CHAPTER 3

Some Classes of Regular Γ-Semigroups

3 Introduction

Orthodox semigroups play an important role in the literature of semigroup theory. In the present chapter, like orthodox semigroup, we define right(resp. left) orthodox Γ-semigroup and intend to give a complete characterization of right(resp. left) orthodox Γ-semigroup. A semigroup $(S, \cdot)$ is called a orthodox semigroup if the set of idempotent elements form a subsemigroup. Naturally, we would ask whether we can extend orthodox semigroup to right(resp. left) orthodox Γ-semigroup so that right(resp. left) orthodox Γ-semigroup would have analogous properties as orthodox semigroup? In order to answer this question, we define right(resp. left) orthodox Γ-semigroup and study their properties in section 1.

In 1974 Nambooripad [41] introduced sandwich sets on a regular semigroup and Gomes [25] studied several results of a congruence on a regular semigroup by using sandwich sets. In section 2, we introduce right(resp. left) sandwich sets and study some properties of such sets.

A semigroup $S$ is an inverse semigroup if and only if every principal left and every principal right ideal of $S$ has a unique idempotent generator. By splitting left-right symmetry in inverse semigroups P. S. Venkatesan [72] defined right inverse semigroup in which every principal left ideal of $S$ has a unique idempotent generator and it is a natural generalization of inverse semigroup and right groups. In [51] Saha and Seth
introduced and studied inverse $\Gamma$-semigroup. In section 3, we introduce right(resp. left) inverse $\Gamma$-semigroups. In this section we investigate the maximum idempotent separating congruence on right inverse $\Gamma$-semigroups.

3.1 Right orthodox $\Gamma$-semigroups

In [59] Saha introduced the notion of orthodox $\Gamma$-semigroups (Definition 1.6.3) and studied some results. In this section we want to introduce the notion of right orthodox $\Gamma$-semigroups and study the structure of these type of $\Gamma$-semigroups.

Definition 3.1.1 A regular $\Gamma$-semigroup $S$ is called a right(resp. left) orthodox $\Gamma$-semigroup if for any $\alpha$-idempotent $e$ and $\beta$-idempotent $f$ of $S$, $eaf$ (resp. $fae$) is a $\beta$-idempotent.

Hence we find that every orthodox $\Gamma$-semigroup is a right orthodox $\Gamma$-semigroup as well as left orthodox $\Gamma$-semigroup. The following example shows that there is a right orthodox $\Gamma$-semigroup which is not an orthodox $\Gamma$-semigroup.

Example 3.1.2 Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. $S$ denotes the set of all mappings from $A$ to $B$. Here members of $S$ will be described by the images of the elements 1, 2, 3. For example the map $1 \rightarrow 4, 2 \rightarrow 5, 3 \rightarrow 4$ will be written as $(4, 5, 4)$ and $(5, 5, 4)$ denotes the map $1 \rightarrow 5, 2 \rightarrow 5, 3 \rightarrow 4$. A map from $B$ to $A$ will be described in the same fashion. For example $(1, 2)$ denotes $4 \rightarrow 1, 5 \rightarrow 2$. Now $S = \{(4, 4, 4), (4, 4, 5), (4, 5, 4), (4, 5, 5), (5, 5, 5), (5, 4, 5), (5, 4, 4), (5, 5, 4)\}$ and let $\Gamma = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$. Let $f, g \in S$ and $\alpha \in \Gamma$. We define $f \alpha g$ by $(f \alpha g)(a) = f \alpha (g(a))$ for all $a \in A$. So $f \alpha g$ is a mapping from $A$ to $B$ and hence $f \alpha g \in S$ and we can show that $(f \alpha g)\beta h = f \alpha (g\beta h)$ for all $f, g, h \in S$ and $\alpha, \beta \in \Gamma$. The operation in the $\Gamma$-semigroup $S$ is described by the following table.
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The box obtained as the intersection of the first row and first column contains the elements,

\[(4,4,5)(3,1) (4,4,4) = (5,5,5), \ (4,4,5)(3,1) (4,4,5) = (5,5,4), \ (4,4,5)(3,1) (4,5,4) = (5,4,5),

From the above table we see that (5, 5, 5) and (4, 4, 4) are \(\alpha\)-idempotents for all \(\alpha \in \Gamma\). (2, 3)-idempotents are (4, 4, 5) and (5, 4, 5). (1, 2)-idempotents are (4, 5, 4) and (4, 5, 5).
and (3,1)-idempotents are (5,4,4) and (5,5,4). Thus all elements are idempotent and hence regular i.e, $S$ is a regular $\Gamma$-semigroup.

For an $\alpha$-idempotent $f$, we see from the composition table that $fag = g$ for all $g \in S$. Thus $S$ is a right orthodox $\Gamma$-semigroup. Again (5,5,4) is (3,1)-idempotent and (5,4,5) is (2,3)-idempotent but (5,5,4)(2,3)(5,4,5) = (4,5,4) which is not (3,1)-idempotent. Hence $S$ is not a left orthodox $\Gamma$-semigroup.

Since every orthodox $\Gamma$-semigroup is a right orthodox $\Gamma$-semigroup, we also like to extend different results of orthodox $\Gamma$-semigroups to right orthodox $\Gamma$-semigroups. We can also develop the same results in left orthodox $\Gamma$-semigroups. So in this chapter we only deal with right orthodox $\Gamma$-semigroups.

Theorem 3.1.3 A regular $\Gamma$-semigroup $S$ is a right orthodox $\Gamma$-semigroup if and only if for any $\alpha$-idempotent $e$ with $V^\beta_\alpha(e) \neq \emptyset$, each member of $V^\beta_\alpha(e)$ is a $\beta$-idempotent.

Proof: Let $S$ be a right orthodox $\Gamma$-semigroup and $e$ be an $\alpha$-idempotent. Let $a \in V^\beta_\alpha(e)$. Then $a\beta e$ is an $\alpha$-idempotent and $e\alpha a$ is a $\beta$-idempotent. Since $S$ is a right orthodox $\Gamma$-semigroup $(a\beta e)\alpha(e\alpha a)$ is a $\beta$-idempotent. Now $a = a\beta e\alpha a = (a\beta e)\alpha(e\alpha a)$ shows that $a$ is a $\beta$-idempotent and hence each member of $V^\beta_\alpha(e)$ is a $\beta$-idempotent.

Conversely, suppose that the given condition holds. Let $e$ be an $\alpha$-idempotent, $f$ be a $\beta$-idempotent and $x \in V^\delta_\gamma(e\alpha f)$ where $\gamma, \delta \in \Gamma$. i.e., we have $x\delta e\alpha f\gamma x = x$ and $e\alpha f\gamma x\delta e\alpha f = e\alpha f$. \[\ldots (3.1.1)\]

Now $(f\gamma x\delta e)\alpha(f\gamma x\delta e) = f\gamma(x\delta e\alpha f\gamma x)\delta e = f\gamma x\delta e$ i.e. $f\gamma x\delta e$ is an $\alpha$-idempotent. Again $(e\alpha f)\beta(f\gamma x\delta e)\alpha(e\alpha f) = e\alpha f\gamma x\delta e\alpha f = e\alpha f$ (by (3.1.1)) and $(f\gamma x\delta e)\alpha(e\alpha f)\beta(f\gamma x\delta e) = f\gamma x\delta e\alpha f\gamma x\delta e = f\gamma x\delta e$. Hence $e\alpha f \in V^\beta_\alpha(f\gamma x\delta e)$. Since $f\gamma x\delta e$ is an $\alpha$-idempotent, we have $e\alpha f$ is a $\beta$-idempotent which implies $S$ is a right orthodox $\Gamma$-semigroup.

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Within a group $G$ we have a very useful property that $(ab)^{-1} = b^{-1}a^{-1}$ and it is reasonable to ask whether some version of this holds in a regular $\Gamma$-semigroup. In general it is not held. The following theorem shows that similar type of result holds in right orthodox $\Gamma$-semigroups.

