alpha)\) which yields \((e, u^\alpha)\) is an \( \alpha \)-idempotent and we have \((e, u^\alpha)\alpha(e, u) = (e, u^\alpha u) = (e, u) \). Since \( S \times_\Phi T \) is a left inverse \( \Gamma \)-semigroup,
\((e, u) = (e, u^\alpha)\alpha(e, u) = (e, u^\alpha)\alpha(e, u^\alpha) = (e, u^\alpha u^\alpha\alpha u^\alpha) = (e, u^\alpha u^\alpha\alpha u^\alpha) = (e, u^\alpha u^\alpha) = (e, u^\alpha) \) i.e., \( u = u^\alpha \). Thus if \((e, u)\) is an \( \alpha \)-idempotent then \( u = u^\alpha \). Now \((e, t) \in S \times_\Phi T \) with \( e \in E(S) \) and let \((e', t') \in V_\Phi^\gamma \{(e, t)\} \) for some \( \gamma, \delta \in \Gamma \). Then we get \( e' \in V(e), t'^{e^*}\gamma(t'^{e^*})^\delta t = t \) i.e., \( t'^{e^*}\gamma(t'^{e^*})^\delta = t'^{e^*} \) which implies \( t'^{e^*}\gamma(t'^{e^*})^\delta = t'^{t'^{e^*}} \). Since \((e', t')^\delta t = (e', t')^\delta = (e', t'')^\delta t = (e', t'^{e^*})^\delta = (e', t'^{e^*})^\delta = t' = (e', t'^{e^*})^\delta = t'^{e^*} \). Thus \( t'^{e^*} = t'^{e^*} \gamma(t'^{e^*})^\delta t'^{e^*} = t'^{e^*} \gamma(t'^{e^*})^\delta t = t \) and hence \( t'^{e^*} = t'^{e^*} \gamma(t'^{e^*})^\delta t'^{e^*} = t'^{e^*} \gamma(t'^{e^*})^\delta = t \).

Conversely suppose that \( S \) and \( T \) satisfy (i) and (ii). Then \( S \times_\Phi T \) is regular. Now let \((e, t)\) be an \( \alpha \)-idempotent and \((g, u)\) be a \( \beta \)-idempotent. Then \( e^2 = e \) and \( t = t^\alpha at = tat \) (by (ii)) and similarly \( g^2 = g \) and \( u^\beta u = u \) i.e., \( e, g \in E(S) \) and \( t \) is an \( \alpha \)-idempotent, \( u \) is an \( \beta \)-idempotent. Hence \( ge \in E(S) \). Thus we have \((e, t)\beta(g, u)\alpha(e, t) = (ege, t^\beta\beta u^\alpha\alpha at) = (ege, t^\beta u^\alpha at) \) (by (ii)) = \((eg, t^\beta u) = (eg, t^\beta)u = (e, t)\beta(g, u) \). Thus \( S \times_\Phi T \) is a left inverse \( \Gamma \)-semigroup.

### 4.2 Semidirect product of a semigroup and a \( \Gamma \)-semigroup

In this section we study the semidirect product of a semigroup and a \( \Gamma \)-semigroup. We take \( S \) as a semigroup and \( T \) as a \( \Gamma \)-semigroup. \( \Phi : S \not\rightarrow End(T) \) be a given
antimorphism. We now define the semidirect product of $S$ and $T$ in same fashion of section 1. We note that Lemma 4.1.2 holds in this context also. If $S \times \Phi T$ is a regular $\Gamma$-semigroup and if $S$ has no identity element then from Lemma 4.1.3 we see that $T$ may not be regular. So the absence of the identity element in $S$ may be the reason of the failure of the results described in section 1. In this section we study the the results in case of absence of the identity element in $S$.

**Theorem 4.2.1** Let $S \times \Phi T$ be a semidirect product of a semigroup $S$ and a $\Gamma$-semigroup $T$. Then $T^e$ is a $\Gamma$-semigroup for all $x \in S$ where $T^e = \{ t^e : t \in T \}$. If moreover $S \times \Phi T$ is a regular $\Gamma$-semigroup then $S$ is a regular semigroup and $T^e$ is a regular $\Gamma$-semigroup for all $e \in E(S)$.

**Proof**: The first part is clear from Lemma 4.1.2. Let $S \times \Phi T$ be regular. For $(s, t) \in S \times \Phi T$, there exist $(s', t') \in S \times \Phi T$ and $\alpha, \beta \in \Gamma$ such that $(s, t) = (s', t')\alpha(s', t')\beta(s, t) = (ss's, t^e\alpha(t')^e\beta t)$ and $(s', t') = (s, t)\beta(s, t)\alpha(s', t') = (ss's, (t')^e\beta t^e\alpha t')$. This implies $s' \in V(s)$. Let $e \in E(S)$, then for $(e, t^e)$, there exist $(s', t') \in S \times \Phi T$ and $\alpha, \beta \in \Gamma$ such that $(e, t^e) = (e, t^e)\alpha(s', t')\beta(e, t^e) = (es'e, t^{e'\alpha t^e\beta t^e})$ and $(s', t') = (s', t')\beta(e, t^e)\alpha(s', t') = (s'es', (t')^{e'\beta t^e \alpha t'})$. Hence $s' \in V(e)$ and we have $t^e = t^e\alpha t^e\beta t^e$ and $t^e = t^e\beta t^e\alpha t^e$. i.e, $t^e \in V_{\alpha t^e}(t^e)$. Hence $T^e$ is a regular $\Gamma$-semigroup.

The following theorem gives a necessary conditions for a semidirect product of a semigroup and a $\Gamma$-semigroup to be a right(resp. left) orthodox $\Gamma$-semigroup and a right(resp. left) inverse $\Gamma$-semigroup.

