CHAPTER 1

Preliminary Ideas

1 Introduction

This chapter is devoted to the presentation of some basic definitions and known results concerning $\Gamma$-semigroup which are required for the development of this thesis.

1.1 $\Gamma$-semigroups

In [58] Sen and Saha introduced $\Gamma$-semigroup as follows.

Definition 1.1.1 Let $S = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two nonempty sets. $S$ is called a $\Gamma$-semigroup if

(i) $aab \in S$, for all $a \in \Gamma$ and $a, b \in S$ and

(ii) $(aab)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Let $S$ be a $\Gamma$-semigroup. If there exists an element $0 \in S$ such that $a00 = 00a = 0$ then that element is said to be the zero element of $S$ and we call $S$, a $\Gamma$-semigroup with zero. There are several examples of $\Gamma$-semigroup, we list some of them.

Example 1.1.2 [58] Let $S$ be the set of all $m \times n$ matrices and $\Gamma$ be the set of all $n \times m$ matrices over a field. Then for $A_{m,n}, B_{m,n} \in S$, the usual matrix multiplication $A_{m,n}B_{m,n}$ can not be defined i.e., $S$ is not a semigroup under the usual matrix multiplication. But for all $A_{m,n}, B_{m,n}, C_{m,n} \in S$ and $P_{n,m}, Q_{n,m} \in \Gamma$, $A_{m,n}P_{n,m}B_{m,n}$ is defined and an element
of $S$. Also we notice that $(A_{m,n}P_{n,m}B_{m,n})Q_{n,m}C_{m,n} = A_{m,n}P_{n,m}(B_{m,n}Q_{n,m}C_{m,n})$. Hence $S$ is a $\Gamma$-semigroup.

Example 1.1.3 [65] Let $T$ be a semigroup, $I, \Lambda$ be two index sets and $\Gamma$ be the collection of some $\Lambda \times I$ matrices over $T$. Then the set $S = I \times T \times \Lambda$ is a $\Gamma$-semigroup with respect to the multiplication $(i, a, \lambda)P(j, b, \mu) = (i, ap\lambda_j b, \mu)$ for $(i, a, \lambda), (j, b, \mu) \in S$ and $P = (p_{\lambda i}) \in \Gamma$. This $\Gamma$-semigroup is called the Rees matrix $\Gamma$-semigroup over $T$ with the set $\Gamma$ of sandwich matrices and it is denoted by $S = M(I, T, \Lambda; \Gamma)$. Let $T^0$ denote the semigroup $T$ with a zero element adjoint. Let $\Gamma$ be a set of some $\Lambda \times I$ matrices over $T^0$. Then the set $S = (I \times T \times \Lambda) \cup \{0\}$ is a $\Gamma$-semigroup with respect to the multiplication

$$(i, a, \lambda)P(j, b, \mu) = \begin{cases} (i, ap\lambda_j b, \mu), & \text{if } p_{\lambda i} \neq 0 \\ 0, & \text{if } p_{\lambda i} = 0 \end{cases}$$

and $0\Gamma(i, a, \lambda) = (i, a, \lambda)0\Gamma = 0\Gamma 0 = \{0\}$ for all $(i, a, \lambda), (j, b, \mu) \in S$ and $P = (p_{\lambda i}) \in \Gamma$. This $\Gamma$-semigroup is called the Rees matrix $\Gamma$-semigroup over $T^0$ with the set $\Gamma$ of sandwich matrices and we denote it by $M^0(I, T, \Lambda; \Gamma)$.

The notion of regularity of a $\Gamma$-semigroup was introduced in [58].

Definition 1.1.4 [58] Let $S$ be a $\Gamma$-semigroup. An element $a \in S$ is said to be regular in the $\Gamma$-semigroup $S$ if $a \in a\Gamma S T a$ where $a\Gamma S T a = \{a\alpha b\beta a \colon b \in S, \alpha, \beta \in \Gamma\}$. $S$ is said to be regular if every element of $S$ is regular.

Example 1.1.5 Let $S$ be the set of all positive integers of the form $4n + 1$ and $\Gamma$ be the set of all positive integers of the form $4n + 3$. If $a \beta b$ is $a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$ then $S$ is a $\Gamma$-semigroup. Since for the element $2 \in S$, there do not exist any $a \in S$ and $\alpha, \beta \in \Gamma$ such that $2 + \alpha + a + \beta + 2 = 2$, 2 is not a regular element in $S$. Hence $S$ is not a regular $\Gamma$-semigroup.
Now if we take \( m = 3 \) and \( n = 2 \) in Example 1.1.2 then \( S \) is a \( \Gamma \)-semigroup. This is a regular \( \Gamma \)-semigroup shown in [58].

Let \( S \) be a \( \Gamma \)-semigroup and \( \alpha \) be a fixed element of \( \Gamma \). We define \( a.b = aab \) for all \( a, b \in S \). We can show that \((S,.)\) is semigroup and we denote this semigroup by \( S_\alpha \). It is evident that for a \( \Gamma \)-semigroup \( S \), if \( S_\alpha \) is regular semigroup for some \( \alpha \in \Gamma \) then \( S \) is a regular \( \Gamma \)-semigroup but regarding this we now give the following observations.

