CHAPTER 2

CLASSICALITY FROM QUANTUM DYNAMICS VIA
ENVIRONMENT-INDUCED DECOHERENCE

2.1 Introduction

In spite of the remarkable success of quantum theory, its measurement aspect remains a profound interpretational difficulty. Quantum mechanics, it seems, fails to provide a natural framework to accommodate our familiar classical perceptions. The state vector, \( |\psi\rangle \), for a quantum system is a mathematical entity which, according to quantum formalism, contains all possible information about the system and evolves in time according to the Schrödinger equation which is linear and deterministic:

\[
\frac{\partial |\psi\rangle}{\partial t} = H |\psi\rangle,
\]

where \( H \) is the Hamiltonian of the system. Dynamical variables or observables in quantum theory are represented by linear hermitian operators, each associated with a complete set of eigenvectors. According to the quantum formalism, the measurement of a dynamical variable \( A \), corresponding to a hermitian operator \( \hat{A} \) with eigenvalues \( a_i \)'s and eigenvectors \( |\alpha_i\rangle \) \( (\hat{A} |\alpha_i\rangle = a_i |\alpha_i\rangle) \), can be described in the following way. The state vector \( |\psi\rangle \) for the system is expanded as a linear superposition of the orthonormal eigenvectors of \( \hat{A} \):

\[
|\psi\rangle = \sum_i C_i |\alpha_i\rangle
\]
A single measurement of \( A \) yields only one of the eigenvalues \( a_i \) but this result is not definite in the sense that it can be different for different measurements on the same system. Quantum theory only predicts relative probabilities for obtaining different eigenvalues. One can write the expectation value of \( \hat{A} \) in the state \( |\psi\rangle \) as:

\[
\langle \hat{A} \rangle = \langle \psi | \hat{A} |\psi\rangle = \sum_i a_i |c_i|^2.
\]

Quantum mechanics predicts that the probability of obtaining the eigenvalue \( a_i \) for a measurement of \( A \) in state \( |\psi\rangle \) is \( |c_i|^2 \). There is, of course, nothing special about the operator \( \hat{A} \) since \( |\psi\rangle \) can well be expanded in any basis set which forms the complete eigenvector set of any other observable of the system. Thus, we have a situation where the evolution of the state vector is completely deterministic but the outcome of a measurement of any dynamical variable is probabilistic. A complete knowledge of \( |\psi\rangle \) does not ensure a complete knowledge about the outcome of any measurement. The state vector contains no more than 'potentialities' in the form of probabilistic information about the various observables of the system. An additional postulate of quantum mechanics is that the measurement of the observable \( A \) which yields one of the eigenvalues \( a_i \) culminates with the reduction or 'collapse' of the state vector \( |\psi\rangle \) to the eigenstate \( |\alpha_i\rangle \). Thus, we have a situation where the state vector evolves according to the Schrödinger equation containing within itself all the possibilities of the system's future but whenever a measurement of a dynamical variable is performed on the
system, the state vector falls or 'collapses' into one particular state. The dynamical evolution brought about by the Schrödinger equation for a closed system is unitary (i.e., the eigenvalues are preserved) and hence cannot obviously contain within itself the irreversible reduction of the state vector. von Neumann (1932) postulated that in addition to the unitary evolution given by the Schrödinger equation, there is a nonunitary reduction (projection) of the state vector when the system comes into contact with a classical apparatus. This adhoc reduction is not contained within conventional quantum dynamics. How does this reduction take place? The crux of the measurement problem is that quantum formalism provides no mechanism for this collapse.

An intriguing consequence of quantum formalism is that the linearity of the Schrödinger equation allows for solutions which are linear superpositions of other solutions. This means that 'entanglements' of states which may correspond to macroscopically different situations in the classical world are perfectly legitimate solutions in quantum theory. A dramatic implication of the superposition theory is the frequently quoted example of 'Schrödinger's cat' (Schrödinger 1935). Quantum formalism allows the 'cat' to exist in a state which is an 'entanglement' of states in which the cat is 'dead' and 'alive'. Such a scenario has simply no classical counterpart. Macroscopic objects are never seen in entanglements of macroscopically distinct states and the whole concept is unacceptable when viewed in the classical context.