**Theorem 4.2.2** Let $S$ be a semigroup and $T$ be a $\Gamma$-semigroup, $\Phi : S \rightarrow End(T)$ be a given antimorphism. If the semidirect product $S \times \Phi T$ is

(i) a right(resp. left) orthodox $\Gamma$-semigroup then $S$ is an orthodox semigroup and $T^e$ is a right(resp. left) orthodox $\Gamma$-semigroup for every idempotent $e \in S$;
(ii) a right (resp. left) inverse \( \Gamma \)-semigroup then \( S \) is a right (resp. left) inverse semigroup and \( T^* \) is a right (resp. left) inverse \( \Gamma \)-semigroup.

Proof: (i) Let \( S \times \_ T \) be a right orthodox \( \Gamma \)-semigroup. Let \( e, g \in E(S) \) and \( t^e \) be an \( \alpha \)-idempotent and \( u^e \) be a \( \beta \)-idempotent in \( T^* \). Then \((e, t^e)\alpha(e, t^e) = (e, t^e\alpha t^e) = (e, t^e) \). i.e, \((e, t^e)\) is an \( \alpha \)-idempotent. Similarly \((e, u^e)\) is a \( \beta \)-idempotent. Again \((g, u^g)\beta(g, u^g) = (g, u^g\beta u^g) = (g, (u^g\beta u^g)^g) = (g, u^g)\). Thus \((g, u^g)\) is a \( \beta \)-idempotent of \( S \times \_ T \). Now

\[
(e, (t^e\alpha u^e)\beta(t^e\alpha u^e)) = (e, (t^e\alpha u^e))\beta(e, (t^e\alpha u^e)) \\
= ((e, t^e)\alpha(e, u^e))\beta((e, t^e)\alpha(e, u^e)) \\
= (e, t^e)\alpha(e, u^e) \text{ (since } S \times \_ T \text{ is right orthodox)} \\
= (e, t^e\alpha u^e).
\]

Which shows that \( t^e\alpha u^e \) is a \( \beta \)-idempotent and hence \( T^* \) is a right orthodox \( \Gamma \)-semigroup.

Again since \( S \times \_ T \) is a right orthodox \( \Gamma \)-semigroup we have

\[
((eg)^2, (t^g\alpha u^g)^g\beta t^g\alpha u^g) = (eg, t^g\alpha u^g)\beta(eg, t^g\alpha u^g) \\
= ((e, t^e)\alpha(g, u^g))\beta((e, t^e)\alpha(g, u^g)) \\
= (e, t^e)\alpha(g, u^g) \text{ (since } S \times \_ T \text{ is right orthodox)} \\
= (eg, t^g\alpha u^g).
\]

Thus \((eg)^2 = eg\) which shows that \( S \) is orthodox. The case of left orthodox \( \Gamma \)-semigroup is similar to that of right orthodox \( \Gamma \)-semigroup.

(ii) Suppose that \( S \times \_ T \) is a right inverse \( \Gamma \)-semigroup. Let \( e, g \in E(S) \) and \( t^e \) be an \( \alpha \)-idempotent and \( u^e \) be a \( \beta \)-idempotent in \( T^* \). Then \((e, t^e)\) is an \( \alpha \)-idempotent, \((e, u^e), (g, u^g)\) are \( \beta \)-idempotents of \( S \times \_ T \). Now since \( S \times \_ T \) is a right orthodox \( \Gamma \)-semigroup, we have \((e, t^e\alpha u^e\beta t^e) = (e, t^e)\alpha(e, u^e)\beta(e, t^e) = (e, u^e)\beta(e, t^e) = (e, u^e\beta t^e)\) and \((ege, t^g\alpha u^g\beta t^g) = (e, t^e)\alpha(g, u^g)\beta(e, t^e) = (g, u^g)\beta(e, t^e) = (ge, u^g\beta t^e)\). So
we have $\alpha t' \beta t' = u' \beta t$ and $ege = ge$. Consequently we have $S$ is a right inverse semigroup and $T^e$ is a right inverse $\Gamma$-semigroup. The case of left inverse $\Gamma$-semigroup is similar to that of right inverse $\Gamma$-semigroup.

The proofs of the following two theorems are almost similar to the proof of the Theorem 4.1.4 and Theorem 4.1.5 of the previous section.

Theorem 4.2.3 Let $S \times T$ be the semidirect product of a semigroup $S$ and a $\Gamma$-semigroup $T$ corresponding to a given antimorphism $\Phi : S \not\rightarrow \text{End}(T)$ and let $(s, t) \in S \times T$, then
(i) if $(s', t') \in V^2_u((s, t))$ then $(s', t') \in V^2_u((s, t^e t))$. In particular if $s \in E(S)$, then $(s, (s^e t^e t)) \in V^2_u((s, t))$ and
(ii) if $t^e$ is an $\alpha$-idempotent and $s' \in V(s)$, then $(s', t^e t') \in V^2_u((s, t'))$.

Theorem 4.2.4 Let $S$ be a semigroup and $T$ be a $\Gamma$-semigroup and $S \times T$ be the semidirect product corresponding to a given antimorphism $\Phi : S \not\rightarrow \text{End}(T)$. Moreover, if $t \in t^e T$ for every $e \in E(S)$ and every $t \in T$, then
(i) $(e, t)$ is an $\alpha$-idempotent if and only if $e \in E(S)$ and $t^e$ is an $\alpha$-idempotent and
(ii) if $(e, t)$ is an $\alpha$-idempotent, then $(e, t^e) \in V^2_u((e, t))$.

We now give a necessary and sufficient condition for the semidirect product of a semigroup and a $\Gamma$-semigroup to be a right orthodox $\Gamma$-semigroup.

