Brief Notes

Oblique wave scattering by undulations on the bed of an ice-covered ocean

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The problem of oblique wave scattering by cylindrical undulations on the bed of an ice-covered ocean is investigated by using a simplified perturbation analysis. The first-order potential function satisfies a boundary value problem (BVP) which is solved by employing the Green integral theorem after constructing an appropriate Green function. Analytical expressions for the first-order reflection and transmission coefficients are then obtained from the solution of this BVP, in terms of the integrals involving the shape function describing undulations. Three particular forms of the shape function are considered for which the reflection and transmission coefficients up to the first-order are obtained exactly.

1. Introduction

When a train of surface water waves is incident on an obstacle situated at the bottom of a laterally unbounded ocean of uniform finite depth, it is partially reflected by and transmitted over the obstacle. For an obstacle of arbitrary shape, the problem of determining the reflection and transmission coefficients is in general a difficult task. However, when the obstacle is in the form of a small deformation of the bottom (long-crested sea-bed undulations), then some approximate methods can be employed to obtain these coefficients approximately. For example, for small cylindrical deformation of the bottom, MILES [7] used a perturbation method followed by the finite cosine transform technique in the mathematical analysis to obtain the reflection and transmission coefficients up to the first order, when a train of surface waves is obliquely incident on the bottom deformation. When the obstacle is in the form of bottom undulations, such as sand ripples, DAVIES [3] considered normal incidence of a surface water wave train and treated the problem on the basis of linear perturbation theory. He introduced a linear friction term in the dynamical condition at the free surface so as to apply the Fourier transform technique in the mathematical analysis. The coefficient of friction was then made to tend to zero in the asymptotic results.
for the velocity potential far away from the undulations, so as to obtain the reflection and transmission coefficients up to the first order analytically. Later Mandal and Basu [6] generalised the problem considered in [7] to include the effect of surface tension at the free surface. They also employed a simplified perturbation analysis followed by an appropriate use of Green's integral theorem in the mathematical analysis to obtain a general representation of the first-order potential function. Its asymptotic forms far away from the deformation at either side produce the first-order reflection and transmission coefficients in terms of the integrals involving the shape function describing the deformation.

All the works in [3, 6, 7] involve an ocean with a free surface. However, there is a considerable interest in recent times to investigate the wave propagation problems in an ice-covered ocean wherein the ocean is covered by a thin sheet of ice, modelled as an elastic plate, the ice-cover being virtually weightless. This has motivated us to consider the problem of oblique wave scattering by small undulations on the bottom of a laterally unbounded ocean with an ice-cover instead of a free surface. The ice-cover is modelled as a thin sheet of elastic plate of infinite extent having a very small thickness $h_0$.

Assuming linear theory and irrotational motion, the velocity potential function describing the time-harmonic motion of angular frequency $\omega$, in water of uniform finite depth $h$ and having an ice-cover at the top, can be represented by $\text{Re} \left( e^{-i\omega t} \phi \right)$, where $\phi$ satisfies the equation

$$\nabla^2 \phi = 0, \quad 0 \leq y \leq h,$$

the linearised ice-cover condition (Goldshtein and Marchenko [5], Chakrabarti [1])

$$K\phi + (D\nabla^4_{x,z} + 1)\phi_y = 0 \quad \text{on} \quad y = 0,$$

and the bottom condition

$$\phi_y = 0 \quad \text{on} \quad y = h.$$

The time-dependent factor $e^{-i\omega t}$ will be dropped throughout the paper from now on. Here the $y$-axis is directed vertically downwards into the fluid region, $(x,z)$-plane is the rest position of the lower part of the ice-cover, $\nabla^2$ denotes the three-dimensional Laplacian operator while $\nabla^4_{x,z}$ denotes the two-dimensional biharmonic operator in the $(x,z)$-plane, $K = \sigma^2/g$ where $g$ is the gravity, $D$ is the flexural rigidity of ice-cover and is given by

$$D = \frac{Eh_0^3}{12(1-\gamma^2)pg}.$$
where \( E \) is the Young's modulus, \( \gamma \) is Poisson's ratio of the elastic material comprising the ice-cover and \( \rho \) is the density of water. In deriving the ice-cover condition (1.2), waves are assumed to be long compared to the thickness of the ice-cover. A possible solution for \( \phi \) representing a train of time-harmonic waves propagating on the ice-cover and making an angle \( \theta \) with the positive \( x \)-direction, is given by

\[
\phi_0(x, y, z) = \cosh k_0(h - y)e^{ik_0(x \cos \theta + z \sin \theta)}
\]

where \( k_0 \) is the unique real positive root of the transcendental equation

\[
\Delta(k) \equiv k(Dk^4 + 1) \sinh kh - K \cosh kh = 0.
\]

This equation has two real roots \( \pm k_0 \), two pairs of complex conjugate roots \( \pm \mu, \pm \bar{\mu}(\mu = \alpha + i\beta, \bar{\mu} = \alpha - i\beta, \alpha > 0, \beta > 0 \) and \( \alpha > \beta \)) and an infinite number of purely imaginary roots \( \pm ik_n(k_n > 0, n = 1, 2, \ldots) \) where \( k_n(n = 1, 2, \ldots) \) are real and satisfy

\[
k_n(Dk_n^4 + 1) \sin k_nh + K \cos k_nh = 0,
\]

and \( k_n \to n\pi \) as \( n \to \infty \) (cf. CHUNG and FOX [2]).

To tackle the problem of oblique wave scattering by small cylindrical undulations of the bottom of an ocean with an ice-cover, here also we apply a perturbation technique directly to the governing partial differential equation and the boundary and infinity conditions for the potential function, after extracting the \( z \)-dependence by exploiting the geometry of the problem, to obtain a boundary value problem (BVP). A suitable use of Green's integral theorem produces the solution of this BVP, from which the first-order reflection and transmission coefficients are obtained in terms of integrals involving the shape function defining the undulations. For three different forms of the shape functions these coefficients are obtained in closed forms.

2. Formulation of the problem

The problem of oblique wave scattering by small cylindrical bottom undulations in an ice-covered ocean, assuming linear theory and irrotational motion, is mathematically equivalent to solving the following BVP. We solve the partial differential equation (PDE)

\[
\nabla^2 \phi = 0
\]

in the region \( 0 \leq y \leq h + ec(x), \) \( -\infty < x, z < \infty, \) with the boundary conditions

\[
K \phi + (D\nabla_{x,z}^4 + 1)\phi_y = 0 \quad \text{on} \quad y = 0,
\]
\((2.3)\) \quad \phi_n = 0 \quad \text{on} \quad y = h + \epsilon c(x)

together with suitable conditions as \(x \to \pm \infty\) which will be stated shortly.

Here \(c(x)\) is a continuous and bounded function describing the shape of the
undulations of the ocean bed and \(c(x) \to 0\) as \(|x| \to \infty\), so that the ocean is of
uniform finite depth far away from the undulations on either side, and \(\epsilon (> 0)\) is
a small parameter giving a measure of the smallness of the undulations. \(\phi_n\) in
\((2.3)\) denotes the normal derivative.

We assume that a water wave train represented by the velocity potential
\(\phi_0(x, y, z)\), given by \((1.4)\), is obliquely incident upon the undulations from a large
distance in the direction of negative \(x\)-axis, then it undergoes partial transmission
and reflection by the undulations. Thus the asymptotic behaviour of \(\phi(x, y, z)\)
is given by

\[(2.4)\] \quad \phi \to \begin{cases} T \phi_0(x, y, z) \\ \phi_0(x, y, z) + R \phi_0(-x, y, z) \end{cases} \quad \text{as} \quad x \to \infty, \quad x \to -\infty,

where \(T\) and \(R\) are the transmission and reflection coefficients respectively and
will have to be determined.

As \(\epsilon\) is very small, we can approximate the bottom condition \((2.3)\) after
neglecting the \(O(\epsilon^2)\) terms as

\[(2.5)\] \quad -\phi_y + \epsilon \left\{ c'(x) \phi_x - c(x) \phi_{yy} \right\} = 0 \quad \text{on} \quad y = h.

In view of the geometry of the problem, we can assume that

\[(2.6)\] \quad \phi(x, y, z) = \psi(x, y) e^{i\nu z}

where \(\nu = k_0 \sin \theta\). Thus the \(z\)-dependence is extracted, and the function \(\psi(x, y)\)
satisfies the BVP described by

\[(2.7)\] \quad \begin{align*}
\psi_{xx} + \psi_{yy} - \nu^2 \psi &= 0, \quad 0 \leq y \leq h, \quad -\infty < x < \infty, \\
K \psi + \left\{ D \left( \frac{\partial^2}{\partial x^2} - \nu^2 \right)^2 + 1 \right\} \psi_y &= 0 \quad \text{on} \quad y = 0, \\
-\psi_y + \epsilon \left\{ \frac{\partial}{\partial x} (c(x) \psi_x) - \nu^2 C(x) \right\} &= 0 \quad \text{on} \quad y = h,
\end{align*}

\(\psi(x, y) \to \begin{cases} T \psi_0(x, y) \\ \psi_0(x, y) + R \psi_0(-x, y) \end{cases} \quad \text{as} \quad x \to \infty, \quad x \to -\infty

where

\[(2.8)\] \quad \psi_0(x, y) = e^{ik_0 x \cos \theta} \cosh k_0 (h - y).
3. Method of solution

Because of the approximate boundary condition (2.7)_3 and the fact that a wave train propagating in an ocean of uniform finite depth \( h \) experiences no reflection, we may assume that \( \psi, T \) and \( R \) in (2.7) can be expanded in terms of the small parameter \( \epsilon \) as

\[
\psi(x, y) = \psi_0(x, y) + \epsilon \psi_1(x, y) + O(\epsilon^2),
\]

\[
T = 1 + \epsilon T_1 + O(\epsilon^2),
\]

\[
R = \epsilon R_1 + O(\epsilon^2).
\]

Using the expansions (3.1) in Eqs. (2.7), we find that \( \psi_1(x, y) \) satisfies the BVP described by

\[
\psi_{1xx} + \psi_{1yy} - \nu^2 \psi_1 = 0, \quad 0 \leq y \leq h, \quad -\infty < x < \infty,
\]

\[
K\psi_1 + \left\{ D \left( \frac{\partial^2}{\partial x^2} - \nu^2 \right) \right\} \psi_{1y} = 0 \quad \text{on} \quad y = 0,
\]

\[
\psi_{1y} = ik_0 \cos \theta \frac{\partial}{\partial x} \left( c(x) e^{ik_0 x \cos \theta} \right) - \nu^2 c(x)
\]

\[
eq q(x) \quad \text{on} \quad y = h,
\]

\[
\psi_1(x, y) \to \begin{cases} T_1 \psi_0(x, y) & \text{as} \quad x \to \infty, \\ R_1 \psi_0(-x, y) & \text{as} \quad x \to -\infty. \end{cases}
\]

We note that \( \psi_1(x, y) \) behaves as an outgoing wave as \( |x| \to \infty \).