(i) \( S_\alpha \) may not be a regular semigroup for some \( \alpha \in \Gamma \) but \( S \) may be a regular \( \Gamma \)-semigroup.

**Example 1.1.6** Let \( S = \{(a, 0) : a \in \mathbb{R} \} \cup \{(0, b) : b \in \mathbb{R} \} \) where \( \mathbb{R} \) denote the field of real numbers. Let \( \Gamma = \{(0,5),(0,1),(3,0),(1,0)\} \). Defining \( S \times \Gamma \times S \to S \) by \((a,b)(\alpha,\beta)(c,d) = (aac, b\beta d)\) for all \((a,b),(c,d) \in S \) and \((\alpha,\beta) \in \Gamma \), we can show that \( S \) is a \( \Gamma \)-semigroup. \( S_\alpha \) is not a regular semigroup for any \( \alpha \in \Gamma \). Let \((a,0) \in S \). If \( a = 0 \), then \((a,0)\) is regular. Suppose \( a \neq 0 \), then \((a,0)(3,0)(\frac{1}{3a},0)(1,0)(a,0) = (a,0)\).

Similarly we can show that \((0,b)\) is also regular for all \( b \in \mathbb{R} \). Hence \( S \) is a regular \( \Gamma \)-semigroup.

(ii) In a \( \Gamma \)-semigroup \( S \), if \( S_\alpha \) is a regular semigroup for some \( \alpha \in \Gamma \) then there may exist a \( \beta \in \Gamma \) such that \( S_\beta \) is not a regular semigroup.

**Example 1.1.7** Let \( S = \{(a,b) : a, b \in \mathbb{R}, \text{ the field of real numbers} \} \) and \( \Gamma = \{(9,7),(0,3)\} \). In this case \( S \) is a \( \Gamma \)-semigroup where the operation is defined by \((a,b)(\alpha,\beta)(c,d) = (aac, b\beta d)\) for \((a,b),(c,d) \in S \) and \((\alpha,\beta) \in \Gamma \). In this case \( S_{(0,3)} \) cannot be a regular semigroup but \( S_{(9,7)} \) is a regular semigroup.

**Definition 1.1.8** [65] The set \( \Gamma \) of sandwich matrices (defined in Example 1.1.3) is regular if for each \( i \in I \) there exists a matrix \( P \in \Gamma \) and for each \( \lambda \in \Lambda \) there exists a
matrix \( Q \in \Gamma \) such that \( P \) has at least one nonzero entry in the \( i \)-th column and \( Q \) has at least one nonzero entry in the \( \lambda \)-th row.

**Theorem 1.1.9** [65] Rees \( I \times \Lambda \) matrix \( \Gamma \)-semigroup \( M^0(G, I, \Lambda; \Gamma) \) over \( G^0 \), a group with zero is regular if and only if \( \Gamma \) is regular.

Idempotent elements play an important role in semigroup theory. The notion of idempotent element in a \( \Gamma \)-semigroup was defined in [58] as follows.

**Definition 1.1.10** Let \( S \) be a \( \Gamma \)-semigroup and \( \alpha \in \Gamma \). Then \( e \in S \) is said to be an \( \alpha \)-idempotent if \( eae = e \). The set of all \( \alpha \)-idempotents is denoted by \( E_\alpha \). We denote \( \bigcup_{\alpha \in \Gamma} E_\alpha \) by \( E(S) \). The elements of \( E(S) \) are called idempotent element of \( S \). If \( S = E(S) \) then \( S \) is called an idempotent \( \Gamma \)-semigroup [66].

In semigroup theory we know that an element \( b \in S \) is said to be an inverse of an element \( a \) of \( S \) if \( a = aba \) and \( b = bab \). This was generalized by Sen and Saha in [58].

**Definition 1.1.11** Let \( a \in S \) and \( \alpha, \beta \in \Gamma \). An element \( b \in S \) is called \((\alpha, \beta)\)-inverse of \( a \) if \( a = a\alpha b\beta a \) and \( b = b\beta a\alpha b \). In this case we write \( b \in V_{\alpha, \beta}(a) \).

In [58], the concept of homomorphism in \( \Gamma \)-semigroup was introduced.

**Definition 1.1.12** Let \( S \) be a \( \Gamma \)-semigroup and \( S_1 \) be a \( \Gamma_1 \)-semigroup. A pair of mappings \( f_1 : S \to S_1 \) and \( f_2 : \Gamma \to \Gamma_1 \) is said to be a homomorphism from \((S, \Gamma)\) to \((S_1, \Gamma_1)\) if \( f_1(axb) = f_1(a)f_1(x)f_1(b) \) for all \( a, b \in S \) and \( x \in \Gamma \). If \( f_1, f_2 \) are both injective then \((f_1, f_2)\) is said to be a monomorphism of \((S, \Gamma)\) into \((S_1, \Gamma_1)\).