**Theorem 3.1.4** A regular $\Gamma$-semigroup $S$ is a right orthodox $\Gamma$-semigroup if and only if for $a, b \in S$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$, $a' \in V_{\alpha_1}^\beta(a)$ and $b' \in V_{\beta_1}^\alpha(b)$, we have $b'\beta_2a' \in V_{\beta_1}^\alpha(a\alpha_1b)$.

**Proof**: Let us assume that $S$ is a right orthodox $\Gamma$-semigroup and $a' \in V_{\alpha_1}^\beta(a)$, $b' \in V_{\beta_1}^\alpha(b)$. Then $aa_1a'a = a$, $a', a_1a'a = a'$, $b_1/b_2b = b$ and $b_1/b_2b = b'$. Now $a'a_2$ is an $\alpha_1$-idempotent and $b_1/b_2$ is a $\beta_2$-idempotent. Hence $(a'a_2)a_1(b_1/b_2)$ is a $\beta_2$-idempotent and $(b_1/b_2)a_1(b_1/a_1a)$ is an $\alpha_1$-idempotent. Now $aa_1a'a_2\alpha_2b = aa_1a'a_2\alpha_2b, b/b_2b = aa_1a'a_2\alpha_2b, b/b_2b = aa_1a'a_2\alpha_2b, b/b_2b = aa_1a'a_2\alpha_2b, b/b_2b = aa_1a'a_2\alpha_2b$.

Conversely, let the given condition hold. Let $e$ be an $\alpha_1$-idempotent and $f$ be a $\beta_2$-idempotent i.e., $e \in V_{\alpha_1}^\alpha(e)$ and $f \in V_{\beta_2}^\beta(f)$. Then by the given condition we have $eaf \in V_{\alpha_1}^\alpha(f\beta e)$ i.e., $eaf \beta \beta \beta \alpha eaf = eaf$ i.e., $(eaf)\beta \beta \beta (eaf) = eaf$ i.e., $eaf$ is a $\beta_2$-idempotent. Therefore $S$ is a right orthodox $\Gamma$-semigroup.

We now give a characterization theorem of a right orthodox $\Gamma$-semigroup within the class of regular $\Gamma$-semigroups.

**Theorem 3.1.5** A regular $\Gamma$-semigroup $S$ is a right orthodox $\Gamma$-semigroup if and only if for $a, b \in S$, $V_{\alpha_1}^\beta(a) \cap V_{\alpha_1}^\beta(b) \neq \emptyset$ for some $\alpha, \alpha_1, \beta \in \Gamma$ implies that $V_{\alpha_1}^\beta(a) = V_{\alpha_1}^\beta(b)$ for all $\delta \in \Gamma$.

**Proof**: Let $S$ be a right orthodox $\Gamma$-semigroup. Assume that $a' \in V_{\alpha_1}^\beta(a) \cap V_{\alpha_1}^\beta(b)$, $\delta \in \Gamma$ and $a* \in V_{\alpha_1}^\beta(a)$. Then we have $a'\beta a\alpha_1a' = a', a\alpha_1a'\beta a = a, a'\beta boxa' = a'$.
\[ b\alpha a'\beta b = b, \]
\[ a^\ast \delta a_{1} a_{*} = a^* \text{ and } a_{1} a_{*} a^\ast = a. \]
Now \( a^* \delta a = a^* \delta a_{1} a_{*} a^\ast = a^* \delta a_{1} a_{*} b_{0} a_{1} \beta a = (a^* \delta a) a_{1} (a' \beta b) a (a' \beta a) \). Thus \( a^* \delta a \mathcal{R} (a^* \delta a) a_{1} (a' \beta b) \). Again \( a^* \delta a \) is an \( \alpha_{1} \)-idempotent and \( a' \beta b \) is an \( \alpha \)-idempotent. Thus \( (a^* \delta a) a_{1} (a' \beta b) \) is an \( \alpha \)-idempotent. Hence by Theorem 1.5.7, we have

\[ (a^* \delta a) a_{1} (a' \beta b) (a^* \delta a) = a^* \delta a. \quad \ldots (3.1.2) \]

Now \( a' \beta a = a' \beta a \alpha_{1} a^* \delta a = a' \beta a \alpha_{1} a^* \delta a_{1} a \beta a a^* \delta a \text{ (by } 3.1.2) = a' \beta a \alpha_{1} a' \beta a a^* \delta a = a' \beta a a^* \delta a \text{ and hence} \]
\[ b a a' \beta a = b a a' \beta a a^* \delta a = b a a^* \delta a. \quad \ldots (3.1.3) \]

Again \( a \alpha_{1} a^* = a \alpha_{1} a' \beta a a^* = a \alpha_{1} a' \beta b a' a a^* = (a \alpha_{1} a') \beta (b a a') \beta (a \alpha_{1} a^*) \)
i.e., \( a \alpha_{1} a^* \mathcal{L} (b a a') \beta (a \alpha_{1} a^*) \). Since \( b a a' \) is a \( \beta \)-idempotent and \( a \alpha_{1} a^* \) is a \( \delta \)-idempotent,
\[ (b a a') \beta (a \alpha_{1} a^*) \text{ is a } \delta \text{-idempotent. Thus by Theorem 1.5.7 we have} \]
\[ (a \alpha_{1} a^*) \delta (b a a') \beta (a \alpha_{1} a^*) = a \alpha_{1} a^*. \quad \ldots (3.1.4) \]

Now \( a \alpha_{1} a' = a \alpha_{1} a^* \delta a_{1} a_{*} a' = a \alpha_{1} a^* \delta b a a' b a a_{1} a_{*} a' \text{ (by } 3.1.4) = a \alpha_{1} a^* \delta b a a' b a a_{1} a_{*} a' = a \alpha_{1} a^* \delta b a a'. \) Hence
\[ a \alpha_{1} a' \beta b = a \alpha_{1} a^* \delta b a a' \beta b = a \alpha_{1} a^* \delta b. \quad \ldots (3.1.5) \]

Now
\[
\begin{align*}
baa^* \delta b &= baa^* \delta a \alpha_{1} a^* \delta b \\
&= baa^* \delta a \alpha_{1} a' \beta b \quad (\text{by } 3.1.5) \\
&= baa' \beta a \alpha_{1} a' \beta b \quad (\text{by } 3.1.3) \\
&= baa' \beta b \\
&= b
\end{align*}
\]

\[ 65 \]
and

\[ a^* \delta b a a^* = a^* \delta a a, a^* \delta a a a^* \delta a a, a^* \]
\[ = a^* \delta a a, a^* \delta b a a^* \beta a a, a^* \quad \text{(by (3.1.3))} \]
\[ = a^* \delta a a, a^* \beta b a a^* \beta a a, a^* \quad \text{(by (3.1.5))} \]
\[ = a^* \delta a a, a^* \beta a a, a^* \]
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\[ = a^*. \]

Thus \( a^* \in V^\theta_a(b) \) i.e., \( V^\theta_a(a) \subseteq V^\theta_a(b) \). Similarly we can show that \( V^\theta_a(b) \subseteq V^\theta_a(a) \). Therefore we have \( V^\theta_a(a) = V^\theta_a(b) \) for all \( \delta \in \Gamma \).