By an appropriate use of Green's integral theorem, the solution of the BVP is obtained as

\[
\psi_1(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x, h; \xi, \eta) q(x) dx,
\]

where \( G(x, h; \xi, \eta) \) is the corresponding Green's function and is given by (cf. Evans and Porter [4]).
\[ G(x, y; \xi, \eta) \]  
\[ = -4\pi \sum_{n=1}^{\infty} \frac{k_n(Dk_n^4 + 1) \cos k_n(h - y) \cos k_n(h - \eta) e^{-(k_n^2 + \nu^2)^{1/2}|x - \xi|}}{2k_n h(Dk_n^4 + 1) + (5Dk_n^4 + 1) \sin 2k_n h} \]
\[ - 4\pi i \left[ \frac{k_0(Dk_0^4 + 1) \cosh k_0(h - y) \cosh k_0(h - \eta) e^{(k_0^2 - \nu^2)^{1/2}|x - \xi|}}{2k_0 h(Dk_0^4 + 1) + (5Dk_0^4 + 1) \sinh 2k_0 h} \right. \]
\[ \left. + \frac{\mu(D\mu^4 + 1) \cosh \mu(h - y) \cosh \mu(h - \eta) e^{4\mu|x - \xi|}}{2\mu h(1 + D\mu^4) + (5D\mu^4 + 1) \sinh 2\mu h} \right] \]
\[ \frac{- \tilde{\mu}(D\tilde{\mu}^4 + 1) \cosh \tilde{\mu}(h - y) \cosh \tilde{\mu}(h - \eta) e^{-4\tilde{\mu}|x - \xi|}}{2\tilde{\mu} h(D\tilde{\mu}^4 + 1) + (5D\tilde{\mu}^4 + 1) \sinh 2\tilde{\mu} h} \]

where \( \mu' = (\mu^2 - \nu^2)^{1/2} \) and \( -\tilde{\mu}' = \{(-\tilde{\mu})^2 - \nu^2\}^{1/2} \), and that branch of the square root has been chosen such that \( \mu' = \mu, -\tilde{\mu}' = -\tilde{\mu} \) when \( \nu = 0 \).

Since \( \mu' \) and \( -\tilde{\mu}' \) have positive imaginary parts, we find that, as \( |x - \xi| \to \infty \),

\[ G(x, y; \xi, \eta) \to -4\pi i \frac{k_0(Dk_0^4 + 1) \cosh k_0(h - y) \cosh k_0(h - \eta)}{2k_0 h(Dk_0^4 + 1) + (5Dk_0^4 + 1) \sinh 2k_0 h} \]
\[ \times \frac{e^{(k_0^2 - \nu^2)^{1/2}|x - \xi|}}{(k_0^2 - \nu^2)^{1/2}} \]

so that \( G \) behaves as an outgoing wave for \( |x - \xi| \to \infty \).

To obtain the first-order transmission and reflection coefficients \( T_1 \) and \( R_1 \) respectively, we note from (3.2) and (3.5) that

\[ \psi_1(\xi, \eta) \to \begin{cases} T_1 \psi_0(\xi, \eta) & \text{as } \xi \to \infty, \\ R_1 \psi_0(-\xi, \eta) & \text{as } \xi \to -\infty \end{cases} \]

and

\[ G(x, 0; \xi, \eta) \to -4\pi i \frac{e^{\pi k_0 z \cos \theta}}{k_0 \cos \theta} A \psi_0(\pm \xi, \eta) \text{ as } \xi \to \pm \infty, \]

where

\[ A = \frac{1}{h + \frac{(1 + 5Dk_0^4) \sinh^2 k_0 h}{K}}. \]
Using the asymptotic results (3.6) and (3.7) in the representation (3.3), we find that

\[ T_1 = -\frac{i}{k_0 \cos \theta} A \int_{-\infty}^{\infty} e^{-ik_0x \cos \theta} q(x) dx \]

(3.9)

\[ = ik_0 \sec \theta A \int_{-\infty}^{\infty} c(x) dx, \]

\[ R_1 = -\frac{i}{k_0 \cos \theta} A \int_{-\infty}^{\infty} e^{ik_0x \cos \theta} q(x) dx \]

(3.10)

\[ = -ik_0 \sec \theta \cos 2\theta A \int_{-\infty}^{\infty} e^{2ik_0x \cos \theta} q(x) dx. \]

The results for an ocean with a free surface are recovered by putting \( D = 0 \) in (3.9) and (3.10) where then, however, \( k_0 \) denotes the unique real positive zero of the transcendental equation

(3.11) \[ k \sinh kh - K \cosh kh = 0. \]

It is also interesting to note that \( R_1 \) vanishes identically for \( \theta = \pi/4 \), independently of the shape function \( c(x) \). This was also observed in \([6, 7]\) in the case of an ocean with a free surface with or without surface tension.

We now consider three special types of undulations.

(i) \( c(x) = a e^{-|x|}(\lambda > 0) \). Here the bottom undulation reaches maximum at \((0, h)\) and decreases exponentially on either side of \((0, h)\). In this case

\[ T_1 = \frac{2iak_0A}{\lambda} \sec \theta, \]

\[ R_1 = -\frac{2iak_0A\lambda}{\lambda^2 + 4k_0^2 \cos^2 \theta} \sec \theta \cos 2\theta. \]

(ii) \( c(x) = a e^{-\lambda x^2}(\lambda > 0) \). Here the undulation is of Gaussian type and has the maximum value at \((0, h)\). In this case

\[ T_1 = i \left( \frac{\pi}{\lambda} \right)^{1/2} k_0 a \lambda \sec \theta, \]

\[ R_1 = i \left( \frac{\pi}{\lambda} \right)^{1/2} k_0 a \lambda \sec \theta \cos 2\theta e^{-\frac{k_0^2 \cosh^2 \theta}{\lambda}}. \]

(iii) \( c(x) = \begin{cases} a \sin \lambda x, & \frac{m\pi}{\lambda} \leq x \leq \frac{m\pi}{\lambda}, \\ 0, & \text{otherwise}. \end{cases} \)
This represents sinusoidal undulations of the bottom, having number \( m \) of patches and is of considerable physical interest. Davies [3] earlier made a somewhat elaborate study on the effect of sinusoidal undulations on the bottom of an ocean with a free surface, upon an incident surface water wave train. In this case

\[
T_i \equiv 0,
\]

\[
R_i = \sec^2 \theta \cos 2\theta B \frac{\alpha}{\alpha^2 - 1} \sin(\alpha m \pi)
\]

where \( B = a A, \quad \alpha = \frac{2k_0}{\lambda} \cos \theta \). It is interesting to note that when \( \alpha \approx 1 \), i.e. \( \lambda \approx 2k_0 \cos \theta \),

\[
(3.12) \quad R_i \approx \frac{\pi}{2} \sec^2 \theta \cos 2\theta B m.
\]

The result (3.11) has the implication that a somewhat large reflection of the incident wave energy occurs when the bed wave number \( \lambda \) is twice the wave number component of the incident wave field along the \( x \)-direction, if the integer \( m \) denoting the number of patches is made large. This phenomenon has a practical application in the construction of an efficient reflector of incident wave energy.

4. Discussion

A simplified perturbation analysis is employed to obtain the first-order transmission and reflection coefficients for the problem of oblique wave scattering by small cylindrical undulations on the bottom of an ocean with an ice-cover modelled as a thin elastic plate. The first-order reflection coefficient vanishes independently of the shape of the undulations if the angle of incidence is \( \frac{\pi}{4} \).

By making \( D \) equal zero, the results for an ocean with a free surface are recovered. For sinusoidal undulations having \( m \) patches, the first-order transmission coefficient vanishes identically, and the reflection coefficient becomes a constant multiple of the number of patches when the ocean-bed wave number is twice the \( x \)-component of the incident field wave number, what suggests that comparatively large reflection of the incident wave energy is possible by making the number of patches somewhat larger.

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References


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This paper is concerned with two-dimensional unsteady motion of water waves generated by an initial disturbance created at an ice sheet covering the water. The ice cover is modeled as a thin elastic plate. Using linear theory, the problem is formulated as an initial value problem for the velocity potential describing the motion in the liquid. In the mathematical analysis, the Laplace and Fourier transform techniques have been utilized to obtain the depression of the ice-covered surface in the form of an infinite integral. For the special case of initial disturbance concentrated at the origin, taken on the ice cover, this integral is evaluated asymptotically by the method of a stationary phase for a long time and large distance from the origin. The form of the ice-covered surface is graphically depicted for two types of initial disturbances.

1. Introduction

The two-dimensional problems concerning generation of water waves due to a prescribed initial displacement or impulse mostly concentrated at a point were discussed in treaties of Lamb [4] and Stoker [6] within the framework of the linearized theory of water waves. Kranzer and Keller [3] considered axially symmetrical initial surface disturbance in water of finite depth and compared the theory with experimental results. Chaudhury [1] extended these results for any initial surface impulse and elevation across arbitrary regions.

All the problems considered by Lamb [4], Kranzer and Keller [3], and Chaudhury [1] involve an ocean with a free surface. However, in polar regions, the ocean is generally covered by ice. Two types of models for the ice cover are usually assumed. In the first model, the ice cover is assumed to consist of a thin but uniform distribution of noninteracting materials with no elastic property, known as an inertial surface (e.g., broken ice). Mandal [5] considered generation of water waves due to initial disturbances at such an inertial surface. In the second model, the ice cover is assumed to consist of a thin ice sheet of small thickness $h$, say, of which still a smaller part is immersed in water, the ice sheet being composed of materials having elastic properties. In this paper, we consider the problem of generation of water waves in an ocean of infinite depth covered by such
an ice sheet due to initial disturbance on the ice cover, the disturbance being taken as an initial depression of the ice cover or an initial impulse on it.

Assuming the linear theory, the problem is formulated as an initial value problem in terms of nondimensional coordinates and nondimensional time. Using the technique of Laplace transform, the problem is reduced to a boundary value problem which is solved by using Fourier transform. The Laplace-Fourier transform of the ice cover depression is then found. After invoking the inverse transforms, the nondimensional form of the ice cover depression is obtained in terms of an integral. In the absence of the ice cover, this reduces to the classical result. This integral is evaluated for large distance and long time for the case when the initial disturbance is concentrated at the origin. The asymptotic form of the depression is displayed graphically for various values of the ice cover parameter, in a number of figures, and compared with the case when there is no ice cover.

2. Mathematical formulation

We consider water as an inviscid, incompressible, homogeneous liquid of volume density \( \rho \). A rectangular cartesian coordinate system \((x,y)\) is chosen in which the \(y\)-axis is chosen vertically downwards into the water and the plane \(y = 0\) coincides with the rest position of the ice cover. The ice cover is modeled as a thin elastic sheet having a uniform surface density \( \epsilon \rho \), where \( \epsilon \) is a constant having the dimension of length. The motion of the water is created by a sudden initial depression of the ice cover or an impulse on the ice cover. Since the motion starts from rest, it is irrotational and can be described by a velocity potential \( \phi \), which satisfies Laplace's equation

\[
\nabla(\hat{x},\hat{y}) \phi = 0 \quad \text{in the fluid region},
\]

and the bottom condition

\[
\nabla \hat{\phi} \rightarrow 0 \quad \text{as } \hat{y} \rightarrow \infty.
\]

The assumption of no cavitation between the ice and water at any time produces the kinematic condition

\[
\hat{\phi}_y = \bar{\eta}_t \quad \text{on } \hat{y} = 0,
\]

where \( \eta \) is the depression of the ice cover below its mean position, assumed small, and \( \bar{\eta} \) denotes the time.

The linearized dynamic condition at the ice cover is (cf. Chung and Fox [2])

\[
(\hat{\phi} - \bar{\epsilon} \hat{\phi}_y)_t = \left(1 + \hat{D} \frac{\partial^2}{\partial \hat{y}^2}\right) \bar{\eta}_t \quad \text{on } \hat{y} = 0,
\]

where \( \hat{D} \) is the flexural rigidity of the ice cover and is given by \( \hat{D} = Eh^3/12(1 - \nu^2)\rho g \), \( E \) being Young's modulus and \( \nu \) being Poisson's ratio.
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The initial conditions at the ice cover are
\[ \phi - \epsilon \phi_y = 0 \quad \text{on} \quad y = 0, \quad t = 0, \]
\[ (\phi - \epsilon \phi_y)_t = \left\{ 1 + D \frac{\partial^4}{\partial x^4} \right\} \eta_0(k) \quad \text{at} \quad t = 0, \quad y = 0, \]  
(2.5)

where \( \eta_0(k) \) is the initial depression of the ice cover.