The following theorem shows that homomorphic image of a regular \( \Gamma \)-semigroup is regular.

**Theorem 1.1.13** [58] Let \( S \) be a regular \( \Gamma \)-semigroup and \( S' \) be a \( \Gamma' \)-semigroup. If \((f, g)\) be a homomorphism from \((S, \Gamma)\) onto \((S', \Gamma')\) then \( S' \) is a regular \( \Gamma \)-semigroup.
1.2 Congruences on Γ-semigroups

The notion of congruence in a Γ-semigroup was introduced by Dutta and Chatterjee in [18].

**Definition 1.2.1** Let \( S \) be a Γ-semigroup. An equivalence relation \( \rho \) on \( S \) is said to be a right (left) congruence on \( S \) if \((a, b) \in \rho \) implies \((aac, bac) \in \rho \) (resp. \((caa, cab) \in \rho \)) for all \( a, b, c \in S \) and for all \( \alpha \in \Gamma \). An equivalence relation \( \rho \) on \( S \), which is both left and right congruence is called a congruence relation on \( S \).

Let \( S \) be a Γ-semigroup and \( \rho \) be a congruence relation on \( S \). Let \( S/\rho \) be the set of all equivalence classes of \( S \). If \( a\rho, b\rho \) be any two elements of \( S/\rho \) and \( \alpha \in \Gamma \) then we define \((a\rho)\alpha(b\rho) = (aab)\rho \). It can be shown that \( S/\rho \) is a Γ-semigroup.

**Definition 1.2.2** Let \( S \) be a Γ-semigroup. A congruence \( \rho \) on \( S \) is said to be idempotent separating if for any two \( \alpha \)-idempotents \( e \) and \( f \) of \( S \), \((e, f) \in \rho \) implies \( e = f \).

We find the following interesting example in [61].

**Example 1.2.3** Let \( S = J/(6) \) = the set of all residue classes modulo 6 = \{0, 1, 2, 3, 4, 5\} and \( \Gamma = \{1, 3\} \). We define \( \overline{a} \overline{\alpha} \overline{b} = \overline{aab} \) where \( \overline{a}, \overline{b} \in S \) and \( \overline{\alpha} \in \Gamma \). Then \( S \) is a regular Γ-semigroup. But \( S_{\beta} \) is not a regular semigroup, because for \( \overline{\beta} = 3 \) there is no \( \overline{p} \) in \( S \) such that \( \overline{1} \overline{3} \overline{p} \overline{3} \overline{1} = \overline{1} \) holds. Now we see that \( \overline{0}, \overline{1}, \overline{3}, \overline{4} \) are \( \overline{1} \)-idempotents and \( \overline{0}, \overline{3} \) are \( \overline{3} \)-idempotents. Let us define a binary relation \( \rho \) on \( S \) by \( \overline{a} \rho \overline{b} \) if and only if \( \overline{a} - \overline{b} = \overline{n} \) where \( n \) is even. Then \( \rho \) is a congruence. Now in \( S_{\beta} \), where \( \overline{\beta} = 3, \overline{3} \) and \( \overline{0} \) are idempotents and \( \overline{3} - \overline{0} = \overline{3} \). Thus \( \overline{3} \) is not \( \rho \)-related to \( \overline{0} \). Consequently \( \rho \) is not idempotent separating congruence in \( S_{\alpha} \).
1.3 Ideals in Γ-semigroups

Ideal theory in Γ-semigroups was developed by Sen and Saha in [58]. Latter, T.K.Dutta and N.C.Adhikari developed this field for both sided Γ-semigroup in [14], [15] and [17]. We recall some definitions and results for ideals which are required for this thesis.

Definition 1.3.1 [58] Let $S$ be a Γ-semigroup. A nonempty subset $B$ of $S$ is said to be a Γ-subsemigroup of $S$ if $B\Gamma B \subseteq B$.

Definition 1.3.2 A right (resp. left) ideal of a Γ-semigroup $S$ is a nonempty subset $I$ of $S$ such that $IT \subseteq I$ (resp. $SI \subseteq I$). If $I$ is both a right ideal and a left ideal then we say that $I$ is an ideal of $S$.

A (right, left) ideal $I$ of a Γ-semigroup $S$ is said to be proper (right, left) ideal of $S$ if $I \neq S$.

For each $a \in S$, the smallest left ideal of $S$ containing $a$ is called the principal left ideal of $S$ generated by $a$ and is denoted by $(a)_l$. Similarly $(a)_r$ and $(a)$ are the principal right ideal and principal ideal of $S$ generated by $a$ respectively. From [49], we find that $(a)_l = STa \cup \{a\}, (a)_r = aTS \cup \{a\}$ and $(a) = \{a\} \cup aTS \cup STa \cup STaTS$.

We now show some examples of ideal in Γ-semigroup.

