Chapter 2

Mathematical Preliminaries

In this chapter a brief description of some mathematical techniques used to handle the various physical problems presented in the thesis are given.

2.1 Integral transforms

The technique of integral transforms is a powerful and indispensable tool for modern applied mathematicians or theoretical physicists for successful investigation of boundary value problems arising in mathematical physics. Some of the well known integral transforms such as Fourier, Laplace and Hankel transforms are utilized to handle some water wave problems considered in the thesis. Laplace or Fourier transforms are generally used to solve a boundary value problem where the governing partial differential equation is the Laplace's equation or the modified Helmholtz equation in the rectangular cartesian coordinate system while Hankel transform is used in cylindrical co-ordinate system. Most of the integral transforms and their inversion formula are available in standard books on integral transform, e.g. Sneddon(1972), Davies(1977), Titchmarsh(1937), Debnath(1995) etc.
Fourier transform

If \( f(x) \) is piecewise continuously differentiable and absolutely integrable on the whole real line, then at a point at which it is continuous

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-i\xi x} d\xi \tag{2.1.1}
\]

where

\[
F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx. \tag{2.1.2}
\]

Then we say that the function \( F(\xi) \) is the Fourier transform of \( f(x) \) and the pair of formulae (2.1.1) and (2.1.2) together is called the Fourier inversion theorem. We also write \( \mathcal{F}[f(x); \xi] \) or \( \mathcal{F}[f] \) for the Fourier transform of \( f(t) \) and \( \mathcal{F} \) is called the Fourier transform operator or the Fourier transformation. Also \( \mathcal{F} \) is a linear operator i.e.

\[
\mathcal{F}(\lambda f + \mu g) = \lambda \mathcal{F}(f) + \mu \mathcal{F}(g) \tag{2.1.3}
\]

where \( \lambda \) and \( \mu \) are constants.

The inverse Fourier transform, denoted by \( \mathcal{F}^{-1}\{F(\xi)\} = f(x) \), is defined by

\[
\mathcal{F}^{-1}\{F(\xi)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} F(\xi) d\xi \tag{2.1.4}
\]

where \( \mathcal{F}^{-1} \) is called the inverse Fourier transform operator.

When \( f(x) \) is defined only for positive values of \( x \), there are two forms of Fourier integral transforms known as Fourier cosine transform and Fourier sine transform.

When \( f(x) \) is an even function of \( x \), then

\[
f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\xi) \cos(\xi x) d\xi \tag{2.1.5}
\]

where

\[
F_c(\xi) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\xi x) dx. \tag{2.1.6}
\]
The function $F_c(\xi)$ is called Fourier cosine transform of $f(x)$ and the formula (2.1.5) and (2.1.6) are together called the Fourier cosine inversion theorem.

If $f(x)$ is an odd function, then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_0(\xi) \sin(\xi x) d\xi$$  \hspace{1cm} (2.1.7)

where

$$F_0(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\xi x) dx.$$  \hspace{1cm} (2.1.8)

The function $F_s(\xi)$ is called the Fourier sine transform and the formula (2.1.7) and (2.1.8) together are called Fourier sine inversion theorem.

**Laplace transform**

Let $f(t)$ be an arbitrary function defined on the interval $0 < t < \infty$, then

$$\mathcal{L}\{f(t)\} = \mathcal{F}(p) = \int_0^\infty e^{-pt} f(t) dt$$  \hspace{1cm} (2.1.9)

is the Laplace transform of $f(t)$, provided that the integral exists.

**The existence of Laplace transform**

A function $f(t)$ defined for $t \geq 0$, is said to be of exponential order if there exists constants $A$ and $b$ such that $|f(t)| < Ae^{bt}$ for $t \in [0, \infty)$. The sufficient condition for the Laplace transform of the function $f(t)$ to exist for Re $p > b$ is that $f(t)$ is a piecewise continuous function of exponential order on $[0, \infty)$.

**The Inversion Formula For the Laplace transform**

If $F(p)$ is an analytic function of the complex variable $p$ and is of order $O(p^{-k})$ in some half-plane Re $p \geq \gamma$ where $\gamma$ and $k$ are real constants and $k > 1$, then the integral

$$\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \int_{c-i\epsilon}^{c+i\epsilon} e^{pt} F(p) dp$$
along any line \( \text{Re} p = c > \gamma \) converges to a function \( f(x) \) which is independent of \( c \) and whose Laplace transform is \( F(p) \), \( \text{Re} \ p > \gamma \).

Furthermore, the function \( f(x) \) is continuous for each \( x \geq 0 \) and is \( O(e^{\alpha x}) \) as \( x \to \infty \).