It is convenient to write the system in a nondimensional form by introducing a characteristic length \( l \) and the characteristic time \( \sqrt{\frac{g}{h}} \), where \( g \) is the gravity. Defining the dimensionless quantities (without the overbar) as

\[ (x, y) = \left( \frac{\xi, \eta}{l} \right), \quad t = \frac{t}{\sqrt{\frac{g}{h}}}, \quad \eta = \frac{\eta}{h}, \]
\[ \phi = \frac{\Phi}{\sqrt{\frac{g}{h}}}, \quad D = \frac{D}{h}, \quad \epsilon = \frac{\epsilon}{h}, \]  
(2.6)

we find that \( \phi \) satisfies

\[ \nabla^2 \Phi = 0 \quad \text{in the fluid region}, \]  
(2.7)
\[ (\phi - \epsilon \phi_y)_t = \left\{ 1 + D \frac{\partial^4}{\partial x^4} \right\} \phi_y \quad \text{on} \quad y = 0, \]
\[ \nabla \phi \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty, \]  
(2.8)
\[ \Phi_y = \eta \quad \text{on} \quad y = 0, \]  
(2.9)
\[ \phi = 0 \quad \text{at} \quad t = 0, \quad y = 0, \]  
(2.10)
\[ (\phi - \epsilon \phi_y)_t = \left\{ 1 + D \frac{\partial^4}{\partial x^4} \right\} \eta_0(x) \quad \text{at} \quad t = 0, \quad y = 0, \]  
(2.11)

where \( \eta_0(x) \) is the initial nondimensional depression of the ice cover.

3. Solution for \( \Phi \)

Defining joint Laplace-Fourier transform of \( \phi(x, y, t) \) as

\[ \Phi(k, y, s) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\xi x} \int_{-\infty}^{\infty} e^{-s\eta} \phi(x, y, t) \, dt \, dx, \]  
(3.1)

it is easy to see that \( \Phi \) satisfies

\[ \hat{\phi}_{yy} - k^2 \hat{\phi} = 0 \quad y \geq 0, \]  
(3.2)
\[ \hat{\phi}_y = s\hat{\phi} - \hat{\eta}_0(k) \quad \text{on} \quad y = 0, \]  
(3.3)
\[ s^2 \hat{\phi} - (\epsilon s^2 + \frac{\partial}{\partial x} + Dk^4) \hat{\phi}_y = (1 + Dk^4) \hat{\eta}_0(k), \]  
(3.4)
\[ \hat{\phi}_y \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty. \]  
(3.5)

The general solution of (3.2) satisfying (3.5) is

\[ \hat{\phi} = A(k, s)e^{-k^2 y}, \]  
(3.6)
where the unknown function $A(k,s)$ is obtained by using (3.4). We are interested in the depression $\eta(x,t)$ of the ice cover. Its Laplace-Fourier transform $\widetilde{\eta}$ is obtained by using (3.3) and is given by

$$\widetilde{\eta} = \frac{\eta_0(k)}{s^2 + c^2},$$

(3.7)

where

$$c^2 = \frac{|k|(1 + Dk^4)}{(1 + e|k|)}.$$  
(3.8)

If the initial depression of the ice cover is concentrated at the origin, then we can choose $\eta_0(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta function and in this case, we find

$$\widetilde{\eta} = \frac{s}{\sqrt{2\pi} s^2 + c^2}.$$  
(3.9)

After taking inversions, we find that the depression of the ice cover, in this case, is obtained in the form

$$\eta(x,t) = \frac{1}{4\pi} \left[ \int_0^\infty e^{i(x-kd)} dk + \int_0^\infty e^{i(x+k(\sigma_1))} dk + \int_0^\infty e^{-i(x-k(\sigma_1))} dk + \int_0^\infty e^{-i(x+k(\sigma_2))} dk \right].$$  
(3.10)

In the next section, we evaluate $\eta$ for large $x$ and $t$ asymptotically by using the method of stationary phase.

4. Asymptotic form of $\eta(x,t)$

We are interested in the waves created by the initial disturbance after a long lapse of time and a large distance from the origin. We use the method of stationary phase to evaluate the integral in (3.10) for large $x$ and $t$ such that $x/t$ remains finite. The second and fourth integrals have no stationary point in the range of integration. Thus these do not contribute to $\eta(x,t)$ for large values of $x$ and $t$.

From the first and third integrals, we note that the stationary points are given by

$$f'(k) = 0,$$  
(4.1)

where

$$f(k) = \left\{ \frac{k(1 + Dk^4)}{1 + e k} \right\}^{1/2} - \frac{kx}{t}.$$  
(4.2)

It is easy to see that (4.1) has two positive roots when $x/t \geq 1$, but no root for $x/t < 1$. Thus the stationary points are given by $k = \alpha_1$ and $k = \alpha_2$, where $\alpha_1$ and $\alpha_2$ are the only two positive real roots of

$$\frac{1}{2} \left\{ \frac{1 + e k}{k(1 + Dk^4)} \right\}^{1/2} \frac{(1 + 5Dk^4 + 4DKe^5)}{(1 + ke)^2} - \frac{x}{t} = 0 \quad \left( \frac{x}{t} \gtrless 1 \right).$$  
(4.3)
Thus we find that for large $x$ and $t$ such that $x/t \geq 1$,

$$
\eta(x,t) \approx \sum_{\alpha_i} \frac{1}{2\pi} \left( \frac{2\pi}{t |f'''(\alpha_i)|} \right)^{1/2} \cos \left( tf(\alpha_i) \pm \frac{\pi}{4} \right),
$$

(4.4)

where the sign $\pm$ is to be chosen to agree with the sign of $f'''(\alpha_i)$. If the disturbance is in the form of an initial impulsive pressure $I(x)$ per unit area applied to the ice-covered surface, then only the initial conditions (2.11) and (2.12) are to be changed and are given, respectively, by

$$
\phi - \epsilon \phi_y = \frac{I(x)}{\rho} \quad \text{at } t = 0, \ y = 0,
$$

$$
(\phi - \epsilon \phi_y)_t = 0 \quad \text{at } t = 0, \ y = 0,
$$

(4.5)

where $\rho = \rho_e/\rho_o$.

If the initial impulse is concentrated at the origin, then by a similar analysis, we obtain for large $x$ and $t$ such that $x \geq \tau$,

$$
\eta(x,t) \approx -\frac{1}{2\pi \rho} \sum_{\alpha_i} \left( \frac{\alpha_i}{(1 + \epsilon \alpha_i)(1 + D\alpha_i)} \right) \left( \frac{2\pi}{t |f'''(\alpha_i)|} \right)^{1/2} \sin \left( tf(\alpha_i) \pm \frac{\pi}{4} \right),
$$

(4.6)

where the sign $\pm$ is to be chosen to agree with the sign of $f'''(\alpha_i)$ as before.

5. Numerical results

To study the effect of the presence of the ice cover on the wave motion generated due to initial disturbances at the upper surface, the asymptotic form of $\eta(x,t)$ is depicted graphically in Figures 5.1 and 5.2 for the case of an initial depression concentrated at the origin and in Figures 5.3 and 5.4 for the case of initial impulse also concentrated at the origin.

An obvious observation is the considerable reduction of the maximum amplitude of the generated waves due to the presence of the ice cover compared to the case of a free surface, and the oscillatory nature of the wave motion is quite prominent in the presence of ice cover. These figures show the variation of $\eta(x,t)$ at a fixed point $x$ when the time changes and at a fixed time $t$ when $x$ changes. The oscillatory nature of the wave motion in the presence of the ice cover compared to the case when there is no ice cover is more prominent in Figure 5.5.

Figures 5.6 and 5.7 depict the wave motion due to an initial depression of the ice cover against $x$ and $t$ (large) together, for two choices of values of $D$, namely, $D = 0.5, D = 0.1$, while Figures 5.8 and 5.9 depict the same due to initial impulsive pressure against $x$ and $t$ again for the same two choices of $D$.

Figures 5.10 and 5.11 depict the wave motion in the absence of the ice cover, against $x$ and $t$ (large).
Water waves generated by disturbances at an ice cover

Figure 5.1. Wave motion due to initial displacement for fixed time $t = 100; \epsilon = 0.01, D = 0.5$.

Figure 5.2. Wave motion due to initial displacement for fixed distance $x = 1000; \epsilon = 0.01, D = 0.5$. 
Figure 5.3. Wave motion due to impulse for fixed time $t = 100$; $\epsilon = 0.01$, $D = 0.5$.

Figure 5.4. Wave motion due to impulse for fixed distance $x = 1000$; $\epsilon = 0.01$, $D = 0.5$. 
Water waves generated by disturbances at an ice cover

Figure 5.5. Wave motion due to initial displacement for fixed time $t = 50$; $\epsilon = 0.01$, $D = 0.5$.

Figure 5.6. Wave motion due to initial displacement when $t = 100$ to 150, $x = 150$ to 200; $\epsilon = 0.01$, $D = 0.5$.

Figure 5.7. Wave motion due to initial displacement when $t = 100$ to 150, $x = 150$ to 200; $\epsilon = 0.01$, $D = 0.1$. 
Figure 5.8. Wave motion due to impulse when $t = 100$ to $150$, $x = 150$ to $200$; $\epsilon = 0.01$, $D = 0.5$.

Figure 5.9. Wave motion due to impulse when $t = 100$ to $150$, $x = 150$ to $200$; $\epsilon = 0.01$, $D = 0.1$.

Figure 5.10. Wave motion due to initial displacement when $t = 100$ to $150$, $x = 150$ to $200$; $\epsilon = 0$, $D = 0$.

Figure 5.11. Wave motion due to impulse when $t = 100$ to $150$, $x = 150$ to $200$; $\epsilon = 0$, $D = 0$. 
Water waves generated by disturbances at an ice cover

These figures clearly show the effect of the presence of the ice cover on the wave motion. The presence of the ice cover reduces the amplitude of the wave motion but considerably increases the oscillatory nature. This may be attributed to the elastic behavior of the ice cover.

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Water waves generated due to initial axisymmetric disturbances in water with an ice-cover

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Abstract This paper is concerned with the generation of water waves due to prescribed initial axisymmetric disturbances in a deep ocean with an ice-cover modelled as a thin elastic plate. The initial disturbances are either in the form of an impulsive pressure distributed over a certain region of the ice-cover or an initial displacement of the ice-cover. Assuming linear theory, the problem is formulated as an initial-value problem in the velocity potential describing the ensuing motion in the fluid. In the mathematical analysis, the Laplace and Hankel transform techniques have been utilised to obtain the deformation of the ice-covered surface as an infinite integral in each case. The method of stationary phase is used to evaluate the integral for large values of time and distance. Figures are drawn to show the effect of the presence of ice-cover on the wave motion.

Keywords Axisymmetric disturbances • Ice-covered surface • Linear theory • Stationary phase method

1 Introduction

Waves are generated by an explosion above or within an ocean. The formulation of the problem associated with the generation of these waves as an initial-value problem is based on the linear theory of surface water waves. If the explosion occurs above the ocean surface, the initial condition on the surface is taken as an initial impulse distributed over a certain region while, for the case when the explosion occurs below the ocean surface, the initial condition is taken as an initial elevation or depression of the same surface. For the case of an initial disturbance in the form of an impulse or displacement concentrated at the origin, the potential functions as well as the free surface elevation were given in [1,2]. Chaudhuri [3] and Wen [4] considered the case when the initial surface disturbance consists of both a surface impulse and surface elevation distributed over an arbitrary region of the surface of an ocean of uniform finite depth. For the three-dimensional unsteady motion, Kranzer and Keller [5] considered the case of an axially symmetric initial surface disturbance and compared the theory with experiments.

In all these problems, the ocean was assumed to be bounded above by a free surface. However, for an ocean covered by an inertial surface composed of a thin layer of uniformly distributed noninteracting particles (e.g. broken ice, floating mat), Mandal [6] considered the two-dimensional problem of generation waves due to prescribed initial disturbances at the inertial surface. Mandal and Mukherjee [7] also considered the generation of surface waves due to an initial axisymmetric surface disturbance on an inertial surface.