The density matrix is a convenient formal tool to compare and contrast quantum and classical systems in terms of probabilities. The conceptual
problems of quantum measurement become more transparent when analyzed in the language of density matrices and we shall use this approach throughout the thesis. The density matrix \( \rho \) (see footnote)\(^1\) corresponding to a quantum state \( |\psi\rangle \) can be written as:

\[
\rho = |\psi\rangle \langle \psi |.
\]  

Consider von Neumann's analysis of a measurement process (von Neumann 1932) which involves a two-state quantum system (say, spin-1/2) and a two-state quantum detector (apparatus). The Hilbert space of the system is spanned by the orthonormal states \( |\uparrow\rangle \) and \( |\downarrow\rangle \), while the states \( |d_{\uparrow}\rangle \) and \( |d_{\downarrow}\rangle \) span the space of the detector. Assume that one can devise a quantum detector that begins in a state \( |d_{\downarrow}\rangle \) and 'clicks', \( |\uparrow\rangle \rightarrow |\uparrow\rangle \rightarrow |d_{\uparrow}\rangle \) when the spin is in the state \( |\uparrow\rangle \), but remains unperturbed otherwise (Zurek 1991). One can easily see that an initial system-detector state, \( (\alpha |\uparrow\rangle + \beta |\downarrow\rangle ) \otimes |d_{\downarrow}\rangle \) (with \( |\alpha|^2 + |\beta|^2 = 1 \)) evolves into the correlated state

\[
|\phi_c\rangle = \alpha |\uparrow\rangle |d_{\uparrow}\rangle + \beta |\downarrow\rangle |d_{\downarrow}\rangle .
\]  

where a one-to-one correlation between the states of the system and the apparatus is established; i.e., if the detector is seen in the state \( |d_{\uparrow}\rangle \), the system is guaranteed to be found in the state \( |\uparrow\rangle \). The density matrix corresponding to this state is:

\[^1\text{Henceforth in this Chapter we will not be using the caret (}^\wedge\text{) to distinguish Hilbert-space operators from complex numbers. The operator character should be obvious from their usage.}\]
\[ \rho^c = |\phi_c\rangle\langle\phi_c| = |\alpha|^2 |\uparrow\rangle\langle\uparrow| + |\beta|^2 |\downarrow\rangle\langle\downarrow| + \alpha^*\beta |\uparrow\downarrow\rangle\langle\downarrow\uparrow| + \alpha\beta^* |\downarrow\uparrow\rangle\langle\uparrow\downarrow| , \]

i.e., in the basis \( |\uparrow\rangle \langle\uparrow| \) \( |\downarrow\rangle \langle\downarrow| \)

\[ \rho^c = \begin{bmatrix} |\alpha|^2 & \alpha^*\beta \\ \alpha\beta^* & |\beta|^2 \end{bmatrix} \quad (2.1.6) \]

One can see that apart from the diagonal terms, \( |\alpha|^2 |\uparrow\rangle\langle\uparrow| \) \( |\downarrow\rangle\langle\downarrow| \) and \( |\beta|^2 |\downarrow\rangle\langle\downarrow| \) \( |\uparrow\rangle\langle\uparrow| \), which can be understood classically as the probabilities corresponding to the system and detector being up and down respectively, \( \rho^c \) contains off-diagonal terms, \( \alpha^*\beta |\uparrow\downarrow\rangle\langle\downarrow\uparrow| \) and \( \alpha\beta^* |\downarrow\uparrow\rangle\langle\uparrow\downarrow| \), which cannot be interpreted as classical probabilities. These off-diagonal terms are related to the entanglements that arise as a consequence of the superposition principle. Moreover, just like the state \( |\phi_c\rangle \), \( \rho^c \) can be represented in any basis leaving the choice of the observable to be measured to the observer long after the system-apparatus interaction has ceased. Such predictions of quantum formalism are counterintuitive and strongly contradict classical perceptions where the choice of what the apparatus measures cannot be made by us long after the system-apparatus interaction has ceased! In sharp contrast to the quantum case, a classical situation would correspond to a completely diagonal density matrix in one particular basis, where the choice of the measured observable is decided by the nature of the system-apparatus interaction and not left to the discretion of the observer. For a measurement process the density matrix must undergo the reduction: \( \rho^c \rightarrow \)
\[
\begin{bmatrix}
|\alpha|^2 & 0 \\
0 & |\beta|^2
\end{bmatrix}
\] (The second stage of reduction to either \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) or \( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \)