**Hankel transform**

If \( \sqrt{x} f(x) \) is piecewise continuous and absolutely integrable on the positive real line, then at a point of continuity, for \( \nu > -\frac{1}{2} \),

\[
f(x) = \int_0^\infty \xi F_\nu(\xi) J_\nu(\xi x) d\xi
\]

where

\[
F_\nu(\xi) = \int_0^\infty x f(x) J_\nu(\xi x) d\xi,
\]

\( J_\nu \) being the Bessel function of order \( \nu \). \( F_\nu(\xi) \) is called the Hankel transform and the formulae (2.1.10) and (2.1.11) together are called the Hankel inversion theorem. It may be noted that \( \nu \) can be 0 or a positive integer.

**2.2 Green's function theory**

A two-point boundary value problem for a Strum-Liouville type second order linear ordinary differential equation is one which satisfies appropriate boundary conditions at two end points of the interval under consideration. The problem occurs often in various physical problems.

It is given by

\[
L[y] = -\frac{d}{dx} \left[p(x) \frac{dy}{dx}\right] + q(x)y = -f(x), \quad p(x) > 0
\]

with boundary conditions

\[
\begin{align*}
U_1[y] &= a_1 y(a) + a_2 y'(a) = 0, \\
U_2[y] &= b_1 y(b) + b_2 y'(b) = 0
\end{align*}
\]

where \( a_1, a_2 \) and \( b_1, b_2 \) are constants but \( a_1, a_2 \) and \( b_1, b_2 \) do not vanish simultaneously. Here the differential expression \( L[y] \) is self-adjoint.
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The Green's function for the differential operator under given homogeneous boundary conditions at the two end points is the function $G(x, \xi)$ satisfying the following conditions:

(i) $G(x, \xi)$ is continuous for all values of $x$, and its first and second derivatives are continuous for all $x \neq \xi$ in $a \leq x, \xi \leq b$.

(ii) At the pt $x = \xi$, the first derivative of $G(x, \xi)$ has a jump discontinuity given by

$$\frac{dG(x, \xi)}{dx} \bigg|_{x=\xi^-}^{x=\xi^+} = -\frac{1}{p(\xi)}. \quad (2.2.3)$$

(iii) For fixed $\xi$, $G(x, \xi)$ satisfies the prescribed boundary conditions. Moreover, $G(x, \xi)$ is the solution of the associated homogeneous equation except at the point $x = \xi$.

The following result is well known.

If $f(x)$ is continuous in $a \leq x \leq b$, then the function

$$y(x) = \int_a^b G(x, \xi)f(\xi)d\xi \quad (2.2.4)$$

is a solution of the boundary value problem given by (2.2.1) and (2.2.2).

It may be noted that if the associated homogeneous boundary value problem has the trivial solution only, then the Green's function exists and is unique.

For water wave problems, appropriate Green's functions are constructed so as to obtain the solutions of the corresponding boundary value problems involving elliptic type partial differential equations such as Laplace and Helmholtz equations.

Green's identities: The Green's first identity is

$$\int_R u \nabla^2 v dV = -\int_R (\nabla u \cdot \nabla v) dV + \int_S u \frac{\partial v}{\partial n} ds \quad (2.2.5)$$
while the second identity is
\[ \int_{\mathcal{R}} (u \nabla^2 v - v \nabla^2 u) dV = \int_{\partial \mathcal{R}} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS \]  
(2.3.6)

where \( \mathcal{S} \) is boundary of the region \( \mathcal{R} \), \( n \) is unit normal outward to \( \mathcal{S} \). \( u \) and \( v \) have partial derivatives of the second order and are continuous in the bounded region \( \mathcal{R} \).

The Green's function plays the same role for solving partial differential equations as it plays in the case of ordinary differential equation.

The Green's function \( G(X, \xi) \) for the Laplace equation in an open, bounded region \( \mathcal{R} \) in two-dimensional space with boundary \( \mathcal{S} \) is the solution of the boundary value problem

\[ \nabla^2 G = -\delta(X - \xi) \]

with

\[ \text{either } G|_{\partial \mathcal{R}} = 0 \text{ or } \frac{\partial G}{\partial n}|_{\partial \mathcal{R}} = 0 \]  
(2.2.7)

where \( X \) and \( \xi \) are in \( \mathcal{R} \) and \( \delta(X - \xi) \) is the Dirac delta function.