Conversely assume that the given condition holds. Let \( e \) be an \( \alpha \)-idempotent and \( f \) be a \( \beta \)-idempotent. Let us consider \( e a f \). Since \( S \) is regular, there exist \( \gamma, \delta \in \Gamma \) and \( x \in S \) such that \( e a f \gamma \delta e a f = e a f \) and \( x \delta e a f \gamma x = x \). Let \( g = f \gamma \delta e \). Then \( g a g = f \gamma \delta e a f \gamma \delta e = f \gamma \delta e = g \) which shows that \( g \) is an \( \alpha \)-idempotent and hence \( f \gamma \delta e \in V^\alpha_a(f \gamma \delta e) \). Let \( h = e a f \gamma \delta e \). Then \( h \) is an \( \alpha \)-idempotent. Also

\[ f \gamma \delta e a e a f \gamma \delta e a f \gamma \delta e = f \gamma \delta e a f \gamma \delta e a f \gamma \delta e = f \gamma \delta e \]

and

\[ e a f \gamma \delta e a f \gamma \delta e a e a f \gamma \delta e = (e a f \gamma \delta e) \alpha (e a f \gamma \delta e) \alpha (e a f \gamma \delta e) = e a f \gamma \delta e \]

i.e. \( f \gamma \delta e \in V^\alpha_a(e a f \gamma \delta e) \). Hence \( V^\alpha_a(g) \cap V^\alpha_a(h) \neq \emptyset \). Then by the given condition \( V^\theta_a(g) = V^\theta_a(h) \) for all \( \theta \in \Gamma \). But \( e a f \beta f \gamma \delta e a e a f = e a f \gamma \delta e a f = e a f \) and \( f \gamma \delta e a e a f \beta f \gamma \delta e = f \gamma \delta e a f \gamma \delta e = f \gamma \delta e \) i.e., \( e a f \in V^\theta_a(f \gamma \delta e) = V^\theta_a(e a f \gamma \delta e) \)

i.e., \( e a f = e a f \beta e a f \gamma \delta e a f = e a f \beta e a f \gamma \delta e a f = e a f \beta e a f \) i.e.,

\( e a f \) is a \( \beta \)-idempotent which shows that \( S \) is a right orthodox \( \Gamma \)-semigroup.
Theorem 3.1.6 In a regular $\Gamma$-semigroup $S$, the following are equivalent:

(i) $S$ is a right orthodox $\Gamma$-semigroup;
(ii) for any $\alpha$-idempotent $e$ and $\beta$-idempotent $f$, $V_\alpha^\delta (e\alpha f) = V_\beta^\delta (f\beta e)$ for all $\delta \in \Gamma$;
(iii) for any $\alpha$-idempotent $e$ and $\beta$-idempotent $f$, if $eRf$ then $V_\alpha^\delta (e) = V_\beta^\delta (f)$ for all $\delta \in \Gamma$.

Proof: (i) $\Rightarrow$ (ii): Let $S$ be a right orthodox $\Gamma$-semigroup in which $e$ be an $\alpha$-idempotent and $f$ be a $\beta$-idempotent. Then $e\alpha f$ and $f\beta e$ are $\beta$-idempotent and $\alpha$-idempotent respectively i.e., we have $e\alpha f \in V_\beta^\delta (e\alpha f)$.

\[
\begin{align*}
(e\alpha f) \beta (f\beta e) \alpha (e\alpha f) &= e\alpha f \beta e \alpha f = e\alpha f \\
(f\beta e) \alpha (e\alpha f) \beta (f\beta e) &= f\beta e \alpha f \beta e = f\beta e
\end{align*}
\]

i.e., $e\alpha f \in V_\beta^\delta (f\beta e)$. Thus $V_\beta^\delta (e\alpha f) \cap V_\beta^\delta (f\beta e) \neq \emptyset$. Hence by Theorem 3.1.5, $V_\beta^\delta (e\alpha f) = V_\beta^\delta (f\beta e)$ for all $\delta \in \Gamma$.

(ii) $\Rightarrow$ (iii): Assume (ii). Let $e$ be an $\alpha$-idempotent and $f$ be a $\beta$-idempotent such that $eRf$. Then $e\alpha f = f$ and $f\beta e = e$. Hence from the given assumption (iii) follows.

(iii) $\Rightarrow$ (i): Assume (iii). Let $e$ be an $\alpha$-idempotent and $a \in V_\alpha^\delta (e)$. Then $e\alpha aRe$.

Now $e\alpha a$ is a $\beta$-idempotent and $e$ is an $\alpha$-idempotent. By (iii), $V_\alpha^\delta (e) = V_\beta^\delta (e\alpha a)$ for all $\delta \in \Gamma$. In particular we have, $V_\alpha^\delta (e) = V_\beta^\delta (e\alpha a)$. Hence $a \in V_\beta^\delta (e\alpha a)$. Therefore $a = a\beta e\alpha a\beta a = a\beta a$ i.e., $a$ is a $\beta$-idempotent and by Theorem 3.1.3, $S$ is a right orthodox $\Gamma$-semigroup.

3.2 Sandwich sets in regular $\Gamma$-semigroups

In this section we introduce the notion of right sandwich set and left sandwich set on regular $\Gamma$-semigroup analogous to the sandwich set introduced by Nambooripad [41] on regular semigroups. We first recall the definition of sandwich set from [41].

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Definition 3.2.1 Let $S$ be a regular semigroup and $E$ be the set of all idempotents in $S$. Let $e, f \in E$. Then the set $S(e, f)$, defined by

$$S(e, f) = \{ g \in V(ef) \cap E : ge = fg = g \}$$

is nonempty and is called the sandwich set of $e$ and $f$.

We now generalize this concept in $\Gamma$-semigroups.

Theorem 3.2.2 Let $S$ be a regular $\Gamma$-semigroup and for $\alpha \in \Gamma$, $E_\alpha$ be the set of all $\alpha$-idempotents in $S$. Let $e \in E_\alpha$ and $f \in E_\beta$. Then the sets

$$RS(e, f)_{(\alpha, \beta)} = \{ g \in V_\beta^\alpha(\alpha f) \cap E_\alpha : gae = f\beta g = g \}$$

and

$$LS(e, f)_{(\alpha, \beta)} = \{ g \in V_\beta^\alpha(\beta f) \cap E_\beta : gae = f\beta g = g \}$$

are non-empty.

Proof: Since $S$ is a regular $\Gamma$-semigroup and $\alpha f \in S$, there exist $b \in S$ and $\gamma, \delta \in \Gamma$ such that $e\alpha f \gamma b = e\alpha f$ and $b \delta e \alpha f \gamma b = b$. Now $(e\alpha f)\beta(f\gamma b)e\alpha f = e\alpha(f\beta f)\gamma \delta (e\alpha f)e\alpha f = e\alpha f\gamma b e\alpha f = e\alpha f$ and $(f\gamma b e \alpha f)e\alpha f = f\gamma b e \alpha f\gamma b e = f\gamma b e$. Hence $f\gamma b e \in V_\beta^\alpha(e\alpha f)$. Thus $V_\beta^\alpha(e\alpha f) \neq \emptyset$. Let $x \in V_\beta^\alpha(e\alpha f)$ and let $g = f\beta xae$. We have $gag = (f\beta xae)e\alpha(f\beta xae) = f\beta(xae\alpha f\beta xae) = f\beta xae = g$. Thus $g \in E_\alpha$.

Again $gaea'f \beta g = f\beta xaeae\alpha f'f\beta xae = f\beta xaeaf'f\beta xae = f\beta xae = g$ and $e\alpha f\beta gae a'f = e\alpha f'f\beta xaeae\alpha f = e\alpha f'f\beta xaeaf = e\alpha f$ imply that $g \in V_\beta^\alpha(e\alpha f)$. Also $gae = f\beta xae = f\beta xae = g$ and $f\beta g = f\beta f\beta xae = f\beta xae = g$. Therefore $RS(e, f)_{(\alpha, \beta)} \neq \emptyset$. Similarly we can show that $LS(e, f)_{(\alpha, \beta)}$ is nonempty.
Definition 3.2.3 Let $S$ be a regular $\Gamma$-semigroup, $e$ and $f$ be $\alpha$-idempotent and $\beta$-idempotent respectively. Then the set $RS(e, f)_{(\alpha, \beta)}$ (resp. $LS(e, f)_{(\alpha, \beta)}$) described in Theorem 3.2.2 is called the right (resp. left) sandwich set of the $\alpha$-idempotent $e$ and the $\beta$-idempotent $f$.

The following theorem gives an alternative structure of sandwich sets.