corresponds to the collapse of a classical probability distribution as a result of acquisition of knowledge about the system. This special basis, in which the density matrix is diagonal, is often referred to in the literature as the 'preferred basis' (Zurek 1991, Zurek et al 1993). If one is to believe that quantum mechanics is indeed the fundamental theory of nature, then there ought to be a logical explanation for the transition from its counterintuitive implications to the 'common sense' physics of the classical world. How do superpositions disappear? How does the pure state density matrix reduce to a diagonal mixture? There have been many attempts to find answers to the conceptual problems of quantum theory through several fascinating ideas.

Among the earliest explanations is the Copenhagen Interpretation proposed by Niels Bohr (1928), which requires the presence of an external classical apparatus to cause the "collapse" and hence a measurement. The theory cannot explain the actual mechanism of the collapse and the ambiguous border between "quantum" and "classical" world makes it unsatisfactory. The Many Worlds Interpretation of H. Everett (1957) and B.S. DeWitt (1973) treats the entire universe as a quantum system, having one state vector which "branches out" with every interaction. The theory does away with classical concepts completely and "potential outcomes" are accommodated by the branches of the wave function, the observer being conscious of only one branch. The theory, however, fails to provide a convincing answer as to when the branching occurs and remains highly controversial. The statistical
interpretation (Ballentine 1990, 1991) of quantum theory treats the quantum mechanical formalism as merely a description of our information about the "ensemble" in question. Although the theory provides an alternative way of looking at quantum mechanics, it fails to explain the behaviour of a single, specific system. Thus, in spite of the emergence of several interesting theories, many relevant issues of the measurement problem remain unresolved. In the following we introduce the 'environment-induced decoherence theory' (Zeh 1970, Zurek 1991) which has, to a large extent, provided convincing explanations for the emergence of classicality from quantum dynamics. Most of the work discussed in this Chapter is based on this approach.

The environment-induced decoherence theory employs the methods developed by several authors to analyse the quantum mechanics of a system in interaction with its environment. It is founded on the realization that the measurement apparatus, being macroscopic, is never isolated from its environment. The closely-spaced levels of a macroscopic apparatus make it very susceptible to the influence of the environment. This is an important observation and forms the basis of the "environment-induced decoherence" theory (Zeh 1970, Joos and Zeh 1985, Zurek 1991) for the measurement problem. The crucial point here is the dissipative role of the environment. When a small quantum system interacts with a large reservoir (environment), its evolution is analyzed in terms of the reduced density matrix of the system which is obtained by tracing over the environmental degrees of freedom (Caldeira and Leggett 1983, 1985), and one sees the monotonic time decay of the off-diagonal matrix elements of the reduced density matrix. These are actually arising due to the superposition of a large number of harmonic
terms, viz., \( \sum_{j=1}^{n} a_j \exp(i \omega_j t) \), which, under the condition that \( \omega_j \) are closely spaced and that the number of terms \( n \) is large, gives rise to apparent decays in the short time limit \( t \ll T \), where \( T \) is the characteristic 'recurrence' time for the periodic function. However, \( T \) can be astronomically large even under mild conditions of \( n \sim 1000 \), and \( \Delta \omega \sim 10^{-3} \). The presence of off-diagonal elements in the density matrix is a signature of quantum coherence in the system and hence the decay of these off-diagonal terms of the density matrix is called 'decoherence'. Thus a quantum system coupled to an environment consisting of a large number of degrees of freedom behaves like a classical system in the sense that at time scales of interest its density matrix is diagonal. This approach thus, provides a way of explaining state reduction within the framework of quantum theory. The system-reservoir approach has been studied in great detail by many authors in the context of quantum dissipative systems and the quantum measurement problem (Dekker 1981, 1977, Barchielli et al 1982, 1983, Kumar 1984, Ghirardi et al 1986, Caldeira and Leggett 1983, 1985).