**Solution of the Dirichlet problem:**

The solution of the boundary value problem

\[ \nabla^2 W(X) = 0, \quad X \in \mathcal{R}, \quad W|_{\partial \mathcal{R}} = f(X) \]  
(2.2.8)

is

\[ W(X) = -\int_{\partial \mathcal{R}} f(\xi) \frac{\partial G}{\partial n} d\xi. \]  
(2.2.9)

**Solution of the Neuman problem:**

If the boundary value problem is

\[ \nabla^2 W(X) = 0, \quad X \in \mathcal{R} \text{ and } \frac{\partial W}{\partial n} = f \]  
(2.2.10)
then the solution is

\[ W(X) = \int_s G(X, \xi)f(\xi)d\xi. \]  

(2.2.11)

2.3 Perturbation method applied to some water wave problems in the linearised theory

For problems concerning scattering or radiation of water waves due to presence of bodies of various configurations such as nearly vertical barriers, nearly circular cylinders, nearly vertical cliffs, small cylindrical bottom undulations etc., a simplified perturbation analysis can be employed directly to the governing partial differential equation and the boundary conditions of the corresponding boundary value problem to solve it approximately. A perturbational analysis when employed directly to the governing partial differential equation and the boundary conditions of the scattering problems involving nearly vertical barriers, produces a boundary value problem of a very special type (Mandal and Chakrabarti (1989), Mandal and Kundu(1990), Vijayabharati and Chakrabarti(1991), Mandal and Banerjea(1992), Mandal and Banerjea (1993)). The water wave scattering by a nearly vertical plate using perturbational analysis was studied by Mandal and Chakrabarti(1989). This problem was previously studied by Shaw (1985). The method adopted by Shaw(1985) is rather difficult compared to that devised by Mandal and Chakrabarti (1989).

In the present thesis, the perturbation method is used for handling the problems concerning scattering or radiation of water waves due to small cylindrical bottom undulations in a single-layer fluid as well as in a two-layer fluid.

Let us consider the problem of wave scattering by small cylindrical bottom undulations. Let the bottom of an ocean be described by \( y = h + \epsilon c(x) \) where \( c(x) \) is a continuous and bounded function and \( c(x) \to 0 \) as \( |x| \to \infty \) and \( \epsilon \) gives a measure of smallness of the bottom deformation.
For two-dimensional motion let the velocity potential $\Phi(x, y; t)$ be represented by

$$\Phi(x, y; t) = \Re\{\phi(x, y)e^{-i\omega t}\} \tag{2.3.1}$$

where $\omega$ is the angular frequency.

Let a train of surface water waves represented by the velocity potential $\phi^{\text{inc}}(x, y) = \cosh k_0(h - y)\exp(ik_0x)$, where $k_0$ is the positive real root of the dispersion equation

$$K = k \tanh kh$$

be incident on the bottom undulations. Due to the presence of the undulations at the bottom, the incident wave train is partially reflected by and partially transmitted over the undulating bottom. If $\phi(x, y)$ is the velocity potential then $\phi$ satisfies the BVP

$$\nabla^2 \phi = 0 \quad \text{in} \quad 0 \leq y \leq h + \varepsilon c(x), \tag{2.3.2}$$

$$K\phi + \phi_y = 0 \quad \text{on} \quad y = 0, \tag{2.3.3}$$

$$\phi_y = 0 \quad \text{on} \quad y = h + \varepsilon c(x). \tag{2.3.4}$$

The asymptotic behavior of $\phi$ is given by

$$\phi \rightarrow \begin{cases} 
T\phi_0(x, y) & \text{as} \quad x \rightarrow \infty, \\
\phi_0(x, y) + R\phi_0(-x, y) & \text{as} \quad x \rightarrow -\infty 
\end{cases} \tag{2.3.5}$$

where $T$ and $R$ are the transmission and reflection coefficients and $\phi_0(x, y)$ is the incident wave potential.

The approximate method is found by observing that the bottom condition (2.3.4) can be approximated up to first order of $\varepsilon$ as

$$-\phi_y + \varepsilon \frac{\partial}{\partial x} \{c(x)\frac{\partial \phi}{\partial x}(x, h)\} + O(\varepsilon^2) = 0 \quad \text{on} \quad y = h. \tag{2.3.6}$$

If there is no bottom undulation, then the incident wave train will propagate without any hindrance and there will be total transmission.
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This suggests that $\phi$, $T$ and $R$ can be expanded as

$$
\phi(x, y) = \phi_0(x, y) + \epsilon \phi_1(x, y) + O(\epsilon^2),
$$

$$
T = 1 + \epsilon T_1 + O(\epsilon^3),
$$

$$
R = \epsilon R_1 + O(\epsilon^2).
$$

Substituting the expansions (2.3.7) in (2.3.4), (2.3.5) and (2.3.6) we find that the first order function $\phi_1(x, y)$ satisfies the boundary value problem described by

$$
\nabla^2 \phi_1 = 0 \quad \text{in} \quad 0 \leq y \leq h,
$$

$$
K \phi_1 + \phi_{1y} = 0 \quad \text{on} \quad y = 0,
$$

$$
\phi_{1y} = \frac{\partial}{\partial x} \left\{ c(x) \frac{\partial \phi_0}{\partial x} (x, h) \right\} \equiv q(x), \text{say} \quad \text{on} \quad y = h,
$$

$$
\phi_1 \rightarrow \begin{cases} 
T_1 \phi_0(x, y) & \text{as} \quad x \rightarrow \infty, \\
R_1 \phi_0(-x, y) & \text{as} \quad x \rightarrow -\infty.
\end{cases}
$$

This BVP can be solved by standard methods and then the first order reflection and transmission coefficients $R_1$ and $T_1$ can be obtained.