Theorem 3.2.4 Let $S$ be a regular $\Gamma$-semigroup, $e$ and $f$ be $\alpha$-idempotent and $\beta$-idempotent respectively. Then the sets

$$RS(e, f)_{(\alpha, \beta)} = \{g \in V_f^\gamma(\alpha f) : gae = g = f \beta g \text{ and } e \alpha g \alpha f = e \alpha f\}$$

and

$$LS(e, f)_{(\alpha, \beta)} = \{g \in V_f^\gamma(e \beta f) : gae = g = f \beta g \text{ and } e \beta g \beta f = e \beta f\}.$$

Proof: Let $P = \{g \in V_f^\gamma(\alpha f) : gae = g = f \beta g \text{ and } e \alpha g \alpha f = e \alpha f\}$ and let $g \in RS(e, f)_{(\alpha, \beta)}$. Then $g \in E_\alpha$, $gae = g = f \beta g$ and $g \in V_f^\gamma(\alpha f)$. Now $e \alpha g \alpha f = e \alpha g \alpha f = e \alpha f \beta g \alpha e \alpha f = e \alpha f$. Hence $RS(e, f)_{(\alpha, \beta)} \subseteq P$. Next let $g \in P$. Now $gag = gae \alpha f \beta g = g$. Hence $g \in E_\alpha$ which shows that $P \subseteq RS(e, f)_{(\alpha, \beta)}$. Thus $P = RS(e, f)_{(\alpha, \beta)}$. Similar proof holds for $LS(e, f)_{(\alpha, \beta)}$.

Theorem 3.2.5 Let $S$ be a regular $\Gamma$-semigroup and $a, b \in S$. If $a' \in V_a^\delta(a), b' \in V_b^\gamma(b)$ and $g \in RS(a'\beta a, b\gamma b')_{(\alpha, \beta)}$ then $b' \delta g \alpha a' \in V_a^\delta(aab)$ and if $g \in LS(a'\beta a, b\gamma b')_{(\alpha, \beta)}$ then $b' \delta g \alpha a' \in V_a^\delta(ab\delta)$.

Proof: Let $e = a'\beta a$ and $f = b\gamma b'$. Then $eae = a'\beta a a' a = a' a a = e$. Hence $e$ is an $\alpha$-idempotent. Similarly $f$ is a $\delta$-idempotent and also by Theorem 3.2.2, $g$ is an $\alpha$-idempotent. Now $(aab) \gamma(b' \delta g \alpha a') \beta(aab) = a \alpha f \delta g \alpha e a b = a a g a b = a a' \beta a a g a b \gamma b' \delta b = a a e a g a f \delta b = a a e a f \delta b = a a a' \beta a a b a b \gamma b' \delta b = a a b$. Again $(b' \delta g \alpha a') \beta(aab) \gamma(b' \delta g \alpha a') = \ldots$
\[ b'\delta gaoa'b = b'\delta gaoa = b'\delta gao. \] Hence \( b'\delta gaoa' \in V_\alpha^{\beta}(aob) \). The second case is similar to the first case.

With the help of the above theorem we get the following interesting result.

**Corollary 3.2.6** For \( a, b \in S \), if \( V_\alpha^{\beta}(a) \) and \( V_\gamma^{\beta}(b) \) are nonempty subsets of \( S \) then \( V_\gamma^{\beta}(aob) \) is a nonempty subset of \( S \).

**Proof:** Let \( a' \in V_\alpha^{\beta}(a) \) and \( b' \in V_\gamma^{\beta}(b) \). Then from Theorem 3.2.2, we have \( RS(a'\beta a, b\gamma b')(\alpha,\beta) \neq \emptyset \). Let \( g \in RS(a'\beta a, b\gamma b')(\alpha,\beta) \). Then by Theorem 3.2.5, we get \( b'\delta gaoa' \in V_\gamma^{\beta}(aob) \). Hence the proof.

**Theorem 3.2.7** Let \( S \) be a regular \( \Gamma \)-semigroup, \( e, f \) be \( \alpha \)-idempotents and \( g \) be a \( \beta \)-idempotent. Then

(i) \( eLf \) implies \( RS(e, g)(\alpha,\beta) = RS(f, g)(\alpha,\beta) \) and \( LS(e, g)(\alpha,\beta) = LS(f, g)(\alpha,\beta) \);

(ii) \( eRf \) implies \( RS(g, e)(\beta,\alpha) = RS(f, g)(\beta,\alpha) \) and \( LS(g, e)(\beta,\alpha) = LS(f, g)(\beta,\alpha) \).

**Proof:** (i) Let \( eLf \). Then by Theorem 1.5.7, we have \( eaf = e \) and \( fae = f \). Let us suppose that \( h \in RS(e, g)(\alpha,\beta) \) i.e., \( h \in V_\beta^\alpha(\alpha g) \cap E_\alpha \) and \( hae = h = g\beta h \). Now \( haf = haeaf = hae = h \). Again \( hafag\beta h = haeafag\beta h = haeag\beta h = h \) and \( fag\beta hafag = fahafag = fahag = faeahag = faeag = fag \). Hence \( h \in V_\beta^\alpha(fag) \cap E_\alpha \) and \( haf = h = g\beta h \). By Theorem 3.2.2, \( h \in RS(f, g)(\alpha,\beta) \) i.e., \( RS(e, g)(\alpha,\beta) \subseteq RS(f, g)(\alpha,\beta) \). Similarly we can show that \( RS(f, g)(\alpha,\beta) \subseteq RS(e, g)(\alpha,\beta) \). Thus \( RS(e, g)(\alpha,\beta) = RS(f, g)(\alpha,\beta) \).

Again let \( eLf \) and \( h \in LS(e, g)(\alpha,\beta) \). Then \( h \in V_\beta^\alpha(\alpha g) \cap E_\beta \) and \( hae = h = g\beta h \). From \( hae = h \) and \( eLf \) we have \( haf = haeaf = hae = h \). Now \( haf\beta g\beta h = haeaf\beta g\beta h = hae\beta g\beta h = h \) and \( f\beta g\beta haf\beta g = f\beta haf\beta g = fae\beta h\beta g = fae\beta g = f\beta g \). Hence \( h \in V_\beta^\alpha(f\beta g) \cap E_\beta \) and \( haf = g\beta h = h \). Thus \( LS(e, g)(\alpha,\beta) \subseteq LS(f, g)(\alpha,\beta) \). Similarly we can show that \( LS(f, g)(\alpha,\beta) \subseteq LS(e, g)(\alpha,\beta) \). Thus \( LS(e, g)(\alpha,\beta) = LS(f, g)(\alpha,\beta) \).
(ii) Let \( e \mathcal{R} f \) and \( h \in \text{RS}(g, e)_{(\beta, \alpha)} \). Then \( h \in V_{\alpha}^\beta(g \beta e) \cap E_{\beta} \) and \( h \beta g = h = e \alpha h \).

Since \( e \mathcal{R} f \), by Theorem 1.5.7 we have \( e \alpha f = f \) and \( f \alpha e = e \). Now \( f \alpha h = f \alpha e a h = e \alpha h = h \). Again \( h \beta g \beta f a h = h \beta f a h = h \beta h = h \) and \( g \beta f a h \beta g \beta f = g \beta h \beta g \beta f = g \beta h \beta f = g \beta h \beta e a f = g \beta e a f = g \beta f \). Thus \( h \in V_{\alpha}^\beta(g \beta f) \cap E_{\beta} \) and \( h \beta g = h = f \alpha h \).

Hence \( h \in \text{RS}(g, f)_{(\beta, \alpha)} \) i.e., \( \text{RS}(g, e)_{(\beta, \alpha)} \subseteq \text{RS}(g, f)_{(\beta, \alpha)} \). Similarly we can show that \( \text{RS}(g, f)_{(\beta, \alpha)} \subseteq \text{RS}(g, e)_{(\beta, \alpha)} \). Hence \( \text{RS}(g, e)_{(\beta, \alpha)} = \text{RS}(g, f)_{(\beta, \alpha)} \). In the same way we can prove that \( \text{LS}(g, e)_{(\beta, \alpha)} = \text{LS}(g, f)_{(\beta, \alpha)} \).