Theorem 4.2.5 Let $S$ be a semigroup and $T$ be a $\Gamma$-semigroup. Let $\Phi : S \not\rightarrow \text{End}(T)$ be a given antimorphism. Then the semidirect product $S \times T$ is a right(resp. left) orthodox $\Gamma$-semigroup if and only if
(i) $S$ is an orthodox semigroup and $T^e$ is a right(resp. left) orthodox $\Gamma$-semigroup for every $e \in E(S)$.
(ii) for every $e \in E(S)$ and every $t \in T$, $t^e$ is a right orthodox $T$-semigroup. Then by Theorem 4.2.2, $S$ is a right orthodox $\Gamma$-semigroup and $T^e$ is a right orthodox $\Gamma$-semigroup for every $e \in E(S)$. For (iii), let $(e, t) \in S \times T$ with $e \in E(S)$ and let $(e', t') \in V^T_\alpha((e, t))$ for some $\alpha, \beta \in \Gamma$. Then by Theorem 4.2.3, $(e, t), (e', t') \in V^T_\alpha((e, t^e))$. Thus $V^T_\alpha((e, t)) \cap V^T_\alpha((e, t^e)) \neq \emptyset$ and hence by Theorem 3.1.5, $V^T_\alpha((e, t)) = V^T_\alpha((e, t^e))$. So $(e, (t')e\beta e\alpha e't) \in V^T_\alpha((e, t))$. Thus $(e, t) = (e, t)\alpha(e, (t')e\beta e\alpha e't)\beta(e, t) = (e, t\alpha(t')e\beta e\alpha e't)\beta(t\alpha(t')e\beta e\alpha e't)$ and hence $t = t\alpha(t')e\beta e\alpha e'(t')e\beta e\alpha e't \in t^eT$. 

For (iii) we shall first show that for an $\alpha$-idempotent $t^e$ of $T$ if $e \in E(S)$, $t^e$ is an $\alpha$-idempotent for any $e' \in V(e)$. If $e \in E(S)$ and $t^e$ is an $\alpha$-idempotent, then by Theorem 4.2.4, $(e, t) \in V^T_\alpha((e, t))$. Again since $t^e$ is an $\alpha$-idempotent $(e, t^e)$ is also an $\alpha$-idempotent and thus $(e, t^e) \in V^T_\alpha((e, t^e))$. Thus $(e, t) = (e, t)\alpha(e, t^e)\alpha(e, t)$ and by Theorem 4.2.4, $(e, t^e) \in V^T_\alpha((e, t^e))$ i.e., $(e, t^e) \in V^T_\alpha((e, t^e))$. Thus $(e, t) = (e, t)\alpha(e', t^e)\alpha(e, t) = (e e' e, t^e e' e \alpha e \alpha) = (e, t^e \alpha e \alpha) = (e, t^e \alpha e \alpha)$ (since $t = t^e \beta u$ for some $\beta \in \Gamma$, $u \in T$, $t^e \alpha e \alpha = t$). So $t = t^e \alpha e \alpha$ and hence $t^e = (t^e \alpha e \alpha)^e = t^e \alpha e \alpha$. Thus $t^e$ is an $\alpha$-idempotent. Let $e, g \in E(S)$ and suppose that $t^e$ is an $\alpha$-idempotent for $t \in T$, then $t^{\alpha e} = (t^e \alpha)^\gamma = t^{e g}$ i.e., $t^{e g}$ is an $\alpha$-idempotent and we have $e g \in E(S)$ and $ge \in V(eg)$ since $S$ is orthodox. Then by the above fact $t^{e g}$ is an $\alpha$-idempotent.

We now prove the converse part. Suppose $S$ and $T$ satisfy (i), (ii) and (iii). Let $(s, t) \in S \times T$. Since $S$ is regular, there exists $s' \in S$ such that $s = s s' s$ and $s' = s' s s'$. We take $e = s' s$, then $e \in E(S)$. By (ii) $t \in t^e T$ which implies $t = t^e \beta u$ for some $\beta \in \Gamma$, $u \in T$. Let $t' = v^e$ where $v^e \in V^T_\gamma(t^e)$ where $\gamma, \delta \in \Gamma$. Existence of $v$ is assured by the
regularity of $T^e$. Now $t^{s\gamma}(t')^s \delta t = t^{s\gamma}u^s \delta t^s \beta u = (t^{s\gamma}u^s \delta t^s) \beta u = t^s \beta u = t \ i.e., (s, t) = (ss's, ss's \gamma(t')^s \delta t) = (s, t) \gamma(s', t'; \delta(s, t)).$ Again $(t')^{s\delta} \delta t^t = (v^{s\delta})^{s\delta} \delta t^t \gamma v^{s\delta} = v^{s\delta} \delta t^t \gamma v^{s\delta} = v^{s\delta} \delta t^t \gamma v^{s\delta} = v^{s\delta} = v^t \ i.e., (s', t') = (ss's, (t')^{s\delta} \delta t^t \gamma t') = (s', t') \delta(s, t) \gamma(s', t').$ Thus we have $(s', t') \in V^t_2(s, t)$ which yields $S \times \Phi T$ is a regular $\Gamma$-semigroup.

Now let $(e, t)$ be an $\alpha$-idempotent and $(g, u)$ be a $\beta$-idempotent. Then by Theorem 4.2.4, $e, g \in E(S), t^e$ is an $\alpha$-idempotent and $u^g$ is a $\beta$-idempotent. By (iii) $t^{o\alpha}$ is an $\alpha$-idempotent, $u^{o\beta}$ is a $\beta$-idempotent and $t^{o\alpha t^{o\beta}} = (t^{o\alpha}t^{o\beta})^g = t^{o\alpha g}$ i.e., $t^{o\alpha g}$ is an $\alpha$-idempotent. By our assumption $e, g \in E(S)$ and $(t^{o\alpha u})^{o\beta} = t^{o\alpha g} \omega^{o\alpha} g$ is a $\beta$-idempotent. Thus by Theorem 4.2.4, $(e, t) \alpha(g, u) = (e, t)$ is a $\beta$-idempotent which shows that $S \times \Phi T$ is a right orthodox $\Gamma$-semigroup. The case of left orthodox $\Gamma$-semigroup is similar to that of right orthodox $\Gamma$-semigroup.

The next theorem gives a necessary and sufficient condition for a semidirect product of a semigroup and a $\Gamma$-semigroup to be a right inverse $\Gamma$-semigroup.