In the present thesis this perturbation method has been successively employed to handle scattering problem due to the presence of small bottom undulations in a single-layer as well as in two-layer fluid with a free surface or with an ice-cover.

2.4 Method of stationary phase

In the mathematical study of wave generation problems due to initial disturbances at the surface of a liquid, the solutions are obtained in terms of integrals involving rapidly oscillating integrals. The method of obtaining the asymptotic expansion of these integrals was developed by Kelvin and is called the principle or the method of stationary phase(cf. Lamb(1932), Stoker(1957),
The asymptotic forms of these integrals are more convenient in certain important applications in oceanography. Now the method of stationary phase is discussed here briefly.

We consider the generalised Fourier integral

$$I(\alpha) = \int_a^b f(k)e^{i\alpha h(k)}dk, \quad b > a,$$

(2.4.1)

for large positive \(\alpha\) when \(f(k)\) and \(h(k)\) are real and the integral exists. The integrand is a complex number expressed in polar form with \(f(k)\) and \(\alpha h(k)\) being its amplitude and argument (phase).

The method of stationary phase is used to obtain an approximate treatment for (2.4.1) valid when the real constant \(\alpha\) is large. When \(\alpha\) is large, the function \(\exp\{i\alpha h(k)\}\) oscillates rapidly as \(k\) changes, unless \(h(k)\) is nearly constant. The positive and negative contributions to the value of \(I(\alpha)\) cancel out, with the major contribution arising from the neighbourhood of those points in the interval from \(a\) to \(b\) at which \(h(k)\), the phase of the oscillatory part of the interval, varies most slowly, i.e. from the neighbourhoods of the points at which the phase is stationary, that is

$$h'(k) = 0 \quad \text{at} \quad k = k_0.$$

(2.4.2)

In the neighbourhood of this stationary point the oscillatory factor of the integrand of (2.4.1) may be written as

$$e^{i\alpha h(k_0)}\exp\{i\alpha[h(k) - h(k_0)]\}.$$

We approximate \(h(k)\) by the first two terms of the Taylor expansion as

$$h(k) \approx h(k_0) + \frac{1}{2}(k - k_0)^2h''(k_0).$$

(2.4.3)

Thus dominant part of (2.4.1) in the neighbourhood of \(k_0\) is given by

$$I \approx f(k_0)e^{i\alpha h(k_0)} \int_{-\infty}^{\infty} \exp\left[\frac{1}{2}i(k - k_0)^2\alpha h''(k_0)\right]dk$$

(2.4.4)
where the limits $(a, b)$ have been approximated by $(-\infty, \infty)$. Using the fact that
\[ \int_{-\infty}^{\infty} e^{\pm ikx} dk = \left( \frac{\pi}{t} \right)^{1/2} e^{\pm im/4} \quad (2.4.5) \]
we finally have
\[ I \approx e^{i\phi(k_0)} f(k_0) \left( \frac{2\pi}{\alpha |h''(k_0)|} \right)^{1/2} e^{\pm im/4} \quad (2.4.6) \]
where the ± sign is to be taken if $h''(k_0) > 0$.

To determine the validity of the approximate expansion of (2.4.1) in the form (2.4.6), we note that the next term of $h(k)$ in (2.4.3) will be $\frac{1}{6}(k - k_0)^3 f'''(k_0)$ so that this approximation is valid when $(k - k_0)f''(k_0)/f'(k_0)$ is small even when $(k - k_0)^2 f''(k_0)$ is a multiple of $2\pi$. It follows that the above method of approximation is valid under the condition that the ratio
\[ \frac{f'''(k_0)}{||f''(k_0)||^{1/2}} \]
should be small.

In the presence of a stationary point, it is obvious that the principal contribution from the neighbourhood of a stationary point is $O(\alpha^{-1/2})$ but in the absence of a stationary point, the principal contribution is $O(\alpha^{-1})$. Hence only the stationary points contribute to the leading term in the asymptotic expansion of $I(\alpha)$. This stationary phase method is used to obtain the asymptotic form of the surface depression for some wave generation problems in Chapter 8 to 10 in this thesis.

2.5 Source potentials

Velocity potentials due to the presence of different types of singularities present in water is generally known as source potentials and have wide applications in the linearised theory of water waves. For radiation or scattering problems involving the presence of bodies of various configurations the resulting motion can be described by a series of singularities placed on the bodies. These singularities are characterised by their giving rise to a velocity potential.
(Green's function) which is a typical singular solution of Laplace's equation in the neighbourhood of the singularity.