Example 1.3.3 Let $S$ be the set of all $2 \times 3$ matrices over the set of all positive integers and $\Gamma$ be the set of all $3 \times 2$ matrices over the same set. Then $S$ is a Γ-semigroup where $acb$ denotes the usual product of matrices $a, \alpha, b$ for $a, b \in S$ and $\alpha \in \Gamma$. Let $I$ be the subset of $S$ consisting of all those $2 \times 3$ matrices whose elements are even positive integers. Then $I$ is an ideal of $S$.

Example 1.3.4 Let $S$ be the set of all $2 \times 3$ matrices and $\Gamma$ be the set of all $3 \times 2$ matrices over the ring of integers. Then $S$ is a Γ-semigroup with respect to usual
product of matrices. Let \( A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \): \( a, b, c, d \) are integers \}. Then \( A \) is a left ideal of \( S \) but not necessarily a right ideal of \( S \). Consider \( B = \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \end{pmatrix} \): \( a, b, c \) are integers \}. Then \( B \) is a right ideal of \( S \) but not necessarily a left ideal of \( S \).

In the theorems given below, the regular \( \Gamma \)-semigroup was characterized by condition on one sided ideals.

**Theorem 1.3.5** [49] Let \( S \) be a \( \Gamma \)-semigroup. If \( a = aab(3a, b \in T, \beta \in \Gamma \) be a regular element of \( S \) then \( (a)_r = a\alpha S \) and \( (a)_l = S\beta a \).

**Theorem 1.3.6** [49] Let \( S \) be a \( \Gamma \)-semigroup. An element \( a \in S \) is regular if and only if \( (a)_r = e\beta S \) for some idempotent \( e = e\beta e \in S \) where \( \beta \in \Gamma \).

**Theorem 1.3.7** [58] A \( \Gamma \)-semigroup \( S \) is regular if and only if for any left ideal \( A \) and for any right ideal \( B \) of \( S \) we have \( A \cap B = B\Gamma A \)

### 1.4 Operator semigroup of a \( \Gamma \)-semigroup

After the introduction of \( \Gamma \)-ring by Nobusawa [44] in the year 1964, the the notion of operator rings of a \( \Gamma \)-ring was introduced by Kyuno ( [35], [36]). In 1986, Booth [3] also studied operator rings and we feel that operator rings are most useful structure for studying a \( \Gamma \)-ring. In [14], T. K. Dutta and N. C. Adhikari introduced this notion in both sided \( \Gamma \)-semigroup and studied several results relating this structure. They defined operator semigroups as follows:

Let \( S \) be a both sided \( \Gamma \)-semigroup and \( \rho \) be a relation on \( S \times \Gamma \) defined as follows :
(\( x, \alpha \rho (y, \beta) \) if and only if \( x\alpha s = y\beta s \) for all \( s \in S \) and \( \gamma x\alpha = \gamma y\beta \) for all \( \gamma \in \Gamma \). Then
\( \rho \) is an equivalence relation on \( S \times \Gamma \). Let \([x, \alpha]\) denote the equivalence class containing \((x, \alpha)\) and let \( L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\} \). Define a multiplication on \( L \) as follows:

\([x, \alpha][y, \beta] = [x \alpha y, \beta]\). It can be shown that the multiplication is well defined and \( L \) forms a semigroup which is called the left operator semigroup of the \( \Gamma \)-semigroup \( S \).

Similarly define a relation \( \xi \) on \( \Gamma \times S \) as follows: \((\alpha, a)\xi(\beta, b)\) if and only if \( saa = s \beta a \) for all \( s \in S \) and \( \alpha a \gamma = \beta b \gamma \) for all \( \gamma \in \Gamma \). Then \( \xi \) is an equivalence relation on \( \Gamma \times S \).

Let \([\alpha, a]\) denote the equivalence class containing \((\alpha, a)\) and \( R = \{[\alpha, x] : \alpha \in \Gamma \text{ and } x \in S\} \). Define a multiplication on \( R \) by \([\alpha, x][\beta, y] = [\alpha, x \beta y]\). Then \( R \) is a semigroup and it is called the right operator semigroup of \( S \).

We are now interested about \( \Gamma \)-semigroup defined by Sen and Saha [58]. Here we can define operator semigroups as follows.

Let \( S \) be a \( \Gamma \)-semigroup and \( \rho \) be a relation on \( S \times \Gamma \) defined as follows: \((x, \alpha)\rho(y, \beta)\) if and only if \( x \alpha s = y \beta s \) for all \( s \in S \). Then \( \rho \) is an equivalence relation on \( S \times \Gamma \). Let \([x, \alpha]\) denote the equivalence class containing \((x, \alpha)\) and let \( L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\} \).

We now define a multiplication on \( L \) as follows: \([x, \alpha][y, \beta] = [x \alpha y, \beta]\). Then the multiplication is well defined and \( L \) forms a semigroup. We call \( L \) the left operator semigroup of the \( \Gamma \)-semigroup \( S \). Similarly we can introduce right operator semigroup of \( S \).

We see that the left operator semigroup of the one sided \( \Gamma \)-semigroup is constructed by omitting one condition of the both sided \( \Gamma \)-semigroup.

We now give some definitions and results which are analogous to that of [14].

**Definition 1.4.1** Let \( S \) be a \( \Gamma \)-semigroup. An element \( e \in S \) is said to be a left (resp. right) \( \gamma \)-unity for some \( \gamma \in \Gamma \) if \( e \gamma a = a \) (resp. \( a \gamma e = a \)) for all \( a \in S \).

We now consider the following examples.
Example 1.4.2 Let $S$ be the set of all $2 \times 3$ matrices and $T$ be the set of all $3 \times 2$ matrices over the ring of integers $\mathbb{Z}$. Then for all $A, B, C \in S$ and $P, Q \in T$ we have $APB \in S$ and since the matrix multiplication is associative, we have $(APB)QC = AP(BQC)$.

Hence $S$ is a $\Gamma$-semigroup. In this $\Gamma$-semigroup \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]
is a left $\alpha$-unity but not a right $\alpha$-unity of $S$ for $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Example 1.4.3 Let $S$ be the set of all integers of the form $4n+1$ and $T$ be the set of all integers of the form $4n+3$ where $n$ is an integer. If $a \circ b$ is $a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$ then $S$ is a $\Gamma$-semigroup. Here 1 is a left $(-1)$-unity and also a right $(-1)$-unity.

Example 1.4.4 Let us consider $N$, the set of all natural numbers. Let $S$ be the set of all mappings from $N$ to $N \times N$ and $T$ be the set of all mappings from $N \times N$ to $N$. Then the usual mapping product of two elements of $S$ cannot be defined. But if we take $f, g$ from $S$ and $\alpha$ from $\Gamma$, the usual mapping product $f \circ g$ can be defined. Also, we find that $f \circ g \in S$ and $(f \circ g) \circ h = f \circ (g \circ h)$. Hence $S$ is a $\Gamma$-semigroup. Now we know that the set $N \times N$ is countable. Hence there exists a bijective mapping $f \in S$. Since $f$ is bijective, there exists $\alpha : N \times N \rightarrow N$ such that $f \circ \alpha$ is the identity mapping on $N \times N$.

Then $f \circ g = g \circ \alpha = g$ for all $g \in S$. Hence $f$ is both left $\alpha$-unity and right $\alpha$-unity of $S$.

We observe that if $e$ is a left (resp. right) $\gamma$-unity of $S$ then $[e, \gamma] \in L$ (resp. $[\gamma, e] \in R$). If $S$ has a left $\gamma$-unity and as well as right $\delta$-unity for some $\gamma, \delta \in \Gamma$ then we say that $S$ is a $\Gamma$-semigroup with left unities as well as right unities.

We can prove the following theorems to show that the unity elements of $S$ play an important role in the operator semigroups.
Theorem 1.4.5 Let $S$ be a $\Gamma$-semigroup. If $e$ is a left $\delta$-unity of $S$ then $[e, \delta]$ is the identity element of $L$.

Dually we have

Theorem 1.4.6 Let $S$ be a $\Gamma$-semigroup. If $f$ is a right $\mu$-unity of $S$ then $[\mu, f]$ is the identity element of $R$.

We now give a relation between the set of all ideals of $S$ and that of the left (resp. right) operator semigroup $L$ (resp. $R$).

Throughout this article we consider a $\Gamma$-semigroup $S$ such that $S$ has left and right unities.

Let $S$ be a $\Gamma$-semigroup with left operator semigroup $L$. For $P \subseteq L$ and $Q \subseteq S$ we define $P^+ = \{x \in S : [x, \alpha] \in P \text{ for all } \alpha \in \Gamma\}$ and $Q^{+'} = \{[x, \alpha] \in L : x\alpha s \in Q \text{ for all } s \in S\}$.

The following theorems (analogous to that of [14]) give the relation between the ideals of a $\Gamma$-semigroup $S$ and the ideals of the left operator semigroup $L$ of $S$.

Theorem 1.4.7 Let $S$ be a $\Gamma$-semigroup with left operator semigroup $L$. Then

(i) if $P$ is a right ideal of $L$ then $P^+$ is a right ideal of $S$ and

(ii) if $P$ is an ideal of $L$ then $P^+$ is an ideal of $S$.

Theorem 1.4.8 Let $S$ be a $\Gamma$-semigroup with left operator semigroup $L$. Then

(i) if $Q$ is a right ideal of $S$ then $Q^{+'}$ is a right ideal of $L$ and

(ii) if $Q$ is an ideal of $S$ then $Q^{+'}$ is an ideal of $L$.

Theorem 1.4.9 Let $S$ be a $\Gamma$-semigroup with left operator semigroup $L$. Then there exists an inclusion preserving bijection $Q \longrightarrow Q^{+'}$ between the set of all right ideals of $S$ and the set of all right ideals of $L$.  

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Proof: Note that $(Q^+)^+ = \{x \in S : [x, \alpha] \in Q^+ \text{ for all } \alpha \in \Gamma\} = \{x \in S : x\alpha s \in Q \text{ for all } s \in S \text{ and for all } \alpha \in \Gamma\}$. Since $Q$ is a right ideal of $S$, $QTS \subseteq Q$. Thus $Q \subseteq (Q^+)^+$. On the other hand, since $S$ has right unity, $(Q^+)^+ \subseteq (Q^+)^+TS \subseteq Q$. Thus $(Q^+)^+ = Q$.

Let $U$ be a right ideal of $L$. Then we have $(U^+)^+ = \{[x, \alpha] \in L : x\alpha s \in U^+ \text{ for all } s \in S\} = \{[x, \alpha] \in L : [x, \alpha][s, \gamma] = [x\alpha s, \gamma] \in U \text{ for all } s \in S \text{ and for all } \gamma \in \Gamma\}$. Since $U$ is right ideal of $L$, $UL \subseteq U$. Thus $U \subseteq (U^+)^+$. Again since $L$ has identity, $(U^+)^+ \subseteq (U^+)^+L \subseteq U$. Thus $(U^+)^+ = U$. Hence the mapping $Q \rightarrow Q^+$ is bijective.

Now we shall show that the mapping is inclusion preserving. Let $P$ and $Q$ be two right ideals of $S$ such that $P \subseteq Q$. We have to show that $P^+ \subseteq Q^+$. Let $[x, \alpha] \in P^+$. Then $x\alpha s \in P$ for all $s \in S$. Since $P \subseteq Q$, $x\alpha s \in Q$ for all $s \in S$. Hence $[x, \alpha] \in Q^+$. Therefore $P^+ \subseteq Q^+$. Hence the theorem.

Let $S$ be a $\Gamma$-semigroup with right operator semigroup $R$. For $P \subseteq R$ and $Q \subseteq S$ we define $P^* = \{x \in S : [\alpha, x] \in P \text{ for all } \alpha \in \Gamma\}$ and $Q^* = \{[\alpha, a] \in R : x\alpha a \in Q \text{ for all } s \in S\}$.

The proofs of the following theorems are similar to those of the above theorems and hence we omit them.

**Theorem 1.4.10** Let $S$ be a $\Gamma$-semigroup with right operator semigroup $R$. Then

(i) if $P$ is a left ideal of $R$ then $P^*$ is a left ideal of $S$;

(ii) if $P$ is an ideal of $R$ then $P^*$ is an ideal of $S$.

**Theorem 1.4.11** Let $S$ be a $\Gamma$-semigroup with right operator semigroup $R$. Then

(i) if $Q$ is a left ideal of $S$ then $Q^*$ is a left ideal of $R$;

(ii) if $Q$ is an ideal of $S$ then $Q^*$ is an ideal of $R$. 

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Theorem 1.4.12 Let $S$ be a $\Gamma$-semigroup with the right operator semigroup $R$. Then there exists an inclusion preserving bijection $Q \rightarrow Q^*$ between the set of all left ideals of $S$ and the set of all left ideals of $R$.

Theorem 1.4.13 Let $S$ be a $\Gamma$-semigroup with left operator semigroup $L$ and right operator semigroup $R$. Then there exists an inclusion preserving bijection between the set of all ideals of $S$ and the set of all ideals of $L$ (resp. $Q \rightarrow Q^*$) where $Q$ is an ideal of $S$.

1.5 Green's relations in $\Gamma$-semigroups

Green's relations play a fundamental role in semigroup theory and it is natural to consider them in the context of $\Gamma$-semigroup theory. These notions were discussed in [49] by N.K. Saha and in [18] by T. K. Dutta and T. K. Chatterjee and in [65] by A. Seth.

Definition 1.5.1 [49] Let $S$ be a $\Gamma$-semigroup. For $a, b \in S$, The binary relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are given by

- $a \mathcal{L} b$ if $S \Gamma a \cup \{a\} = S \Gamma b \cup \{b\}$
- $a \mathcal{R} b$ if $a \Gamma S \cup \{a\} = b \Gamma S \cup \{b\}$
- $a \mathcal{H} b$ if $a \mathcal{L} b$ and $a \mathcal{R} b$
- $a \mathcal{D} b$ if $a \mathcal{L} c$ and $c \mathcal{R} b$ for some $c \in S$
- $a \mathcal{J} b$ if $a \Gamma S \cup S \Gamma a \cup S \Gamma a \Gamma S \cup \{a\} = b \Gamma S \cup S \Gamma b \cup S \Gamma b \Gamma S \cup \{b\}$

Note the following facts about these relations in a $\Gamma$-semigroup $S$.