**Theorem 3.2.8** Let \( S \) be a regular \( \Gamma \)-semigroup, \( e \) be an \( \alpha \)-idempotent and \( f \) be a \( \beta \)-idempotent. Then

(i) \( e \mathcal{R} f \) implies \( \text{RS}(e, f)_{(\alpha, \beta)} = \{e\} \);

(ii) \( e \mathcal{L} f \) implies \( \text{LS}(e, f)_{(\alpha, \beta)} = \{f\} \).

**Proof :** (i) Let \( e \mathcal{R} f \). Then by Theorem 1.5.7 we have \( e = f \beta e \) and \( f = e \alpha f \). Let \( h \in \text{RS}(e, f)_{(\alpha, \beta)} \). Then \( h \in V_{\alpha}^\beta(e \alpha f) \cap E_{\alpha} = V_{\beta}^\alpha(f) \cap E_{\alpha} \) and \( h\alpha e = h = f \beta h \). Now \( e = f \beta e = f \beta h \alpha f \beta e = h \alpha e = h \). Hence \( \text{RS}(e, f)_{(\alpha, \beta)} = \{e\} \).

(ii) Let \( e \mathcal{L} f \) and \( h \in \text{LS}(e, f)_{(\alpha, \beta)} \). Then we have \( e \beta f = e, f \alpha e = f, h \alpha e = h = f \beta h \) and \( h \in V_{\alpha}^\beta(e \beta f) \cap E_{\beta} = V_{\beta}^\alpha(e) \cap E_{\beta} \) i.e \( h \in V_{\beta}^\alpha(e) \cap E_{\beta} \). Now \( f = f \alpha e = f \alpha e \beta h \alpha e = f \beta h \alpha e = h \). Thus \( \text{LS}(e, f)_{(\alpha, \beta)} = \{f\} \).

We now prove the following theorem.

**Theorem 3.2.9** Let \( S \) be a regular \( \Gamma \)-semigroup. Then \( S \) is right orthodox \( \Gamma \)-semigroup if and only if for any \( \alpha \)-idempotent \( e \) and \( \beta \)-idempotent \( f \), \( f \beta e \in \text{RS}(e, f)_{(\alpha, \beta)} \).

**Proof :** Let \( S \) be a right orthodox \( \Gamma \)-semigroup and let \( g = f \beta e \). Then \( g \alpha e = f \beta e \alpha e = f \beta e = g \) and \( f \beta g = f \beta f \beta e = f \beta e = g \). Again \( e \alpha g \alpha f = e \alpha f \beta e \alpha f = e \alpha f \), since \( S \) is a right orthodox \( \Gamma \)-semigroup. Now \( g \alpha e \alpha f \beta g = f \beta e \alpha e f \beta f \beta e = f \beta e \alpha f \beta e = f \beta e = g \)
and $(eaf)\beta\alpha(eaf) = eaf\beta f\beta eaeaf = eaf\beta eaf = eaf$, since $f\beta e$ is an $\alpha$-idempotent.

Hence $g = f\beta e \in RS(e, f)(\alpha, \beta)$. Conversely let the given condition hold. Then for an $\alpha$-idempotent $e$ and a $\beta$-idempotent $f$, we have $eaf \in RS(f, e)(\beta, \alpha)$. Hence by definition of $RS(f, e)(\beta, \alpha)$, $eaf$ is a $\beta$-idempotent. Hence the theorem.

From the above theorems we conclude the following theorem.

**Theorem 3.2.10** Let $S$ be a regular $\Gamma$-semigroup. Then the following are equivalent:

(i) $S$ is a right orthodox $\Gamma$-semigroup;

(ii) for any $\alpha$-idempotent $e$ and $\beta$-idempotent $f$, $f\beta e \in RS(e, f)(\alpha, \beta)$;

(iii) $V_\alpha^b(b)\delta V_\beta^a(\alpha) \subseteq V_\beta^a(acb)$ for all $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$;

(iv) $V_\beta^a(e) \in E_\beta$ for any $\alpha$-idempotent $e$ where $E_\beta$ is the set of all $\beta$-idempotents of $S$.

### 3.3 Right inverse $\Gamma$-semigroups

In this section we introduce the concept of right inverse $\Gamma$-semigroup and study its properties.

We first recall the definition of right inverse semigroup from [72].

**Definition 3.3.1** A regular semigroup $S$ is said to be a right (resp. left) inverse semigroup if for any $e, f \in E(S)$, $efe = fe$ (resp. $efe = ef$).

We now generalize this concept in $\Gamma$-semigroups.

**Definition 3.3.2** A regular $\Gamma$-semigroup $S$ is called a right (resp. left) inverse $\Gamma$-semigroup if for any $\alpha$-idempotent $e$ and $\beta$-idempotent $f$, $eaf\beta e = f\beta e$ (resp. $e\beta f ae = e\beta f$).

From the above definition we observe that in a right inverse $\Gamma$-semigroup for any $\alpha$-idempotent $e$ and $\beta$-idempotent $f$, $eaf$ is a $\beta$-idempotent i.e., every right inverse
Theorem 3.3.3 The following conditions on a regular \( S \) are equivalent:

(i) \( eTS \cap fTS = eafTS(= \beta eTS) \) for any \( e \)-idempotent \( e \) and \( \beta \)-idempotent \( f \);

(ii) \( S \) is a right inverse \( \Gamma \)-semigroup;

(iii) \( a \) is an element of \( S \) and \( a' \in V_a^{\beta_1}(a), a'' \in V_a^{\beta_2}(a) \) then \( a' \beta_1 a = a'' \beta_2 a \);

(iv) for any \( e \)-idempotent \( e \) of \( S \), the elements of the set \( F_\beta(e) = \{ x \in S : x \in V_a^{\beta}(e) \} \) are \( (A) \) \( \beta \)-idempotents and satisfy \( (B) x \beta y = y \) for all \( x, y \in F_\beta(e) \);

(v) for any \( x \in S, e \in E_\alpha, \beta \in \Gamma \) and \( x' \in V_a^{\beta}(x) \), if \( x \in STe \) then \( x' \in eTS \);

(vi) for any two \( e \)-idempotents \( e \) and \( f \), \( STe = STf \) implies \( e = f \).

Proof : (i) \( \Rightarrow \) (ii) : Let \( eTS \cap fTS = eafTS(= \beta eTS) \) for any \( e \)-idempotent \( e \) and \( \beta \)-idempotent \( f \). i.e., we have \( eafTS \subseteq fTS \). Now \( eaf = eaf \beta f \in eafTS \subseteq fTS \) i.e.,
\( eaf \in fTS \). Therefore \( eaf = f \gamma y \) for some \( \gamma \in \Gamma, y \in S \). This implies \( f \beta eaf = f \beta f \gamma y = f \gamma y = eaf \). Thus \( S \) is a right inverse \( \Gamma \)-semigroup.

(ii) \( \Rightarrow \) (iii) : Let \( a' \in V_a^{\beta_1}(a), a'' \in V_a^{\beta_2}(a) \) for some \( a \in S \) and some \( \alpha, \beta, \gamma \in \Gamma \). We have \( a' \beta_1 a \) and \( a'' \beta_2 a \) are \( \alpha \)-idempotents. Now \( (a' \beta_1 a) \alpha (a'' \beta_2 a) = a' \beta_1 a \) and by (ii)
\( (a'' \beta_2 a) \alpha (a' \beta_1 a) = a'' \beta_2 a \). i.e., \( (a'' \beta_2 a) \alpha (a' \beta_1 a) = a' \beta_1 a \).