In the presence of an ice-cover modelled as a thin elastic floating sheet, the boundary condition on the water surface is considerably modified due to the occurrence of fifth-order partial derivatives in the boundary condition while solving a partial differential equation of second order. Because of this modified boundary condition and due to an increase in scientific activities in polar oceans (modelled as water with an ice-cover) in
recent years, water wave problems in the presence of ice-cover have become attractive to researchers on water waves ([8–10]). This has motivated us to generalise some classical water wave problems in the presence of free surface to problems wherein the free surface is replaced by an ice-cover, and study the effect of the ice-cover on the wave motion. Here, we consider the problem of generation of surface waves in water with an ice-cover due to a prescribed initial axially symmetric disturbance to the ice-cover. The problem is formulated as an initial-value problem. Taking the Laplace transform in time, the transformed potential satisfies a boundary-value problem. Due to axially symmetric disturbances, the Hankel transform is employed to solve this boundary-value problem. Finally, the depression of the ice-covered surface at any time is obtained as an integral. For large time and distance, this integral is evaluated asymptotically by the method of stationary phase. The asymptotic results are also depicted graphically to show the effect of the presence of the ice on the upper surface.

2 Formulation of the problem

We consider the potential flow of an ideal incompressible liquid of infinite depth. A cylindrical coordinate system \((r, \theta, y)\) is chosen in which the \(y\)-axis is taken vertically downwards into the water, which is assumed to be homogeneous with density \(\rho\). The upper surface of water is covered by a thin layer of ice modelled as an elastic sheet. The motion is generated by an initial disturbance (impulse or displacement) distributed over the mean horizontal position of the ice-cover, taken as the \(y = 0\) plane. Assuming linear theory and the motion to be irrotational, the motion is described by a velocity potential \(\phi(r, y, t)\), which satisfies the Laplace equation

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad y \geq 0, \quad t \geq 0.
\]

(1)

Also, \(\phi\) satisfies the bottom condition

\[
\nabla \phi \to 0 \quad \text{as} \quad y \to \infty.
\]

(2)

The assumption of no cavitation between the ice and water at any time gives the kinematic condition that

\[
\phi_y = \eta_t \quad \text{on} \quad y = 0,
\]

(3)

where \(\eta\) is the vertical displacement of the ice-sheet and \(t\) is the time.

Also, \(\phi\) satisfies the linearised ice-cover condition

\[
(\phi - \delta \phi_y)_{yt} = (1 + \beta \nabla^2) \phi_y \quad \text{on} \quad y = 0
\]

(4)

where \(\delta = \rho h / \rho_0 \) and \(h\) are, respectively, the density and thickness of the plate and \(\beta = E h^3 / 12 (1 - \nu^2) \rho g\)

\(E\) is the Young's modulus, \(\nu\) is the Poisson's ratio of the plate and \(\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r})\)

The initial conditions at the ice-cover are

\[
\phi - \delta \phi_y = 0 \quad \text{on} \quad y = 0, \quad t = 0,
\]

(5a)

\[
(\phi - \delta \phi_y)_t = (1 + \beta \nabla^2) \phi_y G_1(r) \quad \text{on} \quad y = 0, \quad t = 0
\]

(5b)

when an initial axially symmetric depression \(G_1(r)\) of the ice-covered surface at a distance \(r\) from the origin is prescribed, or

\[
\phi - \delta \phi_y = - \frac{F_1(r)}{\rho}, \quad y = 0, \quad t = 0,
\]

(5a)

\[
(\phi - \delta \phi_y)_t = 0 \quad \text{on} \quad y = 0, \quad t = 0
\]

(5b)

when an axially symmetric impulse \(F_1(r)\) is applied per unit area of the ice-covered surface at a distance \(r\) from the origin.
Let us introduce the dimensionless variables
\[ r' = \frac{r}{l}, \quad y' = \frac{y}{l}, \quad t' = \frac{t\sqrt{g}}{l}, \quad \phi' = \phi/(l\sqrt{g}) \]
where \( l \) is a characteristic length and \( g \) is the acceleration due to gravity. The primes are dropped below for convenience.

Then, the initial-value problem for the nondimensional potential \( \phi \) is
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi'}{\partial r} \right) + \frac{\partial^2 \phi'}{\partial y'^2} = 0, \quad y' \geq 0, \quad t' \geq 0, \quad (7)
\]
\[
\nabla \phi' \rightarrow 0 \quad \text{as} \quad y' = \infty, \quad (8)
\]
\[
\phi' = \eta, \quad \text{on} \quad y' = 0, \quad (9)
\]
\[
(\phi - \epsilon \phi_y)_{\infty} = (1 + D\nabla^4)G(r), \quad \text{on} \quad y' = 0, \quad (10)
\]

where \( \epsilon = \frac{\delta}{l} \) and \( D = \beta/l^4 \) are dimensionless parameters of the problem.

The initial conditions at the ice-cover are
\[
\phi - \epsilon \phi_y = 0, \quad y' = 0, \quad t' = 0 \quad (11a)
\]
\[
(\phi - \epsilon \phi_y)_{\infty} = (1 + D\nabla^4)G(r), \quad y' = 0, \quad t' = 0 \quad (11b)
\]
or
\[
\phi - \epsilon \phi_y = \frac{F(r)}{\rho} \quad \text{on} \quad y' = 0, \quad t' = 0 \quad (12a)
\]
\[
(\phi - \epsilon \phi_y)_{\infty} = 0 \quad \text{on} \quad y' = 0, \quad t' = 0 \quad (12b)
\]

accordingly, as the initial disturbance is axially symmetric depression of the ice-cover or an axially symmetric impulse at the ice-cover (the notations \( G(r) \), \( F(r) \) being obvious).

3 Solution of the problem

Let \( \Phi(r, y; p) \) denote the Laplace transform of \( \phi(r, y; t) \) in time \( t \), defined as
\[
\Phi(r, y; p) = \int_0^\infty \phi(r, y; t)e^{-pt}dt, \quad p > 0. \quad (13)
\]

Then, \( \Phi(r, y; p) \) satisfies the boundary-value problem
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial y^2} = 0, \quad y \geq 0, \quad (14)
\]
\[
p^2 \Phi - [\epsilon p^2 + (1 + D\nabla^4)]\Phi_y = (1 + D\nabla^4)G(r) \quad \text{on} \quad y = 0 \quad (15a)
\]
or
\[
p^2 \Phi - [\epsilon p^2 + (1 + D\nabla^4)]\Phi_y = -\frac{p}{\rho} F(r) \quad \text{on} \quad y = 0 \quad (15b)
\]

and
\[
|\nabla \Phi| \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty. \quad (16)
\]

Equation (15a) holds for an initial surface displacement while (15b) holds for an initial surface impulse.

Let \( \psi(k, y; p) \) be the Hankel transform of \( \Phi(r, y; p) \) defined by
\[
\psi(k, y; p) = \int_0^\infty \Phi(r, y; p)J_0(kr)dr, \quad k > 0. \quad (17)
\]
Then, $\psi$ satisfies
\[
\frac{d^2 \psi}{dy^2} - k^2 \psi = 0, \quad y \geq 0,
\]
\[
p^2 \psi - \left[ e p^2 + 1 + D(D^2 - k^2)^2 \right] \psi_y = \left[ 1 + D(D^2 - k^2)^2 \right] \mathcal{G}(k) \quad \text{on } y = 0
\]
or
\[
p^2 \psi - \left[ e p^2 + 1 + D(D^2 - k^2)^2 \right] \psi_y = \frac{\mathcal{F}(k)}{\rho} \quad \text{on } y = 0
\]
where $D^2 = \frac{p^2}{\rho}$. 

\[
|\nabla \psi| \to 0 \quad \text{as} \quad y \to \infty,
\]
where $\mathcal{F}(k)$ and $\mathcal{G}(k)$ denote the Hankel transform of $F(r)$ and $G(r)$, respectively. Let $\mathcal{H}$ denote the Laplace–Hankel transform of the depression of the ice-cover. Then, $\mathcal{H}$ is given by
\[
\mathcal{H} = \frac{1}{\rho} \left[ \frac{k}{(1 + \epsilon k)} \frac{1 + D(D^2 - k^2)^2 \mathcal{G}(k)}{p^2 + c^2} \right]
\]
or
\[
\mathcal{H} = \frac{\mathcal{F}(k)}{\rho(1 + \epsilon k)(p^2 + c^2)}
\]
where
\[
c^2 = \frac{k(1 + Dk^4)}{1 + \epsilon k}.
\]
Hence, the ice-covered surface depression can be written as
\[
\eta(r, t) = \int_0^\infty \mathcal{H}(k) [\hat{g}(k) - \frac{1 - \cos ct}{(1 + Dk^4)} (1 + D(D^2 - k^2)^2 \mathcal{G}(k))] dk
\]
or
\[
\eta(r, t) = \frac{1}{2\pi} \int_0^\infty k^{3/2} \frac{\sin ct}{\{(1 + \epsilon k)(1 + Dk^4)^{3/2}\}} J_0(kr) \mathcal{F}(k) dk.
\]
In particular, when the displacement is concentrated at the origin, then $\mathcal{G} = 1/2\pi$, so that in this case,
\[
\eta(r, t) = \frac{1}{2\pi} \int_0^\infty k J_0(kr) \cos ct dk.
\]
Similarly, when the initial impulse is concentrated at the origin, $\mathcal{F}(k) = 1/2\pi$, and in this case
\[
\eta(r, t) = \frac{1}{2\pi\rho} \int_0^\infty k^{3/2} \frac{\sin ct}{\{(1 + \epsilon k)(1 + Dk^4)^{3/2}\}} J_0(kr) dk.
\]
In the absence of ice-cover ($\epsilon = 0, D = 0$), the result (24) for the concentrated unit impulse coincides with the result given in Stoker [2, p. 160] after nondimensionalisation.
4 Asymptotic analysis

The method of stationary phase is now applied to obtain an asymptotic form of \( \eta(r, t) \) for the integral in (23) for large \( r \) and \( t \) but finite ratio \( r/t \).