For two-dimensional problems these singularities are either of logarithmic type or multipole type and for three dimensional problems these are point sources or point multipoles. Throne (1953) gave a detailed survey of source potentials due to fundamental line or point singularities submerged in deep water or water of uniform finite depth. These results were modified in the presence of surface tension (cf. Rhodes-Robinson (1970)). For a two-fluid medium, the source potentials due to singularities present in either of the two fluids were obtained by Gorgui and Kassem (1978), Rhodes-Robinson (1980, 1982), Mandal (1981), Mandal and Chakrabarti (1983, 1986a, 1986b), Kassem (1982, 1986, 1987) for a variety of cases where the lower fluid either extends infinitely downwards or is of uniform finite depth.

In this section source potentials due to presence of line singularities in a single-layer fluid or in a two-layer fluid with an ice-cover are constructed by a method similar to that used to construct these when there exists a free surface.

(a) Source potential in single-layer fluid with an ice-cover

We consider irrotational motion under gravity due to the presence of a line source submerged at a point \((\xi, \eta)\) in the water region. Here the upper surface of water is covered by a thin elastic sheet of ice modelled as an elastic plate. If the line source has sinusoidal dependence of the \(z\) co-ordinate, say of the form \(e^{i\omega z} (\nu < K)\), then the source potential \(G\) satisfies the two-dimensional Helmholtz equation

\[
(\nabla^2 - \nu^2)G = 0 \text{ in the fluid region except at } (\xi, \eta)
\]  

(2.5.1)
and in the neighbourhood of the point \((\xi, \eta)\), \(G\) satisfies
\[
G \to K_0(\nu r) \quad \text{as} \quad r = \left\{ (x - \xi)^2 + (y - \eta)^2 \right\}^{1/2} \to 0
\] (2.5.2)
where \(K_0(z)\) is the modified Bessel function of the second kind. The condition at the ice-cover is
\[
KG + \left\{ D \left( \frac{\partial^2}{\partial x^2} - \nu^2 \right) \right\} G_y = 0 \quad \text{on} \quad y = 0,
\] (2.5.3)
while the bottom condition is
\[
G_y = 0 \quad \text{on} \quad y = h.
\] (2.5.4)
Also \(G\) must behave as outgoing waves as \(|x - \xi| \to \infty\).

The source potential \(G\) can be obtained as
\[
G(x, y; \xi, \eta) = K_0(\nu r) - K_0(\nu r')
\]
\[
+ 2\int_\nu^\infty e^{-kh} \sinh k y \sinh k \eta + \frac{k(Dk^4 + 1) \cosh k(h - y) \cosh k(h - \eta)}{\Delta(k)} \cos \zeta(x - \xi) \frac{\cosh \zeta k(y + \eta)}{\zeta \cosh kh} dk
\]
where
\[
\nu, \nu' = \left\{ (x - \xi)^2 + (y \mp \eta)^2 \right\}^{1/2}, \quad \zeta = (k^2 - \nu^2)^{1/2},
\]
\[
\Delta(k) = k(1 + Dk^4) \sinh kh - K \cosh kh
\] (2.5.5)
and the path of integration is indented below the real positive zero of \(\Delta(k)\).

This is due to the outgoing behavior of \(G\) as \(|x - \xi| \to \infty\).

Using the integral representations (Gradshteyn and Ryzhik(1980))
\[
\frac{\partial}{\partial y} K_0(\nu r) = \mp \int_\nu^\infty \frac{k \cos \zeta(x - \xi)}{\zeta} e^{k(y - \eta)} dk \quad \text{for} \quad y \geq \eta,
\] (2.5.7)
\[
\frac{\partial}{\partial y} K_0(\nu r') = - \int_\nu^\infty \frac{k \cos \zeta(x - \xi)}{\zeta} e^{-k(y + \eta)} dk \quad \text{for} \quad y > \eta,
\]
\(G(x, y; \xi, \eta)\) is expressed in the equivalent form
\[
G(x, y; \xi, \eta) = 2\int_\nu^\infty \frac{k(Dk^4 + 1) \cosh k \eta - K \sinh k \eta}{\Delta(k)} \cosh k(h - y) \frac{\cos \zeta(x - \xi)}{\zeta} dk, \quad y > \eta.
\] (2.5.8)
For $y < \eta$, $y$ and $\eta$ in (2.5.8) are to be interchanged.