Again \( a'' \beta_2 a \alpha a' \beta_1 a = a'' \beta_2 a \alpha a' \beta_1 a = a'' \beta_2 a \). This implies \( a'' \beta_2 a = a' \beta_1 a \).

(iii) \( \Rightarrow \) (iv) : Let \( e \) be an \( \alpha \)-idempotent and \( x \in V_a^{\beta}(e) \). Again \( e \in V_a^{\beta}(e) \). Hence by
(iii) we have \( x \beta e = eae = e \) and \( x = x \beta eax = eax \). Thus \( x \beta x = eax \beta eax = eax = x \).

i.e., \( x \) is a \( \beta \)-idempotent. Again let \( y \in V_a^{\beta}(e) \). Then as above \( y \beta e = e \) and \( eay = y \).

This implies \( x \beta y = x \beta eay = eay = y \).

(iv) \( \Rightarrow \) (v) : We first prove (iv) \( \Rightarrow \) (iii). Let \( a' \in V_a^{\beta_1}(a) \) and \( a'' \in V_a^{\beta_2}(a) \) for \( a \in S \). Let \( t = a' \beta_1 a \) and \( u = a'' \beta_2 a \). Now (iv) \( a' \beta_1 a \alpha a'' \beta_2 a = a' \beta_1 a \alpha a'' \beta_2 a = a' \beta_1 a \) and

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(a^\alpha a^\alpha a)\alpha(a^\beta a)\alpha(a^\alpha a) = a^\alpha a. \text{ i.e., } a^\beta a \in V_\alpha^\alpha(a^\alpha a). \text{ Again } a^\alpha a \in V_\alpha^\alpha(a^\alpha a).

Hence by (iv) \ a^\beta a = t = u a t = a^\alpha a a^\alpha a = a^\alpha a \text{ which shows that (iv) } \Rightarrow \text{ (iii).}

Now let e be an \alpha-idempotent, \ x \in S \text{ and } \ x' \in V_\alpha^\beta(x) \text{ for some } \beta \in \Gamma. \text{ Let } \ x \in ST e. \text{ Then } \ x = y \gamma e \text{ for some } y \in S \text{ and } \gamma \in \Gamma. \text{ i.e., we have } \ x a e = x. \text{ Now } (e a x')\beta x a (e a x') = e a x' x a x' = e a x' \text{ and } x a e a x' = x a x' = x. \text{ Thus } e a x' \in V_\alpha^\beta(x) \text{ and } \ x' \in V_\alpha^\beta(x).

Then by (iii) we have \ x' x = e a x' x \text{ i.e., } \ x' = x a x' = e a x' x a x' = e a x' \text{ which shows that } \ x' \in e G S.

(v) \Rightarrow (vi): \text{ Let } e \text{ and } f \text{ be two } \alpha\text{idempotents and } ST e = ST f. \text{ Now } e = e a e \in ST e = ST f. \text{ i.e., } e = x \gamma f \text{ for some } x \in S \text{ and } \gamma \in \Gamma. \text{ This implies } e a f = e \text{ and similarly } f a e = f. \text{ Thus } f = f a e \in ST e \text{ and } f \in V_\alpha^\beta(f) \text{ and hence by (v) } f \in e G S \text{ i.e., } f = e \delta y \text{ for some } y \in S \text{ and } \delta \in \Gamma \text{ and hence } e a f = e a e \delta y = e \delta y = f. \text{ Thus } e = f.

(vi) \Rightarrow (i): \text{ We first show (vi) } \Rightarrow \text{ (iii). Let } a \in S, \ a' \in V_\alpha^\beta(a) \text{ and } \ a'' \in V_\alpha^\beta(a) \text{ for some } \alpha, \beta_1, \beta_2 \in \Gamma. \text{ Now } ST a = S \gamma a a' \beta_1 a \subseteq ST a' \beta_1 a \subseteq ST a. \text{ i.e., } ST a = ST a' \beta_1 a. \text{ Similarly we can show that } ST a = ST a'' \beta_2 a. \text{ Thus } ST a = ST a' \beta_1 a = ST a'' \beta_2 a. \text{ So by (vi) we have } a' \beta_1 a = a'' \beta_2 a. \text{ Hence (iii) } \Rightarrow \text{ (vi) are equivalent. Now let } e \text{ be an } \alpha\text{idempotent and } f \text{ be a } \beta\text{idempotent. We consider } e a f. \text{ By regularity of } S \text{ there exists } x \in S \text{ and } \gamma, \delta \in \Gamma \text{ such that } e a f \gamma x \delta e a f = e a f \text{ and } x \delta e a f \gamma x = x. \text{ Now } f \gamma x \delta e \text{ is an } \alpha\text{idempotent and we have } e a f \in V_\alpha^\beta(f \gamma x \delta e). \text{ Hence by (iv) we have } e a f \text{ is } \beta\text{idempotent and } (f \beta \delta e a f) \beta (f \beta \delta e a f) = f \beta \delta e a f \beta = f \beta \delta e a f. \text{ i.e., } f \beta \delta e a f \text{ is also a } \beta\text{idempotent and } ST e a f = ST e a f \beta e a f = ST f \beta e a f. \text{ Hence by (vi, } e a f = f \beta \delta e a f \text{ which implies } e a f G S \subseteq e G S \cap f G S. \text{ Again let } x \in e G S \cap f G S. \text{ This implies } x = e \gamma f p = f \delta t q \text{ for some } p, q \in S \text{ and for some } \gamma_1, \delta_1 \in \Gamma. \text{ Now } e a f \beta x = e a f \beta f \delta_1 q = e a f \delta_1 q = e a x = e a e \gamma_1 p = e \gamma_1 p = x. \text{ i.e., } x \in e a f G S. \text{ Hence } e G S \cap f G S \subseteq e a f G S. \text{ Thus we have } e G S \cap f G S = e a f G S.
Theorem 3.3.4 Let $S$ be a regular $\Gamma$-semigroup. Then $S$ is a right inverse $\Gamma$-semigroup if and only if for all $\alpha$-idempotent $e$ and $\beta$-idempotent $f$, $f\beta V_{\alpha}(eaf) = V_{\beta}(eaf)$.

Proof: Let $g \in V_{\alpha}(eaf)$ and $S$ is a right inverse $\Gamma$-semigroup. Hence $S$ is a right orthodox $\Gamma$-semigroup and by Theorem 3.1.3, $g$ is an $\alpha$-idempotent. Since $S$ is a right inverse $\Gamma$-semigroup we have $f\beta eaf = eaf$ and $g\alpha(eaf)\beta g = eaf\beta g$. Again we have $f\beta eaf = eaf$. Thus $g = gaeaf\beta g = eaf\beta g$. Now $g = eaf\beta g = f\beta eaf\beta g = f\beta g$. Hence $g \in f\beta V_{\alpha}(eaf)$ i.e., $V_{\alpha}(eaf) \subseteq f\beta V_{\alpha}(eaf)$. Now let $g \in V_{\alpha}(eaf)$.

We shall show that $f\beta g \in V_{\alpha}(eaf)$. Now $f\beta gaeaf\beta g = f\beta gaeaf\beta g = f\beta g$ and $eaf\beta f\beta gaeaf = eaf\beta gaeaf = eaf$. i.e., $f\beta V_{\alpha}(eaf) \subseteq V_{\alpha}(eaf)$ and hence $f\beta V_{\alpha}(eaf) = V_{\alpha}(eaf)$.

Conversely, let the given condition hold and $e$ and $f$ be two $\alpha$-idempotents such that $STe = STf$. Then by Theorem 1.5.7 we have $eaf = e$ and $f\alpha e = f$. Since $eaf \in V_{\alpha}(eaf)$, by hypothesis we have $e = eaf = f\alpha x$ for some $x \in V_{\alpha}(eaf)$. Therefore $f = f\alpha e = f\alpha(f\alpha x) = f\alpha x = e$. Thus by Theorem 3.3.3, $S$ is a right inverse $\Gamma$-semigroup.