Using the result

\[
J_0(kr) = \frac{2}{\pi} \int_0^{\pi/2} \cos(kr \cos \theta) \, d\theta,
\]

Eq. (23) is written as

\[
\eta(r, t) = \frac{1}{2\pi^2} \int_0^{\infty} k \cos \frac{ct}{r} \int_0^{\pi/2} (e^{ikr \cos \theta} + e^{-ikr \cos \theta}) \, d\theta.
\]

By using the method of stationary phase on the \( \theta \)-integral, we find

\[
\eta(r, t) \approx \frac{1}{4\pi^2} \int_0^{\infty} k^{1/2} \left( \frac{2\pi}{r} \right)^{1/2} e^{it(c+\xi)} \, dk + \frac{1}{4\pi^2} \int_0^{\infty} k^{1/2} \left( \frac{2\pi}{r} \right)^{1/2} e^{-it(c+\xi)} \, dk
\]

\[
+ \int_0^{\infty} k^{1/2} \left( \frac{2\pi}{r} \right)^{1/2} e^{-it(c-\xi)} \, dk + \int_0^{\infty} k^{1/2} \left( \frac{2\pi}{r} \right)^{1/2} e^{-it(c-\xi)} \, dk.
\]

Here, only the second and third integrals contribute due to the stationary phase method applied to the \( k \) integral. Let us now denote

\[
f(k) = c - k + \frac{\pi}{4t},
\]

so that

\[
f'(k) = \frac{1}{2} \left( \frac{1 + ek}{k(1 + Dk^2)} \right)^{1/2} \frac{1 + 5Dk^4 + 4Dk^2}{(1 + ke)^2} - \frac{r}{t}.
\]

We see that when \( \xi \geq 1 \), then \( f'(k) = 0 \) has two positive real roots \( \alpha \) and \( \beta \), say. Thus, we find that

\[
\eta(r, t) \approx \frac{1}{2\pi^2} \left( \frac{2\pi}{r} \right)^{1/2} \alpha^{1/2} \left( \frac{2\pi}{1/f'(\alpha)} \right)^{1/2} \cos \left( t f(\alpha) \pm \frac{\pi}{4} \right)
\]

\[
+ \frac{1}{2\pi^2} \left( \frac{2\pi}{r} \right)^{1/2} \beta^{1/2} \left( \frac{2\pi}{1/f'(\beta)} \right)^{1/2} \cos \left( t f(\beta) \pm \frac{\pi}{4} \right),
\]

(27)

the \( \pm \) sign being chosen as the sign of \( f'(\alpha) \) and \( f'(\beta) \). Similarly, when the impulse is concentrated at the origin, then \( P = 1/2\pi \). In this case, the asymptotic form of the ice-covered surface has the following form

\[
\eta(r, t) \approx -\frac{1}{2\pi^2 \rho} \left( \frac{2\pi}{ra} \right)^{1/2} \frac{\alpha}{(1 + \epsilon \alpha)(1 + D\alpha)} \frac{1}{1/f'(\alpha)} \sin \left( t f(\alpha) \pm \frac{\pi}{4} \right)
\]

\[
- \frac{1}{2\pi^2 \rho} \left( \frac{2\pi}{r\beta} \right)^{1/2} \frac{\beta}{(1 + \epsilon \beta)(1 + D\beta)} \frac{1}{1/f'(\beta)} \sin \left( t f(\beta) \pm \frac{\pi}{4} \right),
\]

(28)
5 Discussion

To study the effect of the presence of the ice on the upper surface, the depression $\eta(r, t)$ of the ice-covered surface is presented in Figs. 1–4.

The Figs. 1 and 2 show the variation of $\eta(r, t)$ for a fixed $r$ when the time $t$ varies and the Figs. 3 and 4 show the variation of $\eta(r, t)$ at a fixed time $t$ while $r$ varies. When the upper surface of the fluid is free of ice (i.e. $\epsilon = 0$, $D = 0$), then the asymptotic results for $\eta(r, t)$, i.e. the form of the free surface, are also depicted in the figures by dashed curves in each figure. All these figures show that the effect of the presence of ice-cover is quite significant and it increases the wave amplitude.

Fig. 1 Waves due to an initial disturbance in the form of an initial depression $r = 60$, $\epsilon = 0.01$, $D = 0.5$

Fig. 2 Waves due to an initial disturbance in the form of an impulse $r = 60$, $\epsilon = 0.01$, $D = 0.5$
The results (27) and (28) are obtained in the case of the initial displacement and impulse being concentrated at the origin. For other forms of $F(r)$ and $G(r)$, appropriate asymptotic results can be obtained. Also, for uniform finite depth of water, a similar type of analysis can be employed to obtain the form of the depression of the ice-cover for large values of $r$ and $t$ (but finite $\xi \geq 1$).

References
SCATTERING OF OBLIQUE WAVES BY BOTTOM UNDULATIONS IN A TWO-LAYER FLUID

PARAMITA MAITI AND B. N. MANDAL*

ABSTRACT. Scattering of waves obliquely incident on small cylindrical undulations at the bottom of a two-layer fluid wherein the upper layer has a free surface and the lower layer has an undulating bottom, is investigated here assuming linear theory. There exists two modes of time-harmonic waves propagating at each of the free surface and the interface. Due to an obliquely incident wave of a particular mode, reflected and transmitted waves of both the modes are created in general by the bottom undulations. For small undulations, a simplified perturbation analysis is used to obtain first-order reflection and transmission coefficients of both the modes due to oblique incidence of waves of again both modes, in terms of integrals involving the shape function describing the bottom. For sinusoidal undulations, these coefficients are plotted graphically to illustrate the energy transfer between the waves of different modes induced by the bottom undulations.

AMS Mathematics Subject Classification: 76B15
Key words and phrases: Oblique scattering, bottom undulations, two-layer fluid, reflection and transmission coefficients.

1. Introduction

The linearised theory of small amplitude waves in two superposed inviscid fluids, separated by a common interface and the upper layer of lower density having a free surface, is given in the treatise of Lamb (1932). In such a two-layer fluid region, for a given frequency, time-harmonic gravity waves of two different modes propagate at each of the free surface and the interface. When a train of waves of a particular mode encounters an obstacle, then some of the energy from the incident wave mode is transferred to the other mode due to scattering by the obstacle. This makes the study of wave scattering problems in a two-layer fluid interesting. Linton and McIver (1995) developed the general theory for two-dimensional wave motion in a two-layer fluid in which the lower layer of
heavier density extends infinitely downwards and the upper layer of lower density has a free surface. They also investigated the two-dimensional wave scattering by a long horizontal circular cylinder submerged in either layer by employing multipole expansion method. This problem arose due to the plan to build a pipe bridge across a Norwegian fjord consisting of a layer of fresh water on top of a deep layer of salt water. Linton and Cadby (2002) extended the work of Linton and Mclver (1995) to oblique scattering.

In the present paper, scattering of oblique waves by small cylindrical undulations at the bottom in a two-layer fluid is investigated. For an ocean with bottom undulations, the corresponding oblique scattering problem was considered quite sometime back by Miles (1981) and Mandal and Basu (1990). Using linear theory, the problem is formulated as a coupled boundary value problem for the two potential functions describing the fluid motion in each of the two layers. Exploiting the smallness of the bottom undulations, a simplified perturbation technique is employed to reduce the original boundary value problem to another coupled one up to first order. This problem is solved here by a method, based on the use of Green's integral theorem, to obtain the first-order reflection and transmission coefficients of different modes due to obliquely incident waves of again different modes, in terms of integrals involving the shape function describing the bottom undulations. For the important case of sinusoidal undulations, the first-order coefficients are depicted graphically for various values of the different parameters.

2. Mathematical formulation

We consider a two-layer fluid for which the upper layer has a free surface and the lower layer has small cylindrical undulations at the bottom. A cartesian co-ordinate system is chosen in such a way that \( y = -h \) denotes the undisturbed free surface while \( y = 0 \) denotes the undisturbed interface, the \( y \)-axis pointing vertically downwards. Then the bottom of the lower layer can be represented by \( y = H + c(x) \) where \( c(x) \) is a bounded and continuous function describing the shape of the bottom and \( c(x) \to 0 \) as \( |x| \to \infty \), \( \epsilon \) giving a measure of smallness of the bottom undulations. Thus the lower layer is of uniform finite depth \( H \) below the mean interface far away from the undulations on either side. To see the far-field behaviors of the potential functions under the usual assumptions of linear theory relevant to the present problem, we consider a two-layer fluid wherein the lower layer is of uniform finite depth \( H \) below the mean interface, and \( \rho_1 \) is the density of the lighter fluid occupying the upper layer while \( \rho_2 (> \rho_1) \) is the density of the lower layer. As in Linton and Cadby (2002), the time-harmonic velocity potentials in the upper and lower layers can be described respectively by \( \text{Re}\{ \psi(x, y)e^{iux}e^{-\sigma t}\} \) and \( \text{Re}\{ \phi(x, y)e^{iux}e^{-\sigma t}\} \), \( \sigma \) denoting the angular frequency and \( t \) the time, where \( \psi, \phi \) satisfy

\[
\begin{align*}
(\nabla_x^2 - \nu^2)\psi &= 0 & \text{in the upper layer}, \\
(\nabla_x^2 - \nu^2)\phi &= 0 & \text{in the lower layer}.
\end{align*}
\]
The linearised free surface and interface conditions are

\[ K \psi + \psi_y = 0 \quad \text{on} \quad y = -h, \]  

\[ s(K \psi + \psi_u) = K \phi + \phi_y \quad \text{on} \quad y = 0, \]  

\[ \psi_y = \phi_y \quad \text{on} \quad y = 0, \]  

where \( s = \rho_1/\rho_2(\leq 1) \) and \( K = \sigma^2/g \), \( g \) being the acceleration due to gravity, and the bottom condition is

\[ \phi_n = 0 \quad \text{on} \quad y = H + cc(x), \]  

\( \phi_n \) denoting the derivative normal to the bottom.

In such a two-layer fluid, the progressive gravity waves propagating at each of the free surface and the interface can be expressed by

\[ f(k, y)e^{\pm i(k^2 - \nu^2)^{1/2}y}(-h < y < 0), \]  

\[ \cosh k(H - y)e^{\pm i(k^2 - \nu^2)^{1/2}y}(0 < y < H) \]  

\[ (2.7) \]

with

\[ f(k, y) = \frac{\sinh kH \{ k \cosh k(h + y) - K \sinh k(h + y) \}}{K \cosh k h_2 - k \sinh k h} \]  

\[ (2.8) \]

where \( k \) is real and positive and satisfies the dispersion relation

\[ \Delta(k) \equiv (1 - s)k^2 + K^2(a + \coth kh \coth kH) - kK(\coth kh + \coth kH) = 0. \]  

\[ (2.9) \]

The equation (2.9) has exactly two real and positive roots, \( m \) and \( M \) say, where \( K < m < M \), so that there exist two modes of waves propagating at each of the free surface and the interface.

Progressive waves of mode \( m \) are of the forms

\[ f(m, y)e^{\pm i(m^2 - \nu^2)^{1/2}y}(-h < y < 0), \]  

\[ \cosh m(H - y)e^{\pm i(m^2 - \nu^2)^{1/2}y}(0 < y < H) \]  

\[ (2.10) \]

where we must have \( \nu < m \) for these waves to exist. Similarly progressive waves of mode \( M \) are of the forms

\[ f(M, y)e^{\pm i(M^2 - \nu^2)^{1/2}y}(-h < y < 0), \]  

\[ \cosh M(H - y)e^{\pm i(M^2 - \nu^2)^{1/2}y}(0 < y < H) \]  

\[ (2.11) \]

where we must have \( \nu < M \) for these waves to exist.

An incident plane wave of mode \( m \) making an angle \( \alpha(0 \leq \alpha < \pi/2) \) with the positive \( x \)-axis has the forms

\[ f(m, y)e^{i\nu z \cos \alpha}(-h < y < 0), \]  

\[ \cosh m(H - y)e^{i\nu z \cos \alpha}(0 < y < H) \]  

\[ (2.12) \]

In this case

\[ \nu = m \sin \alpha. \]  

\[ (2.13) \]

Since \( M > m \), we must have \((M^2 - \nu^2)^{1/2}\) to be real, so that scattered waves of mode \( M \) exist for all values \( m \) and all angles \( \alpha \). Thus if a wave train of mode \( m \) is obliquely incident at angle \( \alpha \) with the positive \( x \)-axis on the cylindrical undulations at the bottom of a two-layer fluid, then reflected and transmitted
waves of both the modes $m$ and $M$ for any angle of incidence $\alpha$ occur and the far-field behaviors of $\psi$ and $\phi$ are given by

$$\psi(x, y) \rightarrow f(m, y) \left( e^{imx\cos\alpha} + r_m e^{-imx\cos\alpha} \right) + R_m f(M, y) e^{-i(M^2 - m^2 \sin^2 \alpha)^{1/2} x}$$

as $x \rightarrow -\infty$,

$$\phi(x, y) \rightarrow \cosh m(H - y) \left( e^{imx\cos\alpha} + r_m e^{-imx\cos\alpha} \right) + R_m \cosh M(H - y) e^{-i(M^2 - m^2 \sin^2 \alpha)^{1/2} x}$$

as $x \rightarrow -\infty$ (2.14)

and

$$\psi(x, y) \rightarrow t_m f(m, y) e^{imx\cos\alpha} + T_m f(M, y) e^{i(M^2 - m^2 \sin^2 \alpha)^{1/2} x}$$

as $x \rightarrow \infty$,

$$\phi(x, y) \rightarrow t_m \cosh m(H - y) e^{imx\cos\alpha} + T_m \cosh M(H - y) e^{i(M^2 - m^2 \sin^2 \alpha)^{1/2} x}$$

as $x \rightarrow \infty$. (2.15)

In (2.14) the constants $r_m$ and $R_m$ denote the reflection coefficients associated with reflected waves of modes $m$ and $M$ respectively due to an obliquely incident wave of mode $m$. Similarly in (2.15), $t_m$ and $T_m$ denote transmission coefficients associated with transmitted waves of modes $m$ and $M$ respectively due to an obliquely incident wave of mode $m$.