An alternative representation of $G$, which gives the outgoing wave term as $|x - \xi| \to \infty$ explicitly, can be obtained following a procedure given by Throne (1953), and is given by

$$
G(x, y; \xi, \eta) = -4\pi \sum_{n=1}^{\infty} \frac{k_n(Dk_n^4 + 1) \cos k_n(h - y) \cos k_n(h - \eta) e^{-(k_n^2 + \nu^2)^{1/2}|x - \xi|}}{2k_nh(Dk_n^4 + 1) + (5Dk_n^4 + 1) \sinh 2k_nh (k_n^2 + \nu^2)^{1/2}}
$$

$$
+ 4\pi i \left[ \frac{k_0(Dk_0^4 + 1) \cosh k_0(h - y) \cosh k_0(h - \eta) e^{(k_0^2 - \mu^2)^{1/2}|x - \xi|}}{2k_0h(Dk_0^4 + 1) + (5Dk_0^4 + 1) \sinh 2k_0h (k_0^2 - \nu^2)^{1/2}}
\right]
$$

$$
+ \frac{\mu(D\mu^4 + 1) \cosh \mu(h - y) \cosh \mu(h - \eta) e^{(\mu^2 - \nu^2)^{1/2}|x - \xi|}}{2\mu h(1 + D\mu^4) + (5D\mu^4 + 1) \sinh 2\mu h (\mu^2 - \nu^2)^{1/2}}
$$

$$
- \frac{\bar{\mu}(D\bar{\mu}^4 + 1) (\cosh \bar{\mu}(h - y) \cosh \bar{\mu}(h - \eta) e^{(\bar{\mu}^2 - \nu^2)^{1/2}|x - \xi|}}{2\bar{\mu} h(D\bar{\mu}^4 + 1) + (5D\bar{\mu}^4 + 1) \sinh 2\bar{\mu} h (\bar{\mu}^2 - \nu^2)^{1/2}}
$$

where $\pm k_0(k_0 > 0)$, $\pm \mu$, $\pm \bar{\mu}$ ($\mu$ has positive real and imaginary parts) and $\pm \mu_n, n = 1, 2, \ldots (\mu_n > 0)$ are the roots of the dispersion equation (cf. Chung and Fox (2002))

$$
\Delta(k) \equiv (1 + Dk^4)k \sinh kh - K \cosh kh = 0 \quad (2.5.10)
$$

If there is no $z$-dependence on the line source, then $\nu = 0$ and in this case

$$
G \to -\ln(r) \quad \text{as} \quad r = ((x - \xi)^2 + (y - \eta)^2)^{1/2} \to 0
$$

The source potential due to a line source submerged at $(\xi, \eta)$ can be obtained in this case, and instead of (2.5.9), $G$ is found to be (cf. Mandal and Basu (2004))
\begin{align*}
G(x, y; \xi, \eta) &= -4\pi \sum_{n=1}^{\infty} \frac{(Dk_n + 1) \cos k_n(h-y) \cos k_n(h-\eta)}{2k_n h (Dk_n^4 + 1) + (5Dk_n^4 + 1) \sin 2k_n h} \times e^{-k_n |x-\xi|} \\
&\quad - 4\pi i \left[ \frac{(Dk_0 + 1) \cosh k_0(h-y) \cosh k_0(h-\eta)}{2k_0 h (Dk_0^4 + 1) + (5Dk_0^4 + 1) \sinh 2k_0 h} \times e^{ik_0 |x-\xi|} \\
&\quad + \frac{(D\mu^4 + 1) \cosh \mu(h-y) \cosh \mu(h-\eta)}{2\mu h (D\mu^4 + 1) + (5D\mu^4 + 1) \sinh 2\mu h} \times e^{i\mu |x-\xi|} \right] \\
&\quad - \frac{\mu(D\mu^4 + 1) (\cosh \mu(h-y) \cosh \mu(h-\eta))}{2\mu h (D\mu^4 + 1) + (5D\mu^4 + 1) \sinh 2\mu h} \times e^{-i\mu |x-\xi|}.
\end{align*}

(2.5.11)

In the absence of ice ice-cover i.e., for free surface, the source potential due to a line source submerged at \((\xi, \eta)\) and having sinusoidal dependence on \(z\), is obtained from (2.5.9) in the form

\begin{align*}
G(x, y; \xi, \eta) &= -4\pi \sum_{n=1}^{\infty} \frac{k_n \cos k_n(h-y) \cos k_n(h-\eta) e^{-(k_n^2 + \nu^2)^{1/2} |x-\xi|}}{2k_n h + \sin 2k_n h} \\
&\quad + 4\pi i \frac{k_0 \cosh k_0(h-y) \cosh k_0(h-\eta) e^{i(k_0^2 + \nu^2)^{1/2} |x-\xi|}}{2k_0 h + \sinh 2k_0 h} \\
&\quad - \frac{k_0 (Dk_0^4 + 1)}{2k_0 h + \sinh 2k_0 h} \times e^{-i k_0 |x-\xi|}.
\end{align*}

(2.5.12)

where \(\pm k_0, \pm ik_n (n = 1, 2, \ldots)\) are the roots of the transcendental equation

\begin{equation}
K = k \tanh kh.
\end{equation}

(2.5.13)