Recall that a congruence $\rho$ on a regular $\Gamma$-semigroup is said to be idempotent separating if for any two $\alpha$-idempotents $e$ and $f$, $(e, f) \in \rho$ implies $e = f$. We now describe maximum idempotent separating congruence on a right inverse $\Gamma$-semigroup. We recall the following theorems from [48].

Theorem 3.3.5 Let $S$ be a regular $\Gamma$-semigroup. If $\rho$ is an idempotent separating congruence on $S$ then $\rho \subseteq H$, where $H$ is the Green's $H$-relation on $S$.

Theorem 3.3.6 Let $S$ be a regular $\Gamma$-semigroup. $a, b \in S$. Then $aHb$ if and only if there exist $a' \in V_{\gamma}(a), b' \in V_{\gamma}(b)$ such that $a\gamma a' = b\gamma b'$, $a'\delta a = b\delta b$.

We prove the following lemmas.
Lemma 3.3.7 Let $S$ be a right inverse $\Gamma$-semigroup and $e$ be an $\alpha$-idempotent and let $x', x'' \in V^\alpha_{\gamma}(x)$ where $x \in S$. Then $x'\delta e a x$ is a $\gamma$-idempotent and $x'\delta e a x = x''\delta e a x$.

Proof: Let $a e a x = e a x \gamma x'\delta e a x = (x'\gamma \delta) e a x (x'\gamma \delta) x = x'\gamma \delta e a x$ and $(x'\delta e a x) \gamma (x'\delta e a x) = x'\delta e a x = x'\delta e a x$. i.e., $x'\delta e a x$ is a $\gamma$-idempotent. Since $S$ is a right inverse $\Gamma$-semigroup we get $x'\delta x = x''\delta x$. Again $x'\delta e a x = x'\delta x x'\delta e a x = (x'\delta x) \gamma (x'\delta e a x) = (x''\delta x) \gamma x'\delta e a x = x''\delta (x'\gamma \delta e a x) = x''\delta e a x$. Hence the proof.

Lemma 3.3.8 Let $S$ be a right inverse $\Gamma$-semigroup. Then the binary relation $\mu$ on $S$ defined by

$$\mu = \{(a, b) \in S \times S : \text{there exist } \gamma, \delta \in \Gamma, a' \in V^\alpha_{\gamma}(a) \text{ and } b' \in V^\beta_{\delta}(b) \text{ satisfying } a'\delta e a a = b'\delta e a b \text{ for any } \alpha\text{-idempotent } e \text{ of } S\}$$

is an idempotent separating right congruence on $S$.

Proof: Obviously $\mu$ is reflexive and symmetric. Suppose $(a, b) \in \mu$ and $(b, c) \in \mu$ i.e., we get $\gamma, \delta, \gamma_i, \delta_i \in \Gamma$ and $a' \in V^\alpha_{\gamma}(a), b' \in V^\beta_{\delta}(b), c' \in V^\beta_{\delta}(b), c* \in V^\beta_{\delta}(c)$ such that $a'\delta e a a = b'\delta e a b$ and $b'\delta f c = c'\delta f c$ for any $\alpha$-idemponent $e$ and $\beta$-idemponent $f$.

Now

$$a\gamma b\delta b = a\gamma a'\delta a\gamma a'\delta b = a\gamma(a'\delta(a\gamma a')\delta a)\gamma b\delta b = a\gamma b\delta a\gamma a'db'\delta b = a\gamma b\delta a\gamma a'db = a\gamma b\delta(a\gamma a')\delta b = a\gamma a'da\gamma a'da = a.$$}

i.e., $a\gamma b\delta b = a \quad \ldots (3.3.1)$

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Again
\[ c_\gamma b^d b = c_\gamma, c^* \delta_1, c_\gamma, c^* \delta_1, c_\gamma b^d b \]
\[ = c_\gamma, b^* \delta_1, c_\gamma, c^* \delta_1, b_\gamma b^d b \]
\[ = c_\gamma, b^* \delta_1 (c_\gamma, c^*) \delta_1, b \]
\[ = c_\gamma, c^* \delta_1 (c_\gamma, c^*) \delta_1, c \]
\[ = c \quad \ldots \text{(3.3.2).} \]

i.e., \( c_\gamma b^d b = c \) and \( a' \delta_\gamma b, b^* \delta_1 a = b' \delta_\gamma b, b^* \delta_1 b = b' b. \)

Let us take \( a = b' \delta_\gamma a'db', b^* \) and \( e = b' \delta_\gamma b, c^* \delta_1 b_\gamma, b^* \).

Now
\[ a' \gamma a' a d = a' \gamma b' \delta_\gamma a'db', b^* \delta_1 a \]
\[ = a' \gamma b' \delta_\gamma a'db', b^* \delta_1 b \]
\[ = a' \gamma b' \delta_\gamma b \]
\[ = a' \gamma b' \delta b \]
\[ = a \quad \text{(by (3.3.1)).} \]

and
\[ a' a' \gamma a = b' \delta_\gamma a'db', b^* \delta_1, a' \gamma b' \delta_\gamma a'db', b^* \]
\[ = b' \delta_\gamma a'db', b^* \delta_1, a' \gamma a'db', b^* \quad \text{(by (3.3.1))} \]
\[ = (b' b) \gamma (a' a' (b_\gamma, b^*)) \delta_1 a \gamma a'db', b^* \]
\[ = b' \delta_\gamma b'db', b^* \delta_1, b_\gamma a'db', b^* \]
\[ = b' \delta_\gamma b', b^* \delta_1, b_\gamma a'db', b^* \]
\[ = b' \delta_\gamma a'db', b^* \]
\[ = a. \]

Thus
\[ 77 \]
\( \bar{a} \in V_{\gamma}^{1}(a) \)

Again

\[
c \gamma \delta_{1} c = c \gamma \delta_{2} b_{\gamma_{1}} c^{*} \delta_{1} b_{\gamma_{1}} b^{*} \delta_{1} c
\]
\[
= c \gamma \delta_{2} b_{\gamma_{1}} b^{*} \delta_{1} b_{\gamma_{1}} b^{*} \delta_{1} b
\]
\[
= c \gamma \delta_{2} b_{\gamma_{1}} b^{*} \delta_{1} b
\]
\[
= c \gamma \delta_{2} b
\]
\[
= c \text{ (by (3.3.2)).}
\]

and

\[
\delta_{1} c \gamma e = b \delta_{2} b_{\gamma_{1}} c^{*} \delta_{1} b_{\gamma_{1}} b^{*} \delta_{1} c \gamma \delta_{2} b_{\gamma_{1}} c^{*} \delta_{1} b_{\gamma_{1}} b^{*}
\]
\[
= b \delta_{2} b_{\gamma_{1}} c^{*} \delta_{1} b_{\gamma_{1}} b^{*} \delta_{1} c \gamma \delta_{2} b_{\gamma_{1}} c^{*} \delta_{1} b_{\gamma_{1}} b^{*} \text{ (by (3.3.2))}
\]
\[
= b \delta_{2} b_{\gamma_{1}} b^{*} \delta_{1} b_{\gamma_{1}} b^{*} \delta_{1} b_{\gamma_{1}} c^{*} \delta_{1} b_{\gamma_{1}} b^{*}
\]
\[
= b \delta_{2} b_{\gamma_{1}} b^{*} \delta_{1} b_{\gamma_{1}} c^{*} \delta_{1} b_{\gamma_{1}} b^{*}
\]
\[
= b \delta_{2} b_{\gamma_{1}} c^{*} \delta_{1} b_{\gamma_{1}} b^{*}
\]
\[
= e.
\]

Thus \( \bar{e} \in V_{\gamma}^{1}(c) \)

Again for an \( \alpha \)-idempotent \( e \),

\[
\alpha \delta_{1} e a a = b \delta \gamma_{a} \delta_{b} b_{\gamma_{1}} b^{*} \delta_{1} e a a
\]
\[
= b \delta \gamma_{a} b_{(b_{\gamma_{1}} b^{*} \delta_{1} e)} a a
\]
\[
= b \delta \gamma_{a} \delta_{b} b_{\gamma_{1}} b^{*} \delta_{1} e a b \text{ (since } b_{\gamma_{1}} b^{*} \delta_{1} e \text{ is an } \alpha \text{-idempotent)}
\]
\[
= b \delta \gamma_{a} b^{*} \delta_{1} e a b
\]
\[
= b \delta \gamma_{a} b^{*} \delta_{1} b_{\gamma_{1}} b^{*} \delta_{1} e a b
\]
\[
= b \delta \gamma_{a} c^{*} \delta_{1} b_{\gamma_{1}} b^{*} \delta_{1} e a c \text{ (since } b_{\gamma_{1}} b^{*} \delta_{1} e \text{ is an } \alpha \text{-idempotent)}
\]
\[
= \alpha \delta_{1} e a c.
\]

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Hence \((a, c) \in \mu\) i.e., \(\mu\) is transitive and consequently \(\mu\) is an equivalence relation.