An incident wave of mode $M$ making an angle $\alpha$ with the positive $x$-axis has the forms

$$f(M, y) e^{iMx\cos\alpha} (-h < y < 0), \cosh M(H - y) e^{iMx\cos\alpha} (0 < y < H).$$

(2.16)

In this case

$$\nu = M \sin \alpha, \quad (m^2 - \nu^2)^{1/2} = (m^2 - M^2 \sin^2 \alpha)^{1/2}. \quad (2.17)$$

For a given angle $\alpha$, there may exist a value of $m$, i.e., $K$, for which $m = M \sin \alpha$ and thus $(m^2 - \nu^2)^{1/2} = 0$. This may be termed as the cut-off frequency (in the terminology used by Linton and Cadby (2002)) denoted by $K_c$. In Fig.1, $K_c h$ is depicted against $s = 3.5$ and $H/h = 2, 3, 5, 10, 100$. For $H/h = 100$, the curve for $K_c h$ almost coincides with the curve given in Figure 1 of Linton and Cadby (2002) where the lower layer was taken to be infinitely deep. For an incident wave of mode $M$ making an angle $\alpha$ with the positive $x$-axis for which the point $(K_c h, \alpha)$ lies on the left side of the curve in Fig.1, reflected and transmitted waves of mode $m$ exist. However, if this point lies on the right side of this curve, then there do not exist reflected and transmitted waves of mode $m$.

Thus for a wave train of mode $M$ obliquely incident on the bottom undulations of the two-layer fluid and making an angle $\alpha$ with the positive $x$-axis where $\alpha < \sin^{-1}(\frac{3.5}{H})$, the far field behaviors of $\psi$ and $\phi$ are given by

$$\psi(x, y) \rightarrow f(M, y) \left( e^{iMx\cos\alpha} + R^M e^{-iMx\cos\alpha} \right) + r_M f(m, y) e^{-i(m^2 - M^2 \sin^2 \alpha)^{1/2} x}$$

as $x \rightarrow -\infty$, ...
\[ \phi(x, y) \rightarrow \cosh M(H - y) \left( e^{iMx \cos \alpha} + R_M e^{-iMx \cos \alpha} \right) + r_M \cosh m(H - y) e^{-i\left( (m^2 - M^2 \sin^2 \alpha)^{1/2} \right) x} \quad \text{as } x \to -\infty \] (2.18)

and

\[ \psi(x, y) \rightarrow T_M f(M, y) e^{iMx \cos \alpha} + t_M f(m, y) e^{i\left( (m^2 - M^2 \sin^2 \alpha)^{1/2} \right) x} \quad \text{as } x \to \infty, \]

\[ \phi(x, y) \rightarrow T_M \cosh M(H - y) e^{iMx \cos \alpha} + t_M \cosh m(H - y) e^{i\left( (m^2 - M^2 \sin^2 \alpha)^{1/2} \right) x} \quad \text{as } x \to \infty. \] (2.19)

The constants \( r_M \) and \( R_M \) denote reflection coefficients associated with reflected waves of modes \( m \) and \( M \) respectively due to an obliquely incident wave of mode \( M \). Similarly, the constants \( t_M \) and \( T_M \) denote transmission coefficients associated with transmitted waves of modes \( m \) and \( M \) respectively due to an obliquely incident wave of mode \( M \). The angle of incidence \( \alpha \) must satisfy \( \alpha < \sin^{-1}\left( \frac{M}{m} \right) \) for the coefficients \( r_M, t_M \) to exist.

---

**Fig. 1.** Cut-off frequency \( K_H \) due to an incident wave of wavenumber \( M \): \( s=0.5 \).
If the incident wave has mode $m$, then $\psi, \phi$ satisfy (2.1), (2.2) with $\nu = m \sin \alpha (0 \leq \alpha < \pi/2)$, the conditions (2.3) to (2.5), the bottom condition

$$\phi_m = 0 \quad \text{on} \quad y = H + \varepsilon(x),$$

and the infinity requirements (2.14), (2.15) involving the unknown coefficients $r_m, R_m, t_m, T_m$.

Similarly if the incident wave has mode $M$, then $\psi, \phi$ satisfy (2.1), (2.2) with $\nu = M \sin \alpha (0 \leq \alpha < \sin^{-1}(\frac{1}{2}))$, the conditions (2.3) to (2.5), (2.20) and the infinity requirements (2.18), (2.19) involving the unknown coefficients $r_M, R_M, t_M, T_M$.

Determination of the reflection and transmission coefficients $r_m, M, R_m, t_m, M, T_m$ for a general type of bottom undulation is a difficult task. However, for small bottom undulations, an approximate method will be used here to obtain these coefficients up to first order. The clue for the approximate method is found by observing that the bottom condition (2.20) can be approximated up to first order of $\varepsilon$ as

$$-\phi_y + \varepsilon \frac{\partial}{\partial x} \{ c(x) \phi_x(x, H) \} + O(\varepsilon^2) = 0 \quad \text{on} \quad y = H.$$  

The lower layer $0 < y < H + \varepsilon c(x), -\infty < x < \infty$ reduces to the uniform strip $0 < y < H, -\infty < x < \infty$ in the following mathematical analysis using a perturbation technique.

3. The perturbation method

We consider a wave train of mode $m$ to be obliquely incident at an angle $\alpha$ ($0 \leq \alpha < \pi/2$) on the bottom undulations. If there is no bottom undulation, then the incident wave train will propagate without any hindrance and there will be total transmission. This along with the approximate form (2.21) of the bottom condition suggest that $\psi(x, y), \phi(x, y), r_m, R_m, t_m$ and $T_m$ can be expanded as

$$\psi = \psi_0 + \varepsilon \psi_1 + O(\varepsilon^2), \quad \phi = \phi_0 + \varepsilon \phi_1 + O(\varepsilon^2),$$

$$r_m = \varepsilon r^m_0 + O(\varepsilon^2), \quad R_m = \varepsilon R^m_0 + O(\varepsilon^2),$$

$$t_m = 1 + \varepsilon t^m_1 + O(\varepsilon^2), \quad T_m = \varepsilon T^m_1 + O(\varepsilon^2)$$

where

$$\psi_0(x, y) = f(m, y)e^{im \cos \alpha}, \quad \phi_0(x, y) = \cosh m(H - y)e^{im \cos \alpha}.$$  

Substituting the expansions (3.1) in (2.1), (2.2), with $\nu = m \sin \alpha$, the conditions (2.3), (2.4), (2.5), (2.21) and (2.14), (2.15), and equating the coefficients of $\varepsilon$ in both sides of the equations and the conditions, we find that the first-order functions $\psi_1(x, y), \phi_1(x, y)$ satisfy the coupled boundary value problem described by

$$\begin{align*}
(\nabla^2_x y - \nu^2) \psi_1 &= 0, & -h < y < 0, \\
(\nabla^2_x y - \nu^2) \phi_1 &= 0, & 0 < y < H,
\end{align*}$$

where $\nabla^2_x$ is the Laplacian operator with respect to the $x$ variable.
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$$K\psi_1 + \psi_1 = 0 \quad \text{on } y = -h,$$  
$$s(\psi_1 + \psi_1) = K\phi_1 + \phi_1 \quad \text{on } y = 0,$$  
$$\psi_1 = \phi_1 \quad \text{on } y = 0,$$  
$$\phi_1 = \eta(x) \quad \text{on } y = H$$  

where

$$q(x) = \frac{\im \cos \alpha}{dx} \left[ e^{\im mx \cos \alpha} - m^2 \sin^2 \alpha x \right],$$  

with the infinity requirements

$$\psi_1(x, y) \to r_1^n f(m, y)e^{-\im \pi \cos \alpha} + R_1^n f(M, y)e^{-\im ((M^2 - m^2 \sin^2 \alpha)^{1/2} x}} \text{ as } x \to -\infty,$$

$$\phi_1(x, y) \to r_1^n \cosh m(H - y)e^{-\im \pi \cos \alpha} + R_1^n \cosh M(H - y)e^{-\im ((M^2 - m^2 \sin^2 \alpha)^{1/2} x}} \text{ as } x \to -\infty,$$

$$\psi_1(x, y) \to r_1^n f(m, y)e^{\im \pi \cos \alpha} + T_1^n f(M, y)e^{\im ((M^2 - m^2 \sin^2 \alpha)^{1/2} x}} \text{ as } x \to \infty,$$

$$\phi_1(x, y) \to r_1^n \cosh m(H - y)e^{\im \pi \cos \alpha} + T_1^n \cosh M(H - y)e^{\im ((M^2 - m^2 \sin^2 \alpha)^{1/2} x}} \text{ as } x \to \infty.$$

A method based on the use of Green's integral theorem is now employed to solve this coupled boundary value problem and the first order coefficients \( r_1, R_1 \) and \( \eta, T_1 \) are obtained in terms of integrals involving the shape function.

4. Solution of the problem

To solve the coupled BVP described by (3.3) to (3.8), we construct Green's functions for the modified Helmholtz equation due to a source submerged in either of the two-layers. Let \( G(x, y; \xi, \eta) \) and \( G'(x, y; \xi, \eta) \) be the Green's functions in the upper and lower layers respectively due to a source submerged in the lower layer at \((\xi, \eta)(0 < \eta < H)\), and \( \mathcal{G}(x, y; \xi, \eta), \mathcal{G}'(x, y; \xi, \eta) \) be the same due to a source submerged in the upper layer at \((\xi, \eta)(-h < \eta < 0)\). Then \( G, G' \) satisfy

\[
(\nabla_x^2 - \nu^2)G = 0 \quad \text{for } -h < y < 0,
\]

\[
(\nabla_x^2 - \nu^2)G' = 0 \quad \text{except at } (\xi, \eta)(0 < \eta < H) \text{ for } 0 < y < H,
\]

\[
G' \to K_0(\nu r) \quad \text{as } r = [(x - \xi)^2 + (y - \eta)^2]^{1/2} \to 0
\]

where \( K_0(z) \) denotes the modified Bessel function of second kind,

\[
KG + G_y = 0 \quad \text{on } y = -h,
\]

\[
s(KG + G_y) = KG' + G'_y \quad \text{on } y = 0,
\]

\[
G_y = G'_y \quad \text{on } y = 0,
\]

\[
G'_y = 0 \quad \text{on } y = H,
\]
$G, G'$ behave as outgoing waves as $|x - \xi| \to \infty$, \hspace{1cm} (4.1)

while $G, G'$ satisfy

\[
(\nabla_x^2 - \nu^2)G = 0 \quad \text{except at } (\xi, \eta) (-h < \eta < 0) \text{for } -h < y < 0,
\]

$G \to K_0(\nu r)$ as $r = \sqrt{(x - \xi)^2 + (y - \eta)^2} \to 0$,

$(\nabla_x^2 - \nu^2)G' = 0$ for $0 < y < H$,

$KG + G_y = 0$ on $y = -h$,

$s(KG + G_y) = K G' + G'_y$ on $y = 0$,

$G_y = G'_y$ on $y = 0$,

$G'_y = 0$ on $y = H$.

$G, G'$ behave as outgoing waves as $|x - \xi| \to \infty$. \hspace{1cm} (4.2)

The functions $G, G'$ can be constructed following the method used by Mandal and Chakrabarti (1986). Expressions for $G, G', G, G'$ and their forms as $|x - \xi| \to \infty$ are given in the Appendix.

We now apply the Green's integral theorem to $\psi_1(x, y)$ and $G(x, y; \xi, \eta)$ in the form

\[
\int_C \left( \frac{\partial G}{\partial n} - G \frac{\partial \psi_1}{\partial n} \right) ds = 0 \tag{4.3}
\]

where $C$ is a contour in the $(x, y)$-plane formed by the lines $y = -h, 0 (-X < x < X), x = \pm X (-h < y < 0)$ and make $X \to \infty$ ultimately. There will be no contribution to the integral from the line $y = -h$ due to the free surface conditions satisfied by $\psi_1$ and $G$. Thus we obtain

\[
- \int_{-h}^{0} (\psi_1 G_x - G \psi_1)_y = \infty \, dy + \int_{-\infty}^{0} (\psi_1 G_y - G \psi_1)_{y=0} \, dx \\
+ \int_{-h}^{0} (\psi_1 G_x - G \psi_1)_{y=0} \, dy = 0. \tag{4.4}
\]

Again we apply the Green's integral theorem to $\phi_1(x, y)$ and $G' (x, y; \xi, \eta)$ in the form

\[
\int_{C'} \left( \frac{\partial G'}{\partial n} - G' \frac{\partial \phi_1}{\partial n} \right) ds = 0 \tag{4.5}
\]

where $C'$ is a contour formed by the lines $y = 0, H (-X < x < X), x = \pm X (0 \leq y < H)$ and a circle of small radius $\epsilon$ with centre at $(\xi, \eta) (0 < \eta < H)$ and make $X \to \infty$ and $\epsilon \to 0$ ultimately. We then obtain

\[
-2\pi \phi(\xi, \eta) + \int_{-\infty}^{\infty} (\phi_1 G'_y - G' \phi_1)_y = \infty \, dx + \int_{-h}^{H} (\phi_1 G'_x - G' \phi_1)_{y=0} \, dy \\
- \int_{-\infty}^{\infty} (\phi_1 G'_y - G' \phi_1)_{y=H} \, dx - \int_{0}^{H} (\phi_1 G'_x - G' \phi_1)_{y=\infty} \, dy = 0. \tag{4.6}
\]
Multiplying (4.4) by \(s\) and subtracting (4.6) from this and using the interface conditions and the fact that

\[
s\int_{-h}^{0} (\psi_1 G_x - G' \psi_{1x})_{x=\pm X} dy + \int_{0}^{H} (\phi_1 G'_x - G'' \phi_{1x})_{x=\pm X} dy \to 0 \quad \text{as} \quad X \to \infty,
\]

we obtain the representation for \(\phi_1(\xi, \eta)\) (\(0 < \eta < H\)), after using (3.8) in the form,

\[
\phi_1(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G'(x, y; \xi, \eta) q(x) dx, \quad 0 < \eta < H. \tag{4.7}
\]

A somewhat similar procedure can be used to obtain a representation for \(\psi_1(\xi, \eta)\) \((-h < \eta < 0)\), in the form

\[
\psi_1(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G'(x, y; \xi, \eta) q(x) dx, \quad -h < \eta < 0. \tag{4.8}
\]

The first-order reflection and transmission coefficients \(r_1^m, R_1^m\) and \(t_1^M, T_1^M\) are now obtained by making \(\xi \to -\infty\) and \(\xi \to \infty\) respectively in (4.7) or (4.8) and comparing with (3.10) and (3.11) \((x, y)\) being replaced by \(\xi, \eta\). For this the forms of \(G'\) or \(G''\) as \(\xi \to \pm \infty\), given in the Appendix, have been used. Thus \(r_1^m, R_1^m\) are obtained as

\[
r_1^m = -\frac{i K m (K \cosh mh - m \sinh mh) \cos 2\alpha}{\cos \alpha \sinh mh \sinh^2 mh \Delta'(m)} \int_{-\infty}^{\infty} e^{i m \cos \alpha c(x)} dx, \tag{4.9}
\]

\[
R_1^m = -\frac{i K m (K \cosh mh - M \sinh mh) \{\cos \alpha (M^2 - m^2 \sin^2 \alpha)^{1/2} - m \sin^2 \alpha\}}{(M^2 - m^2 \sin^2 \alpha)^{1/2} \sinh mh \sinh^2 mh \Delta'(M)} 
\times \int_{-\infty}^{\infty} e^{i (m \cos \alpha + (M^2 - m^2 \sin^2 \alpha)^{1/2}) c(x)} dx, \tag{4.10}
\]

while \(t_1^M, T_1^M\) are obtained as

\[
t_1^M = \frac{i K m (K \cosh mh - m \sinh mh)}{\cos \alpha \sinh mh \sinh^2 mh \Delta'(m)} \int_{-\infty}^{\infty} c(x) dx, \tag{4.11}
\]

\[
T_1^M = \frac{i K m (K \cosh mh - M \sinh mh) \{\cos \alpha (M^2 - m^2 \sin^2 \alpha)^{1/2} + m \sin^2 \alpha\}}{(M^2 - m^2 \sin^2 \alpha)^{1/2} \sinh mh \sinh^2 mh \Delta'(M)} 
\times \int_{-\infty}^{\infty} e^{i (m \cos \alpha - (M^2 - m^2 \sin^2 \alpha)^{1/2}) c(x)} dx. \tag{4.12}
\]

For a wave train of mode \(M\) obliquely incident at an angle \(\alpha\) \((0 \leq \alpha < \sin^{-1}(\frac{M}{h}))\), the first-order coefficients \(r_1^M, R_1^M, t_1^M, T_1^M\) can be obtained by following a similar mathematical analysis described above. The results are as follows:

\[
r_1^M = -\frac{i K M (K \cosh mh - m \sinh mh) \{\cos \alpha (M^2 - m^2 \sin^2 \alpha)^{1/2} - M \sin^2 \alpha\}}{(m^2 - M^2 \sin^2 \alpha)^{1/2} \sinh mh \sinh^2 mh \Delta'(m)}
\]
5. Numerical Results

For sinusoidal undulations at the bottom of the two-layer fluid, the shape function $c(x)$ has the form

\[ c(x) = \begin{cases} \sin \gamma x & \text{for } -\frac{\pi}{\gamma} \leq x \leq \frac{\pi}{\gamma}, \\ 0 & \text{otherwise} \end{cases} \]

where $n$ is a positive integer. In a single-layer ocean bed, sinusoidal undulations occur naturally. Davies (1982) studied this case somewhat elaborately and found that an undulating bed has the ability to reflect incident wave energy which has important implications in respect of coastal protection as well as possible ripple growth if the bed is erodible. For this reason, this particular bed form in a two-layer fluid is considered here and the first-order reflection and transmission coefficients are calculated explicitly and the numerical results are presented graphically against the wave number $K h$.

Since $c(x)$ is an odd function, $t_{10}$, $T_{10}$ vanish identically. The coefficients $r_{11}^{m}$, $R_{11}^{m}$, $T_{11}^{m}$ are obtained as

\[ r_{11}^{m} = (-1)^{n+1} \left( \frac{K m (K \cosh M h - m \sinh M h) \cos 2\alpha}{\cos \alpha \sinh M h \sinh^2 M h \Delta'(M)} \right) \left( \frac{2\alpha \gamma \sin \left( \frac{2m\pi}{\gamma} \cos \alpha \right)}{\gamma^2 - 4m^2 \cos^2 \alpha} \right), \]

\[ R_{11}^{m} = (-1)^{n+1} \times \left( \frac{K m (K \cosh M h - m \sinh M h) \left( \cos \alpha (M^2 - m^2 \sin^2 \alpha)^{1/2} - m \sin^2 \alpha \right)}{(M^2 - m^2 \sin^2 \alpha)^{1/2} \sinh M h \sinh^2 M h \Delta'(M)} \right) \left( \frac{2\alpha \gamma \sin \left( \frac{m\pi}{\gamma} \right) \left( m \cos \alpha + (M^2 - m^2 \sin^2 \alpha)^{1/2} \right)}{\gamma^2 - (m \cos \alpha + (M^2 - m^2 \sin^2 \alpha)^{1/2})^2} \right), \]

\[ T_{11}^{m} = (-1)^{n} \times \left( \frac{K m (K \cosh M h - M \sinh M h) \left( \cos \alpha (M^2 - m^2 \sin^2 \alpha)^{1/2} + m \sin^2 \alpha \right)}{(M^2 - m^2 \sin^2 \alpha)^{1/2} \sinh M h \sinh^2 M h \Delta'(M)} \right). \]
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\[
\frac{2a\gamma \sin \left( \frac{\pi}{2} (m \cos \alpha - (M^2 - m^2 \sin^2 \alpha)^{1/2}) \right)}{\gamma^2 - (m \cos \alpha - (M^2 - m^2 \sin^2 \alpha)^{1/2})^2},
\]

(5.3)

\[
\tau^M_1 = (-1)^{n+1} K M (K \cosh mh - m \sinh mh) \left( \cos \alpha (m^2 - M^2 \sin^2 \alpha)^{1/2} - M \sin^2 \alpha \right) \]

\[
\frac{1}{(m^2 - M^2 \sin^2 \alpha)^{1/2} \sinh mh \sinh^2 MH \Delta'(m)}
\]

(5.4)

\[
R^M_1 = (-1)^n K M (K \cosh mh - m \sinh mh) \cos 2\alpha \frac{2a\gamma \sin \left( \frac{\pi}{2} (M \cos \alpha + (m^2 - M^2 \sin^2 \alpha)^{1/2}) \right)}{\gamma^2 - (M \cos \alpha + (m^2 - M^2 \sin^2 \alpha)^{1/2})^2},
\]

(5.5)

In the results (5.4) to (5.6), \( 0 < \alpha < \sin^{-1}(\frac{\pi}{2}) \).

The figures 2-4 depict \(|r^m_n|, |R^m_n|, |T^m_n|\), the first order reflection and transmission coefficients due to a wave train of mode \( m \) obliquely incident at an angle \( \alpha \) with the positive \( z \)-axis on the undulating bed in the lower layer for \( \alpha/h = 0.1, H/h = 0.5, \alpha = 0.262, 0.524, 1.05, 1.31 \) (measured in radians).

The case \( \alpha = 0 \) represents the case of normal incidence. One feature that is common in all these figures is the oscillating nature of the absolute values of the first-order coefficients as functions of the wave number \( K \). For the case of normal incidence, the peak values of \(|r^m_n|, |R^m_n|\) are the largest, while \(|T^m_n|\) assumes the least peak value. The oscillating nature may be attributed due to multiple interaction of the incident wave train with the sinusoidal bottom, the free surface and the interface.
In the figures 5-7, plots of $|r^M|$, $|R^M|$, $|t^M|$ are shown against $Kh$ for $a/h = 0.1$, $H/h = 2$, $s = 0.5$, $\gamma/\alpha = 1$, $n = 3$ and $\alpha = 0, 0.35, 0.36, 0.38, 0.39$. It may be noted that while for $\alpha = 0$ (the case of normal incidence), conversion of incident wave energy of mode $M$ to reflected wave energy of mode $m$ is always possible, for non-zero $\alpha$ (i.e., the case of oblique incidence), there will be no energy conversion if $\alpha$ exceeds a certain critical angle. The value of the critical angle obviously depends on $Kh$ apart from other parameters.