**Theorem 4.2.6** Let $S$ be a semigroup, $T$ be a $\Gamma$-semigroup and $\Phi : S \not\rightarrow \text{End}(T)$ be a given antimorphism. Then the semidirect product $S \times \Phi T$ is a right inverse $\Gamma$-semigroup if and only if

(i) $S$ is a right inverse semigroup and $T^e$ is a right inverse $\Gamma$-semigroup for every $e \in E(S)$ and

(ii) for every $e \in E(S)$ and every $t \in T^e$.

**Proof:** Let $S \times \Phi T$ be a right inverse $\Gamma$-semigroup. Then by Theorem 4.2.2, $S$ is a right inverse semigroup and $T^e$ is a right inverse $\Gamma$-semigroup for every $e \in E(S)$. Again since every right inverse $\Gamma$-semigroup is a right orthodox $\Gamma$-semigroup from the above theorem, condition (ii) holds.
Conversely, suppose that $S$ and $T$ satisfy (i) and (ii). Regularity of $S \times \varphi T$ can be proved by similar way of Theorem 4.2.5. Let $(e, t)$ be an $\alpha$-idempotent and $(g, u)$ be a $\beta$-idempotent in $S \times \varphi T$. Then by Theorem 4.2.4, $e, g \in E(S)$, $t^\varepsilon$ is an $\alpha$-idempotent, $u^\varphi$ is a $\beta$-idempotent. From (ii) $t^\varepsilon = t^\varepsilon \gamma v$ for some $\gamma \in \Gamma$, $v \in T$ and thus $t^\varepsilon \alpha t = t$ and similarly $u^\varphi \beta u = u$. So $u^\varphi \beta u = (u^\varphi \beta u)^{\varphi}$, $u^\varphi \beta u^\varphi$ and $t^\varepsilon \alpha t = (t^\varepsilon \alpha t)^{\alpha} = t^\varepsilon \alpha t^{\varphi}$ since $S$ is a right inverse semigroup. Again since $T^\varepsilon$ is a right inverse $\Gamma$-semigroup, we have $v^\varphi \alpha v^\varphi \beta v^\varphi = v^\varphi \beta v^\varphi$. Now by (ii) we have $u^\varphi \beta t = (u^\varphi \beta t)^{\varphi} \delta v_1$, for some $\delta \in \Gamma$, $v_1 \in T$ and hence $u^\varphi \beta t = u^\varphi \beta t^\varphi \delta v_1 = u^\varphi \beta t^{\varphi} \delta v_1$. Thus we have $(e, t) \alpha (g, u) \beta (e, t) = (e, u^\varphi \alpha u^\varphi \beta u^\varphi \delta v_1) = (g, u^\varphi \alpha u^\varphi \beta u^\varphi \delta v_1) = (g, u^\varphi \beta t) = (g, u \beta (e, t))$ which implies $S \times \varphi T$ is a right inverse $\Gamma$-semigroup.

**Theorem 4.2.7** Let $S$ be a semigroup, $T$ be a $\Gamma$-semigroup and $\Phi : S \not\rightarrow \text{End}(T)$ be a given antimorphism. Then the semidirect product $S \times \varphi T$ is a left inverse $\Gamma$-semigroup if and only if

(i) $S$ is a left inverse semigroup and $T^\varepsilon$ is a left inverse $\Gamma$-semigroup for every $e \in E(S)$ and

(ii) for every $e \in E(S)$ and every $t \in T$, $t = t^\varepsilon$.

**Proof**: Let $S \times \varphi T$ be a left inverse $\Gamma$-semigroup. Then by Theorem 4.2.2, $S$ is a left inverse semigroup and $T^\varepsilon$ is a left inverse $\Gamma$-semigroup. For (ii) let $(e, u)$ be an $\alpha$-idempotent in $S \times \varphi T$. Then $(e, u) = (e, u) \alpha (e, u) = (e, u^\varepsilon \alpha u)$ i.e., $u^\varepsilon \alpha u = u$. Again $(e, u^\varepsilon) \alpha (e, u^\varepsilon) = (e, u^\varepsilon \alpha u^\varepsilon)$ which yields $(e, u^\varepsilon)$ is an $\alpha$-idempotent and we have $(e, u^\varepsilon) \alpha (e, u) = (e, u^\varepsilon \alpha u) = (e, u)$. Since $S \times \varphi T$ is a left inverse $\Gamma$-semigroup, $(e, u) = (e, u) \alpha (e, u) = (e, u) \alpha (e, u) \alpha (e, u) = (e, u \varepsilon u \alpha \alpha u^\varepsilon \alpha u^\varepsilon) = (e, u \varepsilon u \alpha u^\varepsilon) = (e, u^\varepsilon)$ i.e., $u = u^\varepsilon$. Thus if $(e, u)$ is an $\alpha$-idempotent then $u = u^\varepsilon$. Now let $(e, t) \in S \times \varphi T$ with $e \in E(S)$ and let $(e', t') \in V_{\gamma}^\varepsilon((e, t))$ for some $\gamma, \delta \in \Gamma$. Then we get
\( e' \in V(e), \ t^{\delta e} \gamma(t')^\delta t = t \) i.e., \( t^{\delta e} \gamma(t')^\delta t^{\delta e} = t^{\delta e} \) which implies \( t^{e'} \gamma(t')^\delta t^{e'} = t^{e'} \).

Since \((e', (t')^\delta t) = (e', t')^\delta (e, t)\) and \(S \times \varphi T\) is left orthodox (since it is left inverse), \((e', (t')^\delta t)\) is a \(\gamma\)-idempotent and hence \((t')^\delta t = ((t')^\delta t)^{e'} = (t')^\delta t^{e'}\). Thus \(t^{e'} = t^{e'} \gamma(t')^\delta t^{e'} = t^{e'} \gamma(t')^\delta t = t\). Thus \(t^e = (t^e)^e = t^e = t\).