(a) $\mathcal{L}, \mathcal{R}, \mathcal{J}$ and $\mathcal{H}$ are equivalence relations on $S$,
(b) $a \mathcal{L} b$ implies $a a c \mathcal{L} b a c$ for all $c \in S$ and $\alpha \in \Gamma$, 

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(c) $aRb$ implies $caaRcab$ for all $c \in S$ and $\alpha \in \Gamma$,

(d) $aCb$ if and only if either $a = b$ or there exist $\alpha, \beta \in \Gamma$ and $c, d \in S$ such that $a = cab$ and $b = d\beta a$.

(e) $aRb$ if and only if either $a = b$ or there exist $\alpha, \beta \in \Gamma$ and $c, d \in S$ such that $a = bac$ and $b = a\beta d$ and

(f) $D \subseteq \mathcal{J}$.

We now state some results from [18], [49] and [85].

**Theorem 1.5.2** Let $S$ be a regular $\Gamma$-semigroup. Then

(i) $aCb$ if and only if there exist $a, \alpha, \beta \in \Gamma$ and $a', b' \in V_{\alpha}(a), b' \in V_{\alpha}(b)$ such that $a'\beta a = b'\delta b$.

(ii) $aRb$ if and only if there exist $\alpha, \beta, \gamma \in \Gamma$ and $a', b' \in V_{\alpha}(a), b' \in V_{\beta}(b)$ such that $a\alpha a' = b\gamma b'$.

(iii) $aHb$ if and only if there exist $\gamma, \delta \in \Gamma$ and $a', b' \in V_{\gamma}(a), b' \in V_{\delta}(b)$ such that $a\gamma a' = b\gamma b'$ and $a\delta a = b\delta b$.

**Theorem 1.5.3** [65] Let $S$ be a $\Gamma$-semigroup. If $H$ is an $H$-class of $S$ then either $H\alpha H \cap H = \emptyset$ or $H$ is a subgroup of the semigroup $S_{\alpha}$ where $\alpha \in \Gamma$ and hence $H$ contains an $\alpha$-idempotent.

In semigroups, $D$-class of the Green's relation has some interesting property. If one element of a $D$-class is regular then every element of that $D$-class is regular. Here we state the extended results to $\Gamma$-semigroups.

**Theorem 1.5.4** [65] Let $a$ be an element of a regular $D$-class $D$ in a regular $\Gamma$-semigroup $S$. If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contain $\alpha$-idempotent $e$ and $\beta$-idempotent $f$ respectively, then $H_b$ contains $a^* \in V_{\beta}(a)$ such that $a\beta a^* = e$ and $a^*\alpha a = f$. 

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Theorem 1.5.5 [65] In a regular $\mathcal{D}$-class of a $\Gamma$-semigroup $S$ every $\mathcal{L}$-class and every $\mathcal{R}$-class contains at least one $\alpha$-idempotent for some $\alpha \in \Gamma$.

Theorem 1.5.6 [65] For elements $a$ and $b$ of a $\Gamma$-semigroup $S$ and $\alpha \in \Gamma$, we have $a \circ b \in R_a \cap L_b$ if and only if $R_b \cap L_a$ contains an $\alpha$-idempotent. In such a case $a \alpha(H_b) = (H_a) \alpha b = (H_a) \alpha (H_b) = H_{a \circ b} = R_a \cap L_b$.

Theorem 1.5.7 Let $e$ be an $\alpha$-idempotent and $f$ be a $\beta$-idempotent of a $\Gamma$-semigroup $S$. Then

(i) $e \mathcal{L} f$ implies $e \beta f = e$;

(ii) $e \mathcal{R} f$ implies $e \alpha f = f$.

Theorem 1.5.8 [49] Let $S$ be a $\Gamma$-semigroup and $a \in S$. Let $D_a$ denote the $\mathcal{D}$-class of $S$ containing $a$. If $a$ is regular, then every element of $D_a$ is regular.

1.6 Some classes of regular $\Gamma$-semigroups

It is known that the notion of inverses semigroup is the most natural generalization of the notion of groups. This notion was generalized in the theory of $\Gamma$-semigroup. In 1987, Seth and Saha [51] introduced inverse $\Gamma$-semigroup.

Definition 1.6.1 [51] A regular $\Gamma$-semigroup $S$ is called an inverse $\Gamma$-semigroup if $|V^\beta_\alpha(a)| = 1$ for all $a \in S$ and for all $\alpha, \beta \in \Gamma$, whenever $V^\beta_\alpha(a) \neq \emptyset$. That is every element $a \in S$ has a unique $(\alpha, \beta)$-inverse whenever $(\alpha, \beta)$-inverse of $a$ exists.

The following theorem gives a useful necessary and sufficient condition for a regular $\Gamma$-semigroup to be an inverse $\Gamma$-semigroup.
Theorem 1.6.2 [51] Let $S$ be a $\Gamma$-semigroup. $S$ is an inverse $\Gamma$-semigroup if and only if \( (i) \) $S$ is regular and \( (ii) \) if $e$ and $f$ be any two $\alpha$-idempotents of $S$ then $eaf = fae$, where $\alpha \in \Gamma$.

Orthodox semigroups were first studied by Hall [27] and Yamada in [74] and [75]. A regular semigroup is said to be an orthodox semigroup if the set of all idempotents of the semigroup forms a subsemigroup. In 1990 Sen and Saha [59] generalized this notion in $\Gamma$-semigroups.

Definition 1.6.3 A regular $\Gamma$-semigroup $S$ is called an orthodox $\Gamma$-semigroup if for an $\alpha$-idempotent $e$ and a $\beta$-idempotent $f$ of $S$, $eaf$ and $fae$ are $\beta$-idempotents in $S$.

Example 1.6.4 [59] Let $Q^\ast$ denote the set of all nonzero rational numbers. Let $\Gamma$ be the set of all positive integers. Let $a \in Q^\ast$, $\alpha \in \Gamma$ and $b \in Q^\ast$. Define $aab$ by $|a|ab$. For this operation $Q^\ast$ is a $\Gamma$-semigroup. Let $\frac{p}{q} \in Q^\ast$. Now, $|\frac{p}{q}q|q|p| = \frac{p}{q}$. Hence this is a regular $\Gamma$-semigroup. Here $\frac{1}{q}(q \in \Gamma)$ is a $q$-idempotent. These are the only idempotents of $Q^\ast$. Now $|\frac{1}{q}|q|p|$ is a $p$-idempotent. Hence $Q^\ast$ is an orthodox $\Gamma$-semigroup.

In [58] Sen and Saha proved that in a $\Gamma$-semigroup $S$ if $S_\alpha$ is a group for some $\alpha \in \Gamma$, then $S_\alpha$ is a group for all $\alpha \in \Gamma$.

Definition 1.6.5 A $\Gamma$-semigroup $S$ is called $\Gamma$-group if $S_\alpha$ is a group for some (hence for all) $\alpha \in \Gamma$.

Definition 1.6.6 [58] Let $S$ be a $\Gamma$-semigroup. $S$ is said to be right (resp. left) simple if it has no proper right (resp. left) ideal. $S$ is said to be simple if it has no proper ideal.

They also proved the following theorems.
Theorem 1.6.7 Let $S$ be a $\Gamma$-semigroup. $S$ is a $\Gamma$-group if and only if $S$ is both left simple and right simple $\Gamma$-semigroup.

Definition 1.6.8 [64] A congruence $\rho$ in $S$ is called a $\Gamma$-group congruence if $S/\rho$ is a $\Gamma$-group.

1.7 Semihypergroups

The hyperoperation was first introduced in 1934 when Marty defined hypergroups [39]. He applied that to groups, rational algebraic functions. Since then many researchers have studied in this field and developed it. Several papers have been published in hyperalgebraic structures (for instance one can see [1] and [73]). A short review of the theory of semihypergroup and hypergroup appears in [13]. We now give some basic definitions and example which help us to develop the chapter 6 of the thesis.

Definition 1.7.1 [13] Let $H$ be a set and $P^*(H)$ be the set of all nonempty subset of $H$. A mapping $\circ$ from $H$ to $P^*(H)$ is called a hyperoperation on $H$. If $(a, b) \in H \times H$, its image under $\circ$ is denoted by $a \circ b$.

Remark 1.7.2 [13] The hyperoperation is extended to subsets of $H$ in a natural way, so that $A \circ B$ is given by $A \circ B = \cap \{a \circ b : a \in A, b \in B\}$. The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$ respectively. Generally the singleton set $\{a\}$ is identified by its element $a$.

Definition 1.7.3 [13] Let $H$ be a set and $\circ$ be a hyperoperation on $H$. Then the structure $(H, \circ)$ is called a hypergroupoid. A hypergroupoid $(H, \circ)$ which is associative, i.e., $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$ is called a semihypergroup.
Definition 1.7.4 \[13\] A hypergroup is a semihypergroup in which \(x \circ H = H \circ x = H\) for all \(x \in H\).

Example 1.7.5 Let \(S = \{a, b\}\). We define a hyperoperation \(\circ\) on \(S\) by \(a \circ a = \{a\}\), \(a \circ b = \{b\}\), \(b \circ a = \{a, b\}\) and \(b \circ b = \{a, b\}\), then it can be easily verified that:

\[(a \circ a) \circ a = a \circ (a \circ a) = \{a\}, a \circ (a \circ b) = (a \circ a) \circ b = \{b\}, (a \circ b) \circ a = a \circ (b \circ a) = \{a, b\},\]
\[(a \circ b) \circ b = a \circ (b \circ b) = \{a, b\}, (b \circ a) \circ a = b \circ (a \circ a) = \{a, b\}, (b \circ a) \circ b = b \circ (a \circ b) = \{a, b\},\]
\[(b \circ b) \circ a = b \circ (b \circ a) = \{a, b\}, (b \circ b) \circ b = b \circ (b \circ b) = \{a, b\}.\]

Hence \((S, \circ)\) is a semihypergroup. Moreover \(a \circ S = S \circ a = S = b \circ S = S \circ b\). Hence \((S, \circ)\) is a hypergroup.