Again in the absence of ice-cover, i.e., for free surface, the source potential due to a line source submerged at \((\xi, \eta)\) without having any \(z\)-dependence is obtained as in the form

\begin{align*}
G(x, y; \xi, \eta) &= -4\pi \sum_{n=1}^{\infty} \frac{\cos k_n(h-y) \cos k_n(h-\eta)}{2k_n h + \sin 2k_n h} e^{-k_n |x-\xi|} \\
&\quad - 4\pi i \frac{\cosh k_0(h-y) \cosh k_0(h-\eta) e^{i k_0 |x-\xi|}}{2k_0 h + \sinh 2k_0 h}.
\end{align*}

(2.5.14)

Here \(\pm k_0, \pm ik_n (n = 1, 2, \ldots)\) are the roots of the equation \(K = k \tanh kh\).
(b) Source potentials in two-layer fluid

Let $G(x, y; \xi, \eta)$ and $G'(x, y; \xi, \eta)$ be the source potentials in the upper and lower fluids respectively due to a time-harmonic line source submerged at a point $(\xi, \eta)(\eta > 0)$ in the lower fluid region. Here upper layer is of depth $h$ below the mean free surface while the lower layer is depth $H$ below the mean interface. If the line source has sinusoidal dependence on the $z$-co-ordinate, say of the form $e^{i\nu z}(\nu < K)$, then instead of the two-dimensional Laplace equation, $G$, $G'$ satisfy the two-dimensional Helmholtz equation

$$(\nabla^2 - \nu^2)G = 0 \quad \text{for} \quad -h < y < 0,$$  
(2.5.15)

$$(\nabla^2 - \nu^2)G' = 0 \quad \text{except at } (\xi, \eta)(0 < \eta < H) \text{ for } 0 < y < H,$$  
(2.5.16)

$$G' \to K_0(\nu r) \quad \text{as} \quad r = \{z - \xi\}^2 + \{y - \eta\}^2 \to 0,$$  
(2.5.17)

where $K_0(z)$ denotes the modified Bessel function of second kind,

$$s(KG + G_y) = KG' + G'_y \quad \text{on} \quad y = 0,$$  
(2.5.18)

$$G_y = G'_y \quad \text{on} \quad y = 0,$$  
(2.5.19)

$$G'_y = 0 \quad \text{on} \quad y = H,$$  
(2.5.20)

$G$, $G'$ behave as outgoing waves as $|x - \xi| \to \infty$.

The solutions for $G$ and $G'$ are obtained as

$$G(x, y; \xi, \eta) = \frac{K_0(\nu r) - K_0(\nu r')}{s} - 2\int_\nu^{\infty} \frac{F_{11}(k, y, \eta) \cos\{(k^2 - \nu^2)^{1/2}|z - \xi|\}}{(k^2 - \nu^2)^{1/2} \sinh kh \sinh kH \Delta(k)} dk,$$
\( -h < y < 0, 0 < \eta < H, \quad (2.5.22) \)

\[
G'(x, y; \xi, \eta) = K_0(\nu T) - K_0(\nu T') - 2 \int_0^\infty F_{12}(k; y, \eta) \cos \left\{ \left( k^2 - \nu^2 \right)^{1/2} |x - \xi| \right\} dk,
\]

\[
0 < y, \eta < H,
\]

where

\[
r, r' = \{(x - \xi)^2 + (y - \eta)^2\}^{1/2}, r'' = \{(x - \xi)^2 + (y + 2h + \eta)^2\}^{1/2},
\]

\[
\Delta(k) = (1 - s)k^2 + K^2(s + \coth kh \coth kH) - kK(\coth kh + \coth kH), \quad (2.5.24)
\]

\[
F_{11}(k; y, \eta) = e^{-k(h+\eta)} \left\{ k \sinh kh(k \cosh ky - K \sinh ky) + \cosh kh(k \sinh kH - K \cosh kH)\{ k \sinh (h + y) - k \cosh (h + y) \} \right. \\
\left. + \frac{1}{\eta}(k \sinh kh - k \cosh kh)(k \sinh kH - K \cosh kH) \sinh (h + y) \right\} \\
+ Ke^{-k(h+H)} \sinh (h + \eta)\{ k \cosh (h + y) - K \sinh (h + y) \},
\]

\(-h < y < 0, 0 < \eta < H,\)

and the path of integration in the integrals in (2.5.22) and (2.5.23) are indented below the poles at \( k = m, M \) on the real axis so as to ensure the outgoing nature of \( G, G' \) far away from the source \( (\nu < m < M) \).