Conversely suppose that \(S\) and \(T\) satisfy (i) and (ii). Let \((s, t) \in S \times \varphi T\) and \(e \in E(S)\). Since \(S\) is regular, there exists \(s' \in S\) such that \(s' \in V(s)\). From (ii) we have \(t = t^e\). Since \(T^e\) is regular there exists \(v \in T\) such that \(v^e \in V^e(t^e)\). We now take \(t^e = v^e\). Now \(t^{s'} \gamma(t')^{s} \delta t = t^{s'} \gamma(t')^{s} \delta t^{e} = t^{s'} \gamma v^{e} \delta t^{e} = t^{e} = t\) and \((t')^{s'} \delta t^{e} \gamma t' = (v^s)^{s'} \delta t' \gamma v = v^{s'} \delta t' \gamma v^{s'} = v^{s'} \delta t' \gamma v^{s'} = v^{s'} = (v^{s'})^{s'} = v' = t\).

Thus we have \((s', t') \in V^e(s, t)\). Hence \(S \times \varphi T\) is regular. Now let \((e, t)\) be an \(\alpha\)-idempotent and \((g, u)\) be a \(\beta\)-idempotent. Then \(e^2 = e\) and \(t = t^e \alpha t = t \alpha t\) (by (ii)) and similarly \(g^2 = g\) and \(u \beta u = u\) i.e., \(e, g \in E(S)\) and \(t\) is an \(\alpha\)-idempotent, \(u\) is a \(\beta\)-idempotent. Thus we have \((e, t) \beta(g, u) \alpha(e, t) = (e^2 g, t^2 \beta u \alpha t)\) (by (ii)) = \((e, t^2 \beta u) = (e, t) \beta(g, u)\). Thus \(S \times \varphi T\) is a left inverse \(\Gamma\)-semigroup.

### 4.3 Wreath product of a semigroup and a \(\Gamma\)-semigroup

In this section we introduce the notion of wreath product of a semigroup \(S\) and a \(\Gamma\)-semigroup \(T\). Let \(X\) be a nonempty set. Consider the set \(T^X\) of all mappings from \(X\) to \(T\). For \(f, g \in T^X\) and \(\alpha \in \Gamma\), define \(f \cdot g\) such that \(T^X \times \Gamma \times T^X \rightarrow T^X\) by \((f \cdot g)(x) = f(x) \alpha g(x)\).

Before going to establish the relation between \(T\) and \(T^X\) we assume \(\Gamma = \{\alpha\}\), a set consisting of single element. Then \((T, \cdot)\) becomes a semigroup where \(a \cdot b = aab\) and \(T^X\) also becomes a semigroup where \(f \cdot g = f \cdot g\). Suppose \(T\) is a regular \(\Gamma\)-semigroup. Then \((T, \cdot)\) is a regular semigroup. Let \(f \in T^X\) and let \(x \in X\). Now \(f(x) \in T\) and
V(f(x)) \neq \emptyset. We define \( g: X \to T \) so that \( g(x) \in V(f(x)) \). Hence for each \( x \in X \) we can choose a \( g(x) \) such that \( f(x)g(x)f(x) = f(x) \). Hence \( fgf = f \) which implies that \((T^X, \cdot)\) is a regular semigroup and consequently \( T^X \) is a regular \( \Gamma \)-semigroup. In general we cannot extend the process when \( \Gamma \) contains more than one element. To explain this we consider the following example.

**Example 4.3.1** Let \( T = \{(a,0) : a \in Q\} \cup \{(0,b) : b \in Q\} \), \( Q \) denote the set of all rational numbers. Let \( \Gamma = \{(0,5),(0,1),(3,0),(1,0)\} \). Defining \( T \times \Gamma \times T \to T \) by \((a,b)(\alpha,\beta)(c,d) = (a\alpha c,b\beta d)\) for all \((a,b),(c,d) \in T \) and \((\alpha,\beta) \in \Gamma \), we can show that \( T \) is a \( \Gamma \)-semigroup. Now let \((a,0) \in T \). If \( a = 0 \) then \((a,0) \) is regular. Suppose \( a \neq 0 \), then \((a,0)(3,0)(\frac{1}{3a},0)(1,0)(a,0) = (a,0) \). Similarly we can show that \((0,b) \) is also regular. Hence \( T \) is a regular \( \Gamma \)-semigroup. Let us now take a set \( X = \{x,y\} \), the set consisting of two elements and let us define a mapping \( f: X \to T \) by \( f(x) = (2,0) \) and \( f(y) = (0,3) \).

We now show that \( f \) is not regular in \( T^X \). If possible let \( f \) be regular. Then there exists a mapping \( g: X \to T \) and two elements \( \alpha,\beta \in \Gamma \) such that \( f\alpha g\beta f = f \). i.e., \( f(p)\alpha g(p)\beta f(p) = f(p) \) for all \( p \in X \). Now if \( p = x \), then \( \alpha,\beta \notin \{(0,5),(0,1)\} \), since the first component of \( f(x) \) is nonzero. But if \( p = y \), then \( \alpha,\beta \in \{(0,5),(0,1)\} \), since the second component of \( f(y) \) is nonzero. Thus a contradiction arises. Hence \( T^X \) is not a regular \( \Gamma \)-semigroup.

Let \( S \) be a \( \Gamma \)-semigroup and \( e \) be a left \( \alpha \)-unity. Then \( STe \) is a left ideal such that \( e = eae \in STe \). Also we note that the element \( e \) is both left and right \( \alpha \)-unity of \( STe \) in \( STe \).

Suppose \( S \) is a regular \( \Gamma \)-semigroup with a left \( \alpha \)-unity \( e \). Then we show that \( STe \) is a regular \( \Gamma \)-semigroup with a unity. We only show that \( STe \) is regular. Let \( a\gamma e \in STe \). Since \( S \) is regular there exist \( \beta,\delta \in \Gamma \) and \( b \in S \) such that \( a\gamma e = a\gamma e\beta\delta a\gamma e \) i.e.,
\( a\gamma = a\gamma e b\delta e a\gamma e = (a\gamma e)\beta (b\delta e)\alpha (a\gamma e). \) Since \( b\delta e \in STe, a\gamma e \) is regular. Hence \( STe \) is a regular \( \Gamma \)-semigroup.