Let \((a, b) \in \mu\) and \(c \in S, \beta \in \Gamma\). Then by definition of \(\mu\), there exist \(\gamma, \delta \in \Gamma\) and \(a' \in V^k_\gamma(a), b' \in V^k_\gamma(b)\) such that \(a'\delta eaa = b'\delta eab\) for all \(\alpha\)-idempotent \(e\).

Let \(x \in V^k_\gamma(a'\delta abc)\). Then \(x\delta_2 a'\delta abc \gamma_2 x\delta_2 a' = x\delta_2 a'\) and \(\alpha \beta c \gamma_2 x\delta_2 a' \delta abc = \alpha \gamma a' \delta ab\)

\(c\gamma_2 x\delta_2 a' \delta abc = \alpha \gamma a' \delta abc = \alpha \beta c\), i.e., \(x\delta_2 a' \in V^k_\gamma(\alpha \beta c)\).

Similarly we can show that \(x\delta_2 b' \in V^k_\gamma(\beta \beta c)\).

Now,

\[
(x\delta_2 a')\delta eaa(\alpha \beta c) = x\delta_2 (a'\delta eaa)\beta c = x\delta_2 b' \delta eab\beta c = (x\delta_2 b')\delta eaa(\beta \beta c).
\]

Hence \((\alpha \beta c, \beta \beta c) \in \mu\) i.e., \(\mu\) is a right congruence on \(S\).

We now show that \(\mu\) is idempotent separating.

Let \(e, f\) be two \(\alpha\)-idempotents and \((e, f) \in \mu\). Then there exist \(\gamma, \delta \in \Gamma\) and \(e' \in V^k_\gamma(e), f' \in V^k_\gamma(f)\) such that \(e'\delta g be = f'\delta g \theta f\) for any \(\theta\)-idempotent \(g\). Hence we have \(e'\delta e a e = f'\delta e a f\). i.e., \(e'\delta e = f'\delta e a f\) and hence \(e'\delta e a f = f'\delta e a f\alpha f = f'\delta e a f\).

Now

\[
e a f = e \gamma f' \delta e a f
= e \gamma (f' \delta e a f)
= e \gamma (e' \delta e a e)
= e.
\]

Again \(f' \delta f a f = e' \delta f a e\). i.e., \(f' \delta f = e' \delta f a e\) and hence \(f' \delta f a e = e' \delta f a e a e = e' \delta f a e\).

Now

\[
e a f = f e a e f (\text{since } S \text{ is a right inverse } \Gamma\text{-semigroup})
= f \gamma f' \delta f a e a f
\]

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Hence $e = f$. i.e., $\mu$ is idempotent separating right congruence on the $\Gamma$-semigroup $S$.

With the help of the above right congruence we now construct the following maximum idempotent separating congruence on a right inverse $\Gamma$-semigroup.

**Theorem 3.3.9** Let $S$ be a right inverse $\Gamma$-semigroup. Then the binary relation $\delta$ on $S$ defined by

$$\delta = \{(a, b) \in S \times S : (x\alpha a, x\alpha b) \in \mu \text{ for all } x \in S \text{ and for all } \alpha \in \Gamma \}$$

is the maximum idempotent separating congruence on $S$ where

$$\mu = \{(a, b) \in S \times S : \text{there exist } \gamma, \delta \in \Gamma, a' \in V^\gamma_\alpha(a) \text{ and } b' \in V^\delta_\beta(b) \text{ satisfying } a'\delta\gamma a = b'\delta\gamma b \text{ for any } \alpha \text{-idempotent } e \text{ of } S\}.$$

**Proof:** Since $\mu$ is an equivalence relation, $\delta$ is an equivalence relation. Now let $(a, b) \in \delta$ and $c \in S, \beta \in \Gamma$. Then $(x\alpha c\beta a, x\alpha c\beta b) \in \mu$ and $(x\alpha a, x\alpha b) \in \mu$ for all $\alpha \in \Gamma$, $x \in S$ and hence $(c\alpha a, c\alpha b) \in \delta$. Again since $\mu$ is a right congruence, we have $(x\alpha a\beta c, x\alpha b\beta c) \in \mu$ for all $x \in S$ and $\alpha \in \Gamma$. This implies that $(a\beta c, b\beta c) \in \delta$. Hence $\delta$ is a congruence.

We now show that $\delta$ is an idempotent separating congruence on the $\Gamma$-semigroup $S$. Let $e, f$ be two $\alpha$-idempotents and $(e, f) \in \delta$. Then we have $(e\alpha e, e\alpha f) \in \mu$ i.e., $(e, e\alpha f) \in \mu$. Since $e$ and $e\alpha f$ are both $\alpha$-idempotents and $\mu$ is idempotent separating, we have $e = e\alpha f$. Again $(f\alpha e, f\alpha f) \in \mu$ and since $\mu$ is right congruence we have $(f\alpha e\alpha f, f\alpha f\alpha f) \in \mu$ i.e., $(e\alpha f, f) \in \mu$ since $S$ is a right inverse $\Gamma$-semigroup i.e., we
have $eaf = f$. Thus $e = f$ and consequently $\delta$ is an idempotent separating congruence on $S$.

We now show that $\delta$ is the maximum idempotent separating congruence on $S$. Let $(a, b) \in \rho$ and $x \in S, \alpha \in \Gamma$. Since $\rho$ is a congruence $(x\alpha a, x\alpha b) \in \rho$. From Theorem 3.3.5 we have $x\alpha a \sim x\alpha b$. Let $x\alpha a = c$ and $x\alpha b = d$. Then by Theorem 3.3.6, there exist $c' \in V^\gamma_1(c)$ and $d' \in V^\delta_1(d)$ such that $c\gamma' = d\gamma'$ and $c\delta c = d\delta d$ for some $\gamma, \delta \in \Gamma$. i.e. we have $c' = c\delta c\gamma' = c\delta d\gamma'$ and $d' = d\delta d\gamma' = c\delta c\gamma'$. Since $(c, d) \in \rho$, $(c\delta c, c\delta d) \in \rho$ and accordingly $(c', d') \in \rho$. Let $e$ be an $\alpha \gamma$-idempotent in $S$. Then $(c\delta c\alpha_1, c\delta d\alpha_1, c) \in \rho$. Since $(c, d) \in \rho$ we have $(c\delta c\alpha_1, c\delta d\alpha_1, d) \in \rho$. By transitivity of $\rho$ we have $(c\delta c\alpha_1, c\delta d\alpha_1, d) \in \rho$. Since both of $c\delta c\alpha_1, c\delta d\alpha_1, d$ are $\gamma$-idempotents, we have $c\delta c\alpha_1, c = c\delta d\alpha_1, d$ for all $\alpha \gamma$-idempotent $e$ which implies $(c, d) \in \mu$ and consequently $(a, b) \in \delta$. Thus $\rho \subseteq \delta$. Hence the proof.