As \( |x - \xi| \to \infty \), it can be shown that

\[
G(x, y; \xi, \eta) \to -2\pi i \left[ \frac{K \{ m \cosh m(h + y) - K \sinh m(h + y) \} \cosh m(H - \eta)}{\left( m^2 - \nu^2 \right)^{1/2} \sinh mh \sinh mH \Delta'(m)} \right]
\]
\begin{align*}
& \times e^{i(m^2-\nu^2)^{1/2}|x-\xi|} \\
& + \frac{K\left\{ M \cosh (h+y) - K \sinh (h+y) \right\} \cosh (H-\eta)}{(M^2-\nu^2)^{1/2} \sinh M \sinh MH\Delta'(M)} \\
& \times e^{i(M^2-\nu^2)^{1/2}|x-\xi|}, \quad -h < y < 0, 0 < \eta < H, \quad (2.5.25)
\end{align*}
and
\begin{align*}
G'(x, y; \xi, \eta) & \rightarrow -2\pi i K \left[ \frac{K \cosh mh - m \sinh mh \cosh m(H-\eta) \cosh m(H-y)}{(m^2-\nu^2)^{1/2} \sinh mh \sinh^2 mh\Delta'(m)} \\
& e^{i(m^2-\nu^2)^{1/2}|x-\xi|} \right], \quad -h < y, \eta < H. \quad (2.5.26)
\end{align*}

If we consider \( \nu = 0 \) i.e. if the line source is independent of \( z \), then
\begin{align*}
G' & \rightarrow -\log r \quad \text{as} \quad r = \{(x-\xi)^2 + (y-\eta)^2\}^{1/2}. \quad (2.5.27)
\end{align*}

Construction of \( G, G' \) in this case is obvious, and their forms can be obtained from above by putting \( \nu = 0 \). Also the behavior of \( G \) and \( G' \) as \( |x-\xi| \rightarrow \infty \) can be obtained from the expressions (2.5.25) and (2.5.26) by putting \( \nu = 0 \).

If a line source is present on the upper fluid at \((\xi, \eta)(-h < \eta < 0)\), then the source potentials \( G(x, y : \xi, \eta) \) and \( \overline{G}(x, y : \xi, \eta) \) for the upper and lower regions are given by
\begin{align*}
G(x, y; \xi, \eta) &= K_0(\nu r) - K_0(\nu r') - 2 \int_{\nu}^{\infty} F_{21}(k; y, \eta) \cos\{(k^2-\nu^2)^{1/2}|x-\xi|\} \\
& \frac{1}{(k^2-\nu^2)^{1/2} \sinh k h \sinh kH\Delta(k)} dk, \quad -h < y, \eta < 0. \quad (2.5.28)
\end{align*}
\begin{align*}
\overline{G}(x, y; \xi, \eta) &= s\left\{ K_0(\nu r') - K_0(\nu r'') \right\} - 2 \int_{\nu}^{\infty} F_{22}(k; y, \eta) \cos\{(k^2-\nu^2)^{1/2}|x-\xi|\} \\
& \frac{1}{(k^2-\nu^2)^{1/2} \sinh k h \sinh kH\Delta(k)} dk, \quad 0 < y < H, -h < \eta < 0. \quad (2.5.29)
\end{align*}
where $r, r'$ are the same as above,

$$F_{21}(k; y, \eta) = ke^{-k(h+\eta)} \left\{ (K \cosh kH - k \sinh kH) \cosh ky + s \sinh kH(k \cosh ky - K \sinh ky) \right\} + e^{-kh} \left\{ (1 - s) (k \sinh kH - k \cosh kH) - s Ke^{-kh} \right\} \sinh k(y + \eta)$$

$$F_{21}(k; y, \eta) = \left\{ K \sinh k(h + y) - k \cosh k(h + y) \right\}, -h < y, \eta < 0,$$

and the path of integration in the integrals in (2.5.28) and (2.5.29) are indented below the poles at $k = m, M$ on the real axis as before.

As $|x - \xi| \to 0$, it can be shown that

$$\mathcal{G}(x, y; \xi, \eta) \to -2\pi isK \left[ \frac{(m \cosh m(h + \eta) - m \sinh m(h + \eta))}{(m^2 - \nu^2)^{1/2} \sinh mh \sinh mH \Delta'(m)} \right]$$

$$+ \frac{(K \sinh m(h + y) - m \cosh m(h + y))}{(m^2 - \nu^2)^{1/2} \sinh mh \sinh mH (K \cosh mh - M \sinh mh) \Delta'(m)}$$

$$e^{i(m^2 - \nu^2)^{1/2}|x-\xi|}, -h < y, \eta < 0$$

and

$$\mathcal{G}'(x, y; \xi, \eta) \to -2\pi isK \left[ \frac{(m \cosh m(h + \eta) - K \sinh m(h + \eta)) \cosh m(H - y)}{(m^2 - \nu^2)^{1/2} \sinh mh \sinh mH \Delta'(m)} \right]$$

$$e^{i(m^2 - \nu^2)^{1/2}|x-\xi|} + \frac{(M \cosh M(h + \eta) - K \sinh M(h + \eta)) \cosh M(H - y)}{(M^2 - \nu^2)^{1/2} \sinh mh \sinh MH \Delta'(M)}$$

$$e^{i(M^2 - \nu^2)^{1/2}|x-\xi|}, 0 < y < H, -h < \eta < 0.$$