In this Chapter our goal is to further this line of inquiry by examining the process of measurement keeping in mind the two objectives of a measurement process: (i) the establishment of a one-to-one correlation between the states of the system and that of the apparatus, and (ii) the reduction of the density matrix of the apparatus to a diagonal form in a preferred basis. We consider a canonical spin-1/2 quantum system interacting with an apparatus, which in turn is interacting with an environment modeled by a large number of degrees of freedom. The interaction between the
apparatus and the environment causes decoherence in the quantum evolution of
the former. We consider two kinds of apparata. In the first case the
apparatus is a quantum two-level system interacting with an environment which
consists of a large number of quantum two-level systems. This scheme was
first studied in a very illuminating manner by Zurek (1983). Zurek's
treatment allows the system to interact with the apparatus for a fixed amount
of time in which a definite correlation state of the system and the apparatus
is established. Following this, the system and the apparatus are decoupled
and the latter is allowed to interact with the environment. The success of
this scheme depends on the precise time (Zurek 1983) for which the apparatus
interacts with the system. We, however, consider the evolution of the entire
assembly together and then calculate the reduced density matrix of the system
and the apparatus. We find that the reduced density matrix, though diagonal
in the system-apparatus space, does not contain correlations between the
states of the system and the states of the apparatus. So, in general, no
measurement is made except in the special circumstances considered by Zurek
(1983). We then analyse a Stern-Gerlach apparatus, i.e., a spin-1/2 particle
in the presence of an inhomogeneous magnetic field (Gerlach and Stern 1922).
A measurement of spin is made by studying the position or momentum of the
particle. Such a model has been considered in some detail by Bohm (1951). The
effect of the environment on the translational degrees of freedom is taken
into account via the density matrix equation which incorporates both the
quantum evolution and the stochastic Fokker-Planck type evolution arising due
to the environmental interaction. This equation has been obtained in a
variety of ways in recent literature (Dekker 1981, Kumar 1984, Ghirardi et al
1986, Caldeira and Leggett 1983, 1985). We solve this equation exactly by
including an appropriate system-apparatus interaction and show that in this case the reduced density matrix is driven diagonal at long times in the momentum space and has the desired correlation with a component of spin (Venugopalan et al 1993, 1994). We feel that the main point of difference in the two examples considered is that in the first case the apparatus, which is a quantum two-level system, does not have a classical limit while in the second case the trajectory of the particle certainly has a well-understood classical limit.

Decoherence as a result of environmental influence has interesting implications when one looks at recombination interference experiments with spin-1/2 particles (Bohm 1951, Wigner 1963). In such a setup there is one Stern-Gerlach apparatus (SGA) which splits an incident beam of spin-1/2 particles with a particular spin-polarization into two beams of opposite spin eigenstates, i.e., with polarization directions parallel and antiparallel to the magnetic field. A second SGA then recombines the split beam in a reversed magnetic field. Ideally, according to the predictions of quantum mechanics, one would expect that the resulting beam of spin-1/2 particles would have a spin polarization in the same direction as that of the initial beam sent into the first SGA, i.e., if an initially x-polarized beam is sent in, it would split into two z-polarized beams and then recombine to give back the original x-polarized beam. As a result of environmental influence, however, there will be some loss of coherence (Scully et al 1989) and the final recombined beam will be no longer an x-polarized beam but a statistical mixture of up and down z-polarized beams. We look at the effect of environmental influence on an initial beam of spin-1/2 particles, the quantum state of each particle
being a coherent superposition of two opposite spin eigenstates (up and down) with the spatial parts of the up and down spin eigenstates corresponding to Gaussian wave packets centered around two different mean values.

It seems that regardless of the initial state of a quantum system, the transition from 'quantum' to 'classