PART B

Wave Scattering Problems in an Undulating Bottom Topography
Chapter 3

Oblique wave scattering by undulations on the bed of an ice-covered ocean

3.1 Introduction

When a train of surface water waves is incident on an obstacle situated at the bottom of a laterally unbounded ocean of uniform finite depth, it is partially reflected by and transmitted over the obstacle. For an obstacle of arbitrary shape, the problem of determining the reflection and transmission coefficients is in general a difficult task. However, when the obstacle is in the form of a small deformation of the bottom (undulations on an ocean bed), then some approximate methods can be employed to obtain these coefficients approximately. For example, for small cylindrical deformation of the bottom, Miles (1981) used a perturbation method followed by the finite cosine transform technique in the mathematical analysis to obtain the reflection and transmission

\[1\] The content of this chapter is based on the paper "Oblique wave scattering by undulations on the bed of an ice-covered ocean", Arch. Mech., 61 (2004) 485-493.
coefficients up to first order when a train of surface waves is obliquely incident on the bottom deformation. When the obstacle is in the form of bottom undulations, such as sand ripples, Davies (1982) considered normal incidence of surface water wave train and treated the problem on the basis of linear perturbation theory. He introduced a linear friction term in the dynamical condition at the free surface so as to apply the Fourier transform technique in the mathematical analysis. The coefficient of friction was then made to tend to zero in the asymptotic results for the velocity potential far away from the undulations so as to obtain the reflection and transmission coefficients up to first order analytically. Later Mandal and Basu (1990) generalised the problem considered by Miles (1981) to include the effect of surface tension at the free surface. They also employed a simplified perturbation analysis followed by an appropriate use of Green’s integral theorem in the mathematical analysis to obtain a general representation of the first-order potential function. Its asymptotic forms far away from the deformation at either side produce the first-order reflection and transmission coefficients in terms of integrals involving the shape function describing the deformation.

All the aforesaid works involve an ocean with a free surface. However, there is a considerable interest in recent times to investigate wave propagation problems in an ice-covered ocean wherein the ocean is covered by a thin sheet of ice, modelled as an elastic plate. This has motivated us to consider the problem of oblique wave scattering by small undulations on the bottom of a laterally unbounded ocean with an ice-cover instead of a free surface. The ice-cover is modelled as a thin sheet of elastic plate of infinite extent having a very small thickness $h_0$ of which a still smaller part is immersed in water.

Assuming linear theory and irrotational motion, the velocity potential function describing the time-harmonic motion of angular frequency $\omega$, in water of uniform finite depth $h$ and having an ice-cover at the top, can be represented
by Re ($\phi e^{-i\omega t}$), where $\phi$ satisfies

$$\nabla^2 \phi = 0, \quad 0 \leq y \leq h,$$  \hspace{1cm} (3.1.1)

the linearised ice-cover condition

$$K \phi + (D \nabla^2_{x,z} + 1) \phi_y = 0 \quad \text{on} \quad y = 0,$$  \hspace{1cm} (3.1.2)

and the bottom condition

$$\phi_y = 0 \quad \text{on} \quad y = h.$$  \hspace{1cm} (3.1.3)

The time-dependent factor $e^{-i\omega t}$ will be dropped throughout from now on.

Here $y$-axis is chosen vertically downwards into the fluid region, $(x, z)$-plane is the rest position of the lower part of the ice-cover, $\nabla^2$ denotes the three-dimensional Laplacian operator while $\nabla^2_{x,z}$ denotes the two-dimensional biharmonic operator in the $(x, z)$-plane, $K = a^2/g$ where $g$ is the gravity, $D$ is the flexural rigidity of ice-cover and is given by

$$D = \frac{E h_0^3}{12(1 - \gamma^2) \rho g}$$

where $E$ is the Young's modulus, $\gamma$ is the Poisson's ratio of the elastic material comprising the ice-cover and $\rho$ is the density of water. In deriving the ice-cover condition (3.1.2), waves are assumed to be long compared to the thickness of the ice-cover (cf. Gol'dshtein and Marchenko (1989) and Chakrabarti (2000a)).