The fig. 5 depicts $|R^M|$ for normal incidence ($\alpha = 0$) as well as for oblique incidence with $\alpha$ slightly less than the critical angle (viz. $\alpha = 0.35, 0.36, 0.38, 0.39$). As $\alpha$ increases, the range of $Kh$ for which $|R^M|$ exists, decreases. The fig. 7 depicting $|t^M|$ also displays the same characteristics. However, the fig. 6, depicting $|R^M|$ does not show such behavior since reflection at mode $M$ of incident wave energy of mode $M$ always occurs whatever be the incident wave angle.
The figures 8-10 show the curves for $|r_1^M|$, $|R_1^M|$, $|T_1^M|$ as a function of $K\theta$ for $a/h = 0.1$, $H/h = 2$, $\alpha = 0.35$, $\gamma h = 1$, $\eta = 3$ and different values of $s = 0.1, 0.4, 0.5, 0.56, 0.6$. As $s$ increases, the curves for $|R_1^M|$ and $|T_1^M|$ become more oscillatory (cf. figs. 9 and 10) while the curves for $|r_1^M|$ show that the peak values decrease (cf. fig. 8).

The figures 11-13 depict the curves for $|r_1^M|, |R_1^M|$, and $|T_1^M|$ for the same values of different parameters as in figs. 8-10.
The figs. 11 and 13 show that conversion of the incident wave energy at mode $M$ to reflected and transmitted wave energies at mode $m$ is limited up to appropriate cut-off values of $K_h$ for different values of the density ratio $s$. The fig 10 depicting $|R_M^m|$ however does not show any cut-off values of $K_h$, which is obvious since there is no question of energy conversion from one mode to another here. However, increase of $s$ results in more oscillatory nature for these coefficients.

Effect of $n$, the number of patches of the sinusoidal undulations, on the various first-order coefficients is depicted in figures 14-19. The figs. 14, 15, 16, 18 show that peak values of $|r^m|$, $|R^m|^0$, $|T^m|$ and $R^M_l$ increase with $n$ while the figs. 17 and 19 show the cut-off values of $K_h$ for which energy conversion from mode $M$ to mode $m$ (reflected and transmitted) takes place for different values of $n$.

6. Conclusion
Scattering of oblique waves by small cylindrical undulations on the bottom of a two-layer fluid is investigated. Using a simplified perturbation analysis, the problem is reduced up to first-order to a coupled boundary value problem.

The boundary value problem is solved by a method based on use of Green's integral theorem and the first-order reflection and transmission coefficients of two different modes due to oblique incidence of waves of again two different modes are obtained in terms of integrals involving the shape function describing the bottom undulations. While transfer of incident wave energy at the lower mode to the higher mode is possible for all possible angles of incidence, transfer from higher mode to the lower mode is restricted to a range for the angle of incidence depending on the various parameters involved in the problem as well as the incident wave frequency.

For sinusoidal undulations, the first-order coefficients are evaluated numerically and presented graphically in a number of figures. These figures illustrate...
the role played by various parameters such as the angle of incidence, the density ratio, number of patches of the sinusoidal undulations etc. on the energy transfer between the waves of different modes by the undulating bottom topography. One feature which is common in most of these figures is the oscillatory nature of the first-order coefficients as functions of the wave number.

This is attributed to the interaction of the incident wave field with the free surface, the interface and the undulating bottom. Also, the peak values of these constants increase with the number of patches of the sinusoidal undulations at the bottom. This phenomenon may be significant in coastal hydrodynamics involving a two-layer ocean wherein the upper layer consists of fresh water while the lower layer consists of salt water.

**APPENDIX**

**Construction of Green's functions**

The Green's functions in the upper and lower layers respectively for the modified Helmholtz equation due to a source submerged in the lower or the upper layer can be constructed following a somewhat similar method used by Mandal and Chakrabarti (1986) who constructed the Green's functions for the modified Helmholtz equation in a two-layer fluid wherein the upper layer extends infinitely upwards and the lower layer is of uniform finite depth below the mean interface or extends infinitely downwards. Expressions for the Green's functions and their forms as $|x - \xi| \to \infty$ are given below.

**Source submerged in the lower layer**

Let $(\xi, \eta) \ (0 < \eta < H)$ be the source, $G(x, y; \xi, \eta)$ and $G'(x, y; \xi, \eta)$ be the Green's functions for the modified Helmholtz equation in the upper and lower layers respectively satisfying (4.1). These are obtained as
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\[ G(x, y; \xi, \eta) = \frac{K_0(\nu r) - K_0(\nu r')}{s} - 2 \int_{\nu}^{\infty} F_{11}(k; y, \eta) \cos\left\{ \left( k^2 - \nu^2 \right)^{1/2} |x - \xi| \right\} dk, \]
\[ -h < y < 0, 0 < \eta < H \quad (A1) \]

\[ G'(x, y; \xi, \eta) = K_0(\nu r) - K_0(\nu r') - 2 \int_{\nu}^{\infty} F_{12}(k; y, \eta) \cos\left\{ \left( k^2 - \nu^2 \right)^{1/2} |x - \xi| \right\} dk, \]
\[ 0 < y, \eta < H \quad (A2) \]

where

\[ r = ((x - \xi)^2 + (y - \eta)^2)^{1/2}, \quad r' = ((x - \xi)^2 + (y + 2h + \eta)^2)^{1/2} \]

\[ F_{11}(k; y, \eta) = e^{-k(h+\eta)} \left\{ k \sinh kH (k \cosh ky - K \sinh ky) + \cosh kH (k \sinh kH - K \cosh kH) \right\} \]
\[ + s (k \sinh kH - K \cosh kH) (k \sinh kH - K \cosh kH) \sinh k(h + y) \]
\[ + K e^{-k(h+\eta)} \sinh k(h + \eta) (k \cosh k(h + y) - K \sinh k(h + y)), \]
\[ -h < y < 0, 0 < \eta < H \]

\[ F_{12}(k; y, \eta) = e^{-k(h+\eta)} \left\{ kK + (1 - s) (K^2 - k^2) \sinh kH \cosh kH \right\} + \sinh k(h + \eta) \]
\[ \times e^{-k(h+\eta)} \left\{ (k \cosh kH - K \sinh kH) (K \cosh kH - k \sinh kH) \right\} \cosh k(H - y) \]
\[ - s (K^2 - k^2) \sinh kH \cosh kH \right\} \cosh k(H - y) \]
\[ + e^{-k(h+\eta)} \Delta(k) \sinh kH \cosh k(h + \eta) \sinh k(H - y), \]
\[ 0 < y, \eta < H \]

and the path of integration in the integrals in (A1) and (A2) are indented below the poles at \( k = m, M \) on the real axis so as to ensure the outgoing nature of \( G, G' \) far away from the source (\( \nu < m < M \)).

As \( |x - \xi| \to \infty \), it can be shown that

\[ G(x, y; \xi, \eta) \to -2\pi i \left[ K \left\{ m \cosh m(h + y) - K \sinh m(h + y) \right\} \cosh m(H - \eta) \right] \]
\[ \frac{e^{(m^2 - \nu^2)^{1/2}|s-\xi|}}{(m^2 - \nu^2)^{1/2} \sinh mH \sinh mH \Delta'(m)} \]
\[ \times e^{(M^2 - \nu^2)^{1/2}|s-\xi|} \quad -h < y < 0, 0 < \eta < H, \quad (A3) \]
and

\[ G'(x, y; \xi, \eta) \rightarrow -2\pi i K \left[ \begin{array}{c} \frac{(K \cosh mh - m \sinh mh) \cosh m(H - \eta) \cosh m(H - y)}{(m^2 - \nu^2)^{1/2} \sinh mh \sinh^2 mH \Delta'(m)} \\ \times e^{(m^2 - \nu^2)^{1/2} |x - \xi|} \\ \end{array} \right] \\
+ \frac{(K \cosh Mh - M \sinh Mh) \cosh M(H - \eta) \cosh M(H - y)}{(M^2 - \nu^2)^{1/2} \sinh Mh \sinh MH \Delta'(M)} \\
\times e^{(M^2 - \nu^2)^{1/2} |y - \xi|}, \quad -h < y, \eta < H. \]

(A4)

**Source submerged in the upper layer**

Let \( G(x, y; \xi, \eta) \) and \( G'(x, y; Z, T) \) denote the Green's functions for the modified Helmholtz equations in the upper and lower layers satisfying (4.2) due to a source at \((\xi, \eta)\) submerged in the upper layer \((-h < \eta < 0)\). These are obtained as

\[ G(x, y; \xi, \eta) = K_0(\nu r) - K_0(\nu r') - 2 \int_0^\infty F_{21}(k; y, \eta) \cos\left(\left(\frac{k^2 - \nu^2}{1 - \nu^2}\right)^{1/2} |x - \xi|\right) dk, \]

\[-h < y, \eta < 0 \quad (A5)\]

\[ G'(x, y; \xi, \eta) = s \left[ K_0(\nu r') - K_0(\nu r') \right] - 2 \int_0^\infty F_{22}(k; y, \eta) \cos\left(\left(\frac{k^2 - \nu^2}{1 - \nu^2}\right)^{1/2} |x - \xi|\right) dk, \]

\[ 0 < y < H, -h < \eta < 0, \quad \text{where } r, r' \text{ are the same as above,} \]

\[ F_{21}(k; y, \eta) = k e^{-k(h+\eta)} \left[ \begin{array}{c} (K \cosh kH - k \sin h kH) \cosh ky + \\ s \sinh kH (k \cosh kH - k \sinh kH) e^{-k(h+\eta)} \\ \end{array} \right] \\
\]

\[ \left( k \sinh kH - K \cosh kH \right) \sinh k\eta \sinh kH \Delta(k), \]

\[-h < y, \eta < 0, \quad (A5)\]

\[ F_{22}(k; y, \eta) = s \left[ k^{-1}(k^2 - \nu^2) \sinh k(h + \eta) \right] \\
\left( K \cosh kH - k \sinh kH \right) \sinh k\eta \sinh kH \Delta(k), \]

\[ + k e^{-k(h+\eta)} \left( K^2 - k^2 \right) \sinh kH \Delta(k), \]

\[ 0 < y < H, -h < \eta < 0, \]

and the path of integration in the integrals in (A4) and (A5) are indented below the poles at \( k = m, M \) on the real axis as before.

As \( |x - \xi| \rightarrow 0 \), it can be shown that

\[ G(x, y; \xi, \eta) \rightarrow -2\pi i K \left[ \begin{array}{c} \frac{(K \sinh m(h + \eta) - m \cos m(h + \eta))}{(m^2 - \nu^2)^{1/2} \sinh mh \sinh MH \Delta'(m)} \\ \times e^{(m^2 - \nu^2)^{1/2} |x - \xi|} \\ \end{array} \right] \\
+ \frac{(K \sinh M(h + \eta) - M \cos m(M(h + \eta))}{(M^2 - \nu^2)^{1/2} \sinh MH \sinh MH(M \cosh Mh - M \sinh Mh) \Delta'(M)} \\
\times e^{(M^2 - \nu^2)^{1/2} |y - \xi|}, \quad -h < y, \eta < H. \]
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\[ Q'(x,y) = -2\pi i s K \text{cosh}(m(h + \eta) - K \sinh(m(h + \eta))) \text{cosh}(m(H - y)) \]
\( \times e^{(m^2 - \nu^2)^{1/2}(y-\xi)} \), \(-h < \eta < 0\) \(\cdots\) \(A7\)

\[ G'(x,y;\xi,\eta) = -2\pi i s K \left[ \frac{m \cosh(m(h + \eta) - K \sinh(m(h + \eta))) \cosh(m(H - y))}{(m^2 - \nu^2)^{1/2} \sinh(m(h + \eta)) \sinh(mH \Delta'(m))} \right] \]
\[ + e^{(m^2 - \nu^2)^{1/2}(y-\xi)} \frac{M \cosh(M(h + \eta) - K \sinh(M(h + \eta))) \cosh(M(H - y))}{(M^2 - \nu^2)^{1/2} \sinh(M(h + \eta)) \sinh(MH \Delta'(M))} \]
\[ \times e^{(m^2 - \nu^2)^{1/2}(y-\xi)} \], \(0 < y < H, -h < \eta < 0\). \(\cdots\) \(A8\)

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