Let us now consider \( T \) with a left \( \gamma \)-unity \( e \) and a right \( \delta \)-unity \( g \). Then the constant mapping \( C_e : X \rightarrow T \) which is defined by \( C_e(x) = e \) for all \( x \in X \) is a left \( \gamma \)-unity of \( T^X \). Similarly the constant mapping \( C_g \) is a right \( \delta \)-unity of \( T^X \).

**Theorem 4.3.2** Let \( T \) be a \( \Gamma \)-semigroup with a left \( \gamma \)-unity and a right \( \delta \)-unity for some \( \gamma, \delta \in \Gamma \). Then

(i) \( T^X \) is a regular \( \Gamma \)-semigroup if and only if \( T \) is a regular \( \Gamma \)-semigroup,

(ii) \( T^X \) is a right (resp. left) orthodox \( \Gamma \)-semigroup if and only if \( T \) is so and

(iii) \( T^X \) is a right (resp. left) inverse \( \Gamma \)-semigroup if and only if \( T \) is a right (resp. left) inverse \( \Gamma \)-semigroup.

**Proof:** \( C_t, t \in T \) denote the mapping in \( T^X \) such that \( C_t(x) = t \) for all \( x \in X \). Then it is clear that \( (C_t)\alpha(C_a) = C_{(ta)} \) which shows that \( C_t \) is an \( \alpha \)-idempotent if and only if \( t \) is an \( \alpha \)-idempotent. Again we have that if \( f \) is an \( \alpha \)-idempotent in \( T^X \) then \( f(x) \) is an \( \alpha \)-idempotent in \( T \) for all \( x \in X \).

(i) Assume that \( T^X \) is a regular \( \Gamma \)-semigroup. Then for each \( t \in T \) there exist \( f \in T^X \) and \( \alpha, \beta \in \Gamma \) such that \( C_t\alpha f \beta C_t = C_t \) so that \( t\alpha f(x)\beta t = t \) for all \( x \in X \) which shows that \( t \) is regular in \( T \). Consequently \( T \) is a regular \( \Gamma \)-semigroup. Conversely let \( T \) be regular and let \( e \) be a left \( \gamma \)-unity and \( g \) be a right \( \delta \)-unity of \( T \). Then for each \( f \in T^X \) and for each \( x \in X \), \( f(x) \in T \) is a regular element and hence there exists a triplet \( (\alpha_t, t_e, \beta_e) \in \Gamma \times T \times \Gamma \) such that \( f(x)\alpha_t t_e \beta_e f(x) = f(x) \).

i.e., \( f(x) = (f(x)\delta g)\alpha_t t_e \beta_e (e\gamma f(x)) = f(x)\delta (g\alpha_t t_e \beta_e e)\gamma f(x) \). Define \( h : X \rightarrow T \) by \( h(x) = g\alpha_t t_e \beta_e e \). Then for all \( y \in X \), we have
\[(f \delta h \gamma f)(y) = f(y)\delta h(y)\gamma f(y)\]
\[= f(y)\delta g \alpha t \gamma f(y)\]
\[= f(y)\alpha_t \gamma f(y)\]
Hence \(g\) is regular in \(T^X\). Consequently \(T^X\) is a regular \(\Gamma\)-semigroup.

(ii) Let \(t, u \in T\) such that \(t\) be an \(\alpha\)-idempotent and \(u\) be a \(\beta\)-idempotent. Then \(C_t\) is an \(\alpha\)-idempotent and \(C_u\) is a \(\beta\)-idempotent in \(T^X\). Now if \(T^X\) is a right orthodox \(\Gamma\)-semigroup then \((C_t \alpha C_u) \beta (C_t \alpha C_u) = C_t \alpha C_u\) i.e., \(tou\) is a \(\beta\)-idempotent in \(T\) which implies \(T\) is also a right orthodox \(\Gamma\)-semigroup. Similarly we can show that if \(T^X\) is a left orthodox \(\Gamma\)-semigroup then \(T\) is so. Let \(f\) be an \(\alpha\)-idempotent and \(h\) be a \(\beta\)-idempotent in \(T^X\). Let us now suppose that \(T\) is a right (resp. left) orthodox \(\Gamma\)-semigroup. Then \(f(x)\alpha h(x)(\text{ resp. } f(x)\beta h(x))\) is a \(\beta\)-idempotent ( resp. \(\alpha\)-idempotent ). Hence \(T^X\) is a right (resp. left) orthodox \(\Gamma\)-semigroup.

(iii) Let \(T^X\) be a right (resp. left) inverse \(\Gamma\)-semigroup and let \(t, u \in T\) such that \(t\) is an \(\alpha\)-idempotent and \(u\) be a \(\beta\)-idempotent. Then \(C_t\) is an \(\alpha\)-idempotent and \(C_u\) is a \(\beta\)-idempotent in \(T^X\) and \(C_t \alpha C_u \beta C_t = C_u \beta C_t\) (resp. \(C_t \beta C_u \alpha C_t = C_t \beta C_u\)). Thus we have \(tou \alpha t = u \beta t\) (resp. \(t \beta u \alpha t = t \beta u\)) which implies that \(T\) is a right (resp. left) inverse \(\Gamma\)-semigroup. Again let \(T\) be a right (resp. left) inverse \(\Gamma\)-semigroup. Let \(f\) be an \(\alpha\)-idempotent and \(h\) be a \(\beta\)-idempotent in \(T^X\). Then \(f(x)\alpha h(x) \beta f(x) = h(x) \beta f(x)\) (resp. \(f(x)\beta h(x) \alpha f(x) = f(x) \beta h(x)\)) for all \(x \in X\) i.e., \(f \alpha h \beta f = h \beta f\) (resp. \(f \beta h \alpha f = f \beta h\)). Thus \(T^X\) is a right (resp. left) inverse \(\Gamma\)-semigroup.

Let us now suppose that the semigroup \(S\) acts on \(X\) from the left i.e., \(sx \in X, s(rx) = (sr)x\) and \(1x = x\) if \(S\) is a monoid, for every \(r, s \in S\) and every \(x \in X\). If \(S\) acts on \(X\) from left we call it left \(S\) set \(X\).

For every \(\Gamma\)-semigroup \(T\), it is known that \(End(T)\) is a semigroup. Hence \(End(T^X)\)
is also a semigroup.

Let $S$ be a semigroup, $T$ a $\Gamma$-semigroup and $X$ a nonempty set. Suppose $S$ acts on $X$ from left. Define $\Phi : S \to \text{End}(T^X)$ by $(\Phi(s))(f)(x) = f(sx)$ for all $s \in S$, $f \in T^X$ and $x \in X$. We now verify that $\Phi(s) \in \text{End}(T^X)$. For this, let $f, g \in T^X$, $\alpha \in \Gamma$ and $x \in X$. Then $((\Phi(s))(fag))(x) = (fag)(sx) = f(sx)g(sx) = ((\Phi(s))(f))(\alpha((\Phi(s))(g))(x) = ((\Phi(s))(f))\alpha((\Phi(s))(g))(x)$. Hence $\Phi(s)(fag) = ((\Phi(s))(f))\alpha((\Phi(s))(g))$, which implies that $\Phi(s) \in \text{End}(T^X)$.

Let us now verify that $\Phi : S \to \text{End}(T^X)$ is a semigroup antimorphism. For this let $s_1, s_2 \in S$, $f \in T^X$ and $x \in X$. Then $((\Phi(s_1))(\Phi(s_2))(f))(x) = (\Phi(s_1))(\Phi(s_2))(f))(x) = (\Phi(s_2))(f)(s_1x) = f((s_2(s_1)(x))) = f((s_2s_1)x) = (\Phi(s_2s_1))(f)(x)$. Hence $\Phi(s_2s_1) = \Phi(s_1)\Phi(s_2)$.

For this antimorphism $\Phi : S \not\to \text{End}(T^X)$ we can define the semidirect product $S \ltimes T^X$ of the semigroup $S$ and the $\Gamma$-semigroup $T^X$. We call this semidirect product the wreath product of the semigroup $S$ and the $\Gamma$-semigroup $T$ relative to the left $S$-set $X$. We denote it by $SW_XT$. We also denote $\Phi(s)(f)(x)$ by $f^s(x)$. Hence $f^s(x) = f(sx)$.

If $|T| = 1$, then $|T^X| = 1$ and hence throughout the section we assume that $|T| \geq 2$.

We now give the relation between $T$ and $(T^X)^e$ for all $e \in E(S)$.

The proof of the following theorems are similar to the proof of Theorem 4.2.5 and Theorem 4.2.6 respectively. So we omit the proofs.

**Theorem 4.3.3** Let $S$ be a semigroup acting on the set $X$ from the left and $T$ be a $\Gamma$-semigroup with a left $\gamma$-unity and a right $\delta$-unity for some $\gamma, \delta \in \Gamma$. Then

(i) $T$ is a regular $\Gamma$-semigroup if and only if $(T^X)^e$ is a regular $\Gamma$-semigroup,
(ii) $T$ is a right (resp. left) orthodox $\Gamma$-semigroup if and only if $(T^X)^e$ is so and
(iii) $T$ is a right (resp. left) inverse $\Gamma$-semigroup if and only if $(T^X)^e$ is a right (resp.
Theorem 4.3.4 Let $S$ be a semigroup acting on the set $X$ from the left and $T$ be a \( r \)-semigroup with a left \( \gamma \)-unity and a right \( \delta \)-unity for some \( \gamma, \delta \in \Gamma \). Then the wreath product \( SW_X T \) is a right (resp. left) orthodox \( \Gamma \)-semigroup if and only if

(i) $S$ is an orthodox semigroup and \((T^X)^e\) is a right (resp. left) orthodox \( \Gamma \)-semigroup for every \( e \in E(S)\)

(ii) for every \( x \in X, f \in T^X \) and \( e \in E(S), f(x) \in f(ex)\Gamma T \) and

(iii) \( f(ex) \) is an \( \alpha \)-idempotent for every \( x \in X \), implies that \( f(gex) \) is an \( \alpha \)-idempotent for every \( g \in E(S) \) where \( e \in E(S), f \in T^X \).

We now prove the following theorem.

**Theorem 4.3.5** Let $S$ be an orthodox semigroup acting on the set $X$ from the left and $T$ be a right orthodox \( \Gamma \)-semigroup with a left \( \gamma \)-unity and a right \( \delta \)-unity for some \( \gamma, \delta \in \Gamma \). Then the following statements are equivalent:

(a) $S$ and \( T^X \) satisfy (ii) and (iii) of Theorem 4.3.4;

(b) $S$ permutes $X$ or $T$ is a \( T \)-group and \( geX \subseteq eX \) for every \( e, g \in E(S) \).

**Proof**: (a) \( \Rightarrow \) (b): Let us suppose that $T$ is not a \( \Gamma \)-group. Then there exists $z \in T$ such that $z\Gamma T \neq T$. Let $e_\gamma$ be a left \( \gamma \)-unity in $T$. For $x \in X$, define $f_x : X \to T$ by $f_x(y) = e_\gamma$ if $y = x$ and $f_x(y) = z$ if $y \neq x$. Then by (ii), $e_\gamma = f_x(x) \in f_x(gx)\Gamma T$ for every $g \in E(S)$. If $f_x(gx) = z$ then $e_\gamma \in z\Gamma T$. Thus $e_\gamma = zau$ for some $u \in T$ and $\alpha \in \Gamma$. This implies that $u = e_\gamma u = zauu_\gamma u$ for all $u \in T$. Hence $T = z\Gamma T$ which is a contradiction. Hence $f_x(gx) = e_\gamma$. Thus we can conclude that $gx = x$ for all $g \in E(S)$. Let $a \in S$ and $x, y \in X$ such that $ax = ay$. For $a' \in V(a), a'a \in E(S)$ and $x = (a'a)x = (a'a)y = y$. Again $(aa')x = x$ implies that $a(a'x) = x$. Hence for each
a ∈ S, the mapping \( f_a : X \rightarrow X \) defined by \( f_a(x) = ax \) is a permutation on \( X \). This means that \( S \) permutes \( X \).

Now \( T \) is a \( \Gamma \)-group. Note that \( e_\gamma \) is a \( \gamma \)-idempotent and since \( T \) is a \( \Gamma \)-group, \( E_\gamma(T) = \{e_\gamma\} \). Let \( t \neq e_\gamma \in T \) and \( e \in E(S) \). Define \( h : X \rightarrow T \) by \( h(x) = e_\gamma \)
if \( x \in eX \), otherwise \( h(x) = t \). Now \( h(ex) = e_\gamma \) for every \( x \in X \) and hence by (iii), \( h(gex) = e_\gamma \). This implies that \( geX \subseteq eX \) and hence \( geX \subseteq eX \) for all \( e, g \in E(S) \).

(b) ⇒ (a) : Let \( S \) permutes \( X \). Then for every \( e, g \in E(S) \) we have \( geX = ex = x \)
for every \( x \in X \) and hence \( f(x) = f(ex) = f(ex)de_\delta \) where \( e_\delta \) is the right \( \delta \)-unity of \( T \).
Hence (a) holds. We now suppose that \( T \) is a \( \Gamma \)-group and \( geX \subseteq eX \) for every \( e, g \in E(S) \). Let \( f \in T^X \), \( x \in X \) and \( e \in E(S) \). Since \( T \) is a \( \Gamma \)-group, \( f(x) \in T = f(ex)\Gamma T \).
Again by the given condition, for \( g \in E(S) \), \( gex = ey \) for some \( y \in X \). Now if \( f(ex) \) is
an \( \alpha \)-idempotent for every \( x \in X \) then \( f(gex) = f(ey) \) \( \in E_\alpha \). Hence the proof.

From Theorem 4.3.3 and 4.3.4 we conclude that

**Theorem 4.3.6** Let \( S \) be a semigroup acting on the set \( X \) from the left and \( T \) be a \( \Gamma \)-semigroup with a left \( \gamma \)-unity and a right \( \delta \)-unity for some \( \gamma, \delta \in \Gamma \). Then the wreath product \( SW_XT \) is a right orthodox \( \Gamma \)-semigroup if and only if

(i) \( S \) is an orthodox semigroup and \( T \) is a right orthodox \( \Gamma \)-semigroup and
(ii) \( S \) permutes \( X \) or \( T \) is a \( \Gamma \)-group and \( geX \subseteq eX \) for every \( e, g \in E(S) \).

The following theorem gives the necessary and sufficient condition for the wreath product to be right (resp. left) inverse \( \Gamma \)-semigroup.

**Theorem 4.3.7** Let \( S, T \) and \( X \) be as in Theorem 4.3.6. Then the wreath product \( SW_XT \) is a right inverse \( \Gamma \)-semigroup if and only if

(i) \( S \) is a right inverse semigroup and \( T \) is a right inverse \( \Gamma \)-semigroup and
(ii) \( S \) permutes \( X \) or \( T \) is a \( \Gamma \)-group.

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Proof: Suppose that $SW_xT$ is a right inverse $\Gamma$-semigroup. Then by Theorem 4.2.6 and Theorem 4.3.3, we have $S$ is a right inverse semigroup and $T$ is a right inverse $\Gamma$-semigroup and by Theorem 4.3.6 we have $S$ permutes $X$ or $T$ is a $\Gamma$-group.

Conversely suppose that $S, T$ and $X$ satisfy (i) and (ii). Then by Theorem 4.3.2 $T^X$ is a right inverse $\Gamma$-semigroup. If $T$ is a $\Gamma$-group, then $f(x) \in f(ex)T$ for every $f \in T^X$, $e \in E(S), x \in X$. If $S$ permutes $X$, then by regularity of $T$ we have $f(x) \in f(x)T = f(ex)T$ since $ex = x$ for every $e \in E(S)$. Hence by Theorem 4.2.6, $S \times_a T^X = SW_xT$ is a right inverse $\Gamma$-semigroup.

Theorem 4.3.8 Let $S, T$ and $X$ be as in Theorem 4.3.6. Then the wreath product $SW_xT$ is a left inverse $\Gamma$-semigroup if and only if $S$ is a left inverse semigroup and $T$ is a left inverse $\Gamma$-semigroup and $S$ permutes $X$.

Proof: By Theorem 4.2.7 and Theorem 4.3.3, we have $SW_xT$ is a left inverse $\Gamma$-semigroup if and only if $S$ is a left inverse semigroup and $T$ is a left inverse $\Gamma$-semigroup and $f(ex) = f(x)$ for every $f \in T^X, e \in E(S), x \in X$. To prove the theorem we have to prove that $f(ex) = f(x)$ for every $f \in T^X, e \in E(S), x \in X$ if and only if $S$ permutes $X$. Let $f(ex) = f(x)$ for every $f \in T^X, e \in E(S), x \in X$. Then we define $f_x : X \to T$ by $f_x(y) = t$ if $y = x$, otherwise $f_x(y) = t'$ where $t \neq t'$. Then for every $e \in E(S)$, $f_x(ex) = f_x(x) = t$. Hence $ex = x$ for every $e \in E(S)$ i.e., $S$ permutes $X$. Conversely suppose that $S$ permutes $X$. Then $ex = x$ for every $x \in X$ and $e \in E(S)$ and hence $f(ex) = f(x)$ for every $f \in T^X$. Hence the proof.