A possible solution for $\phi$ representing a train of time-harmonic waves propagating on the ice-cover and making an angle $\theta$ with the positive $x$-direction, is given by

$$\phi_0(x, y, z) = \cosh k_0(h - y) e^{i k_0 (x \cos \theta + z \sin \theta)}$$  \hspace{1cm} (3.1.4)

where $k_0$ is the unique real positive root of the transcendental equation

$$\Delta(k) \equiv k(D k^4 + 1) \sinh kh - K \cosh kh = 0.$$  \hspace{1cm} (3.1.5)

This equation has two real roots $\pm k_0$, two pair of complex conjugate roots $\pm \mu, \pm \overline{\mu} (\mu = \alpha + i\beta, \overline{\mu} = \alpha - i\beta, \alpha > 0, \beta > 0$ and $\alpha > \beta)$ and an infinite number
of purely imaginary roots $\pm i k\alpha (k_n > 0, n = 1, 2, \ldots)$ where $k_n(n = 1, 2, \ldots)$ are real, satisfy

$$k_n(Dk_n^4 + 1) \sin k_n h + K \cos k_n h = 0, \quad (3.1.6)$$

and $k_n \to \frac{m\pi}{h}$ as $n \to \infty$ (cf. Chung and Fox (2000)).

To tackle the problem of oblique wave scattering by small cylindrical undulations of the bottom of an ocean with an ice-cover, here also we employ a perturbation technique directly to the governing partial differential equation and the boundary and infinity conditions for the potential function, after extracting out the $z$-dependence by exploiting the geometry of the problem, to obtain a BVP. A suitable use of Green’s integral theorem produces the solution of this BVP, from which the first-order reflection and transmission coefficients are obtained in terms of integrals involving the shape function defining the undulations. For three different forms of the shape functions these coefficients are obtained in closed forms.

### 3.2 Formulation of the problem

The problem of oblique wave scattering by small cylindrical bottom undulations in an ice-covered ocean, assuming linear theory and irrotational motion, is mathematically equivalent to solving following BVP. To solve the PDE

$$\nabla^2 \phi = 0 \quad (3.2.1)$$

in the region $0 \leq y \leq h + \epsilon c(x), -\infty < x, z < \infty$, with the boundary conditions

$$K \phi + (D\nabla^1_{x,z} + 1)\phi_y = 0 \text{ on } y = 0, \quad (3.2.2)$$

$$\phi_n = 0 \text{ on } y = h + \epsilon c(x) \quad (3.2.3)$$

together with suitable conditions as $x \to \pm \infty$ which will be stated shortly.

Here $c(x)$ is a continuous and bounded function describing the shape of the undulations of the ocean bed and $c(x) \to 0$ as $|x| \to \infty$, so that the ocean is of
uniform finite depth far away from the undulations on either sides, and \(\epsilon(>0)\) is a small parameter giving a measure of the smallness of the undulations. \(\phi_n\) in (3.2.3) denotes the normal derivative.

We assume that a water wave train represented by the velocity potential \(\phi(x,y,z)\), given by (3.1.4), is obliquely incident upon the undulations from a large distance in the direction of negative \(x\)-axis, then it undergoes partial transmission and reflection by the undulations. Thus the asymptotic behavior of \(\phi(x,y,z)\) is given by

\[
\phi \rightarrow \begin{cases} 
T\phi_0(x,y,z) & \text{as } x \to \infty, \\
\phi_0(x,y,z) + R\phi_0(-x,y,z) & \text{as } x \to -\infty 
\end{cases}
\]  

(3.2.4)

where \(T\) and \(R\) are the transmission and reflection coefficients respectively and will have to be determined.

As \(\epsilon\) is very small, we can approximate the bottom condition (3.2.3) after neglecting \(O(\epsilon^2)\) terms as

\[-\phi_y + \epsilon \{c'(x)\phi_x - c(x)\phi_{yy}\} = 0 \text{ on } y = h.\]  

(3.2.5)

In view of the geometry of the problem, we can assume that

\[\phi(x,y,z) = \psi(x,y)e^{iuz}\]  

(3.2.6)

where \(\nu = k_0 \sin \theta\). Thus the \(z\)-dependence is extracted out, and the function \(\psi(x,y)\) satisfies the BVP described by

\[
\begin{align*}
\psi_{xx} + \psi_{yy} - \nu^2 \psi &= 0, \quad 0 \leq y \leq h, \quad -\infty < x < \infty, \\
K\psi + \left\{ D \left( \frac{\partial^2}{\partial x^2} - \nu^2 \right)^2 + 1 \right\} \psi_y &= 0 \text{ on } y = 0, \\
-\psi_y + \epsilon \left\{ \frac{\partial}{\partial x} (c(x)\psi_x) - \nu^2 C(x) \right\} &= 0 \text{ on } y = h, \\
\psi(x,y) &= \begin{cases} 
T\psi_0(x,y) & \text{as } x \to \infty, \\
\psi_0(x,y) + R\psi_0(-x,y) & \text{as } x \to -\infty 
\end{cases}
\end{align*}
\]  

(3.2.7)
where 
\[ \psi_0(x, y) = e^{i k_0 x \cos \theta} \cosh k_0 (h - y). \] (3.2.8)

This BVP is solved approximately up to first order of \( \varepsilon \) by using a perturbation analysis applied to the governing PDE, the boundary conditions and infinity conditions.

3.3 Method of solution

Because of the approximate boundary condition (3.2.7) and the fact that a wave train propagating in an ocean of uniform finite depth \( h \) experiences no reflection, we may assume that \( \psi, T \) and \( R \) in (3.2.7) can be expanded in terms of the small parameter \( \varepsilon \) as

\[ \psi(x, y) = \psi_0(x, y) + \varepsilon \psi_1(x, y) + O(\varepsilon^2), \]
\[ T = 1 + \varepsilon T_1 + O(\varepsilon^2), \]
\[ R = \varepsilon R_1 + O(\varepsilon^2). \] (3.3.1)

Using the expansions (3.3.1) in Eqns. (3.2.7), we find that \( \psi_1(x, y) \) satisfies the BVP described by

\[ \psi_{1xx} + \psi_{1yy} - \nu^2 \psi_1 = 0, \quad 0 \leq y \leq h, \quad -\infty < x < \infty, \]
\[ K \psi_1 + \left\{ D \left( \frac{\partial^2}{\partial x^2} - \nu^2 \right)^2 + 1 \right\} \psi_1 = 0 \text{ on } y = 0, \]
\[ \psi_{1y} = i k_0 \cos \theta \frac{\partial}{\partial x} \left( c(x) e^{i k_0 x \cos \theta} \right) - \nu^2 c(x) \]
\[ \equiv q(x) \text{ on } y = h, \]

\[ \psi_1(x, y) \rightarrow \begin{cases} \left( T_1 \psi_0(x, y) \right) & \text{as } x \rightarrow -\infty, \\ \left( R_1 \psi_0(-x, y) \right) & \text{as } x \rightarrow \infty. \end{cases} \] (3.3.2)

We note that \( \psi_1(x, y) \) behaves as an outgoing wave as \( |x| \to \infty \).

To solve the BVP described by (3.3.2), we need to construct the Green's function \( G(x, y; \xi, \eta) \) which satisfies

\[ G_{xx} + G_{yy} - \nu^2 G = 0, \quad 0 \leq y \leq h, \quad -\infty < x < \infty, \quad \text{except at } (\xi, \eta), \]}
\((-\infty < \xi < \infty, \ 0 < \eta < h),\)

\[ G \to -K_0(\nu r) \text{ as } r = \left\{ (x - \xi)^2 + (y - \eta)^2 \right\}^{1/2} \to 0 \]

where \(K_0(z)\) is the modified Bessel function of order zero,

\[ KG + \left\{ D \left( \frac{\partial^2}{\partial x^2} - \nu^2 \right) + 1 \right\} G_y = 0 \text{ on } y = 0, \quad (3.3.3) \]

\[ G_y = 0 \text{ on } y = h, \]

\(G\) behaves as an outgoing wave as \(|x - \xi| \to \infty\).

The construction of \(G(x, y; \xi, \eta)\) is given in the Appendix.

We now apply the Green's theorem to \(\psi_1(x, y)\) and \(G(x, y; \xi, \eta)\) in the form

\[ \int_C \left( \psi_1 \frac{\partial G}{\partial n} - G \frac{\partial \psi_1}{\partial n} \right) ds = 0 \quad (3.3.4) \]

where \(C\) is a contour in the \((x, y)\)-plane formed by the lines \(y = 0, h(-X \leq x \leq X), x = \pm X (0 \leq y \leq h)\) and a circle of small radius \(\delta\) with centre at \((\xi, \eta)\), and ultimately make \(\delta \to 0\) and \(X \to \infty\). The contribution to the integral in (3.3.4) from the circle of radius \(\delta\) as \(\delta \to 0\) is \(2\pi \psi_1(\xi, \eta)\). There will be no contributions from the lines \(x = \pm X (0 \leq y \leq h)\) as \(X \to \infty\) due to the outgoing nature of \(\psi_1\) and \(G\) as \(|x| \to \infty\). The contribution from the line \(y = h(-X \leq x \leq X)\) as \(X \to \infty\) is

\[ \int_{-\infty}^{\infty} G(x, h; \xi, \eta) q(x) dx. \]

The contribution from the line \(y = 0(-X \leq x \leq X)\) as \(X \to \infty\) is also zero. This can be shown as follows. The integral in (3.3.4) along the line \(y = 0(-X \leq x \leq X)\) is

\[ \int_{-X}^{X} (G \psi_1 - \psi_1 G)_{y=0} dx \]

\[ = \frac{D}{K} \int_{-X}^{X} \frac{\partial F}{\partial x} (x, 0) dx \]

\[ = \frac{D}{K} [F(X, 0) - F(-X, 0)] \]
where
\[ F(x, y) = \psi_{1xxy}G_y - G_{xxy}\psi_{1y} + G_{xxy}\psi_{1y} - \psi_{1xxy}G_y - 2\nu^2(\psi_{xy}G_y - G_{xy}\psi_{1y}). \]

Using the far field behavior of \( \psi_1 \) given in (3.3.2) and \( G \) given in (A9), it is easy to see that \( F(x, 0) - F(-x, 0) \to 0 \) as \( X \to \infty \). Thus we finally obtain the solution for \( \psi_1(x, \eta) \) as
\[ \psi_1(x, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x, h, \xi, \eta)q(x)dx. \quad (3.3.5) \]

To obtain the first-order transmission and reflection coefficients \( T_1 \) and \( R_1 \) respectively, we note from (3.3.2) and (A9) that
\[ \psi_1(\xi, \eta) \to \begin{cases} T_1\psi_0(\xi, \eta) & \text{as } \xi \to \infty, \\ R_1\psi_0(-\xi, \eta) & \text{as } \xi \to -\infty \end{cases} \quad (3.3.6) \]
and
\[ G(x, 0; \xi, \eta) \to -4\pi i e^{\pm ik_0x\cos \theta}/k_0\cos \theta A \psi_0(\pm \xi, \eta) \text{ as } \xi \to \pm \infty \quad (3.3.7) \]
where
\[ A = \frac{1}{h + (1+5Dk_0)\sin \theta k_0 h}. \quad (3.3.8) \]

Using the asymptotic results (3.3.6) and (3.3.7) in the representation (3.3.5) we find that
\[ T_1 = -\frac{i}{k_0 \cos \theta} A \int_{-\infty}^{\infty} e^{-ik_0x\cos \theta} q(x)dx \quad (3.3.9) \]
\[ = \frac{i}{k_0 \cos \theta} A \int_{-\infty}^{\infty} c(x)dx, \]
\[ R_1 = -\frac{i}{k_0 \cos \theta} A \int_{-\infty}^{\infty} e^{ik_0x\cos \theta} q(x)dx \quad (3.3.10) \]
\[ = -\frac{i}{k_0 \sec \theta \cos 2\theta} A \int_{-\infty}^{\infty} e^{ik_0x\cos \theta} q(x)dx. \]

The results for an ocean with a free surface are recovered by putting \( D = 0 \) in (3.3.9) and (3.3.10) where then, however, \( k_0 \) denotes the unique real positive...
zero of the transcendental equation
\[ k \sinh kh - K \cosh kh = 0. \]

It is also interesting to note that \( R_1 \) vanishes identically for \( \theta = \pi/4 \), independently of the shape function \( c(x) \). This was also observed Mandal and Basu (1990) and Miles (1981) in the case of ocean with a free surface with or without surface tension.

We now consider three special types of undulations.

(i) \( c(x) = ae^{-|ax|}(\lambda > 0) \). Here the bottom undulation is maximum at \((0, h)\) and decreases exponentially on either side of \((0, h)\). In this case
\[
T_1 = \frac{2iak_0 A}{\lambda} \sec \theta, \\
R_1 = -\frac{2iak_0 A \lambda}{\lambda^2 + 4k_0^2 \cos^2 \theta} \sec \theta \cos 2\theta.
\]

(ii) \( c(x) = ae^{-\lambda x^2}(\lambda > 0) \). Here the undulation is of Gaussian type having a maximum value at \((0, h)\). In this case
\[
T_1 = i \left( \frac{\pi}{\lambda} \right)^{1/2} k_0 a A \sec \theta, \\
R_1 = i \left( \frac{\pi}{\lambda} \right)^{1/2} k_0 a A \sec \theta \cos 2\theta e^{-\frac{\lambda^2 x^2}{2}}.
\]

(iii) \( c(x) = \begin{cases} 
    a \sin \frac{\pi x}{\lambda}, & -\frac{m\pi}{\lambda} \leq x \leq \frac{m\pi}{\lambda} \\
    0, & \text{otherwise}
\end{cases} \)

This represents sinusoidal undulations of the bottom, having \( m \) number of patches and is of considerable physical interest. Davies (1982) earlier made somewhat elaborate study on the effect of sinusoidal undulations on the bottom of an ocean with a free surface, on an incident surface water wave train. In this case
\[ T_1 = 0, \]
\[ R_1 = \sec^2 \theta \cos 2\theta B (-1)^m \frac{\alpha}{\alpha^2 - 1} \sin(\alpha m \pi) \]

where \( B = a A, \) \( a = \frac{2k_0 \cos \theta}{\lambda} \). It is interesting to note that when \( \alpha \approx 1, \) i.e. \( \lambda \approx 2k_0 \cos \theta, \)

\[ R_1 \approx \frac{\pi}{2} \sec^2 \theta \cos 2\theta B m. \] (3.3.11)

The result (3.3.11) has the implication that somewhat large reflection of the incident wave energy occurs when the bed wave number \( \lambda \) is twice the wave number component of the incident wave field along the \( x \)-direction, if the integer \( m \) denoting the number of patches is made large. This phenomenon has practical application in the construction of an efficient reflector of incident wave energy.

### 3.4 Discussion

A simplified perturbation analysis is employed to obtain the first-order transmission and reflection coefficients for the problem of oblique wave scattering by small cylindrical undulations on the bottom of an ocean with an ice-cover modelled as a thin elastic plate. The first-order reflection coefficient vanishes independently of the shape of the undulations if the angle of incidence is \( \frac{\pi}{4} \). By making \( D \) to be zero, the results for an ocean with a free surface are recovered. For sinusoidal undulations having \( m \) number of patches, the first-order transmission coefficient vanishes identically, and the reflection coefficient becomes a constant multiple of the number of patches when the ocean-bed wave number is twice the \( x \)-component of the incident field wave number, which suggests that comparatively large reflection of the incident wave energy is possible by making the number of patches somewhat large.

### APPENDIX

The Green's function \( G(x, y; \xi, \eta) \) satisfying Eqns. (3.3.3) can be constructed...
as follows. By noting the integral representations

\[
\frac{\partial}{\partial \nu} K_0(\nu r) = \mp \int_\nu^\infty \frac{k \cos (\nu - \xi)}{\xi} e^{\mp k(\nu - \eta)} dk \text{ for } y > \eta, \tag{A1}
\]

\[
\frac{\partial}{\partial \nu} K_0(\nu r') = - \int_\nu^\infty \frac{k \cos (\nu - \xi)}{\xi} e^{-k(\nu + \eta)} dk \text{ for } y > \eta
\]

where

\[
r, r' = \{ (x - \xi)^2 + (y - \eta)^2 \} \text{ and } \zeta = (k^2 - \nu^2)^{1/2},
\]

that branch of square root in \( \zeta \) being chosen such that \( \zeta = k \) for \( \nu = 0 \), we can represent \( G \) as

\[
G(x, y; \xi, \eta) = K_0(\nu r') - K_0(\nu r) - \int_\nu^\infty \frac{\cos (\nu - \xi)}{\xi} A(k) \cosh k(h - y) + B(k) \sinh ky \, dk \tag{A2}
\]

where \( A(k) \) and \( B(k) \) are unknown functions of \( k \) and are such that the integral is meaningful. The representation (A2) for \( G \) satisfies (3.3.3)_1 and (3.3.3)_2. The bottom condition is satisfied if we choose

\[
B(k) = 2e^{-kh} \sinh k\eta \tag{A3}
\]

while the ice-cover condition is satisfied by choosing

\[
A\Delta(k) - k(Dk^4 + 1)B = 2k(Dk^4 + 1) \cosh khe^{-k\eta} \tag{A4}
\]

where \( \Delta(k) \) is given by (3.1.5). Thus (A3) and (A4) give

\[
A(k) = \frac{2k(Dk^4 + 1) \cosh k(h - \eta)}{\Delta(k)}. \tag{A5}
\]

Using (A3) and (A5) in (A2) we find that

\[
G(x, y; \xi, \eta) = K_0(\nu r') - K_0(\nu r) - 2 \int_\nu^\infty \left[ e^{-kh} \sinh ky \sinh k\eta + \frac{k(Dk^4 + 1) \cosh k(h - y) \cosh k(h - \eta)}{\Delta(k)} \right] \frac{\cos (\nu - \xi)}{\zeta \cosh kh} \, dk \tag{A6}
\]

where the contour is indented below the pole \( k = k_0 \) which accounts for the satisfaction of the condition (3.3.3)_5. It may be noted that the zeros of \( \cosh kh \)}
do not contribute to (A6). This is evident by using the following representation of $G$ (obtained after using integral representations of $K_0(\nu r)$ and $K_0(\nu' r')$)

$$G(x, y; \xi, \eta) = -2 \int_0^\infty \frac{k(Dk^4 + 1) \cosh k\eta - K \sinh k\eta \cosh k(h-y)}{\Delta(k)} \frac{\cos \zeta(x - \xi)}{\zeta} \, dk, \, y > \eta. \quad (A7)$$

For $y < \eta, y$ and $\eta$ in (A7) are be interchanged. An alternative representation of $G$, which gives the outgoing wave term as $|x - \xi| \to \infty$ explicitly, can be obtained following a procedure given in Rhodes-Robinson (1974), and is given by

$$G(x, y; \xi, \eta) = -4\pi \sum_{n=1}^\infty \frac{k_n(Dk_n^4 + 1) \cos k_n(h-y) \cos k_n(h-\eta) e^{-(k_n^2+\nu^2)^{1/2}|x-\xi|}}{2k_n h(Dk_n^4 + 1) + (5Dk_n^4 + 1) \sinh 2k_n h} \frac{(k_n^2 + \nu^2)^{1/2}}{2k_n h(Dk_n^4 + 1) + (5Dk_n^4 + 1) \sinh 2k_n h} \frac{\mu(D\mu^4 + 1) \cosh \mu(h-y) \cosh \mu(h-\eta) e^{\mu'|x-\xi|}}{2\mu h(1 + D\mu^4) + (5D\mu^4 + 1) \sinh 2\mu h} \mu'$$

$$(A8)$$

where $\mu' = (\mu^2 - \nu^2)^{1/2}$ and $-\mu' = \{(-\mu)^2 - \nu^2\}^{1/2}$, and that branch of the square root has been chosen such that $\mu' = \mu, -\mu' = -\mu$ when $\nu = 0$.

Since $\mu'$ and $-\mu'$ have positive imaginary parts, we find that, as $|x - \xi| \to \infty$,

$$G(x, y; \xi, \eta) \to -4\pi \frac{k_0(Dk_0^4 + 1) \cosh k_0(h-y) \cosh k_0(h-\eta) e^{(k_0^2+\nu^2)^{1/2}|x-\xi|}}{2k_0 h(Dk_0^4 + 1) + (5Dk_0^4 + 1) \sinh 2k_0 h} \frac{(k_0^2 + \nu^2)^{1/2}}{(k_0^2 - \nu^2)^{1/2}}$$

$$(A9)$$

so that $G$ behaves as an outgoing wave as $|x - \xi| \to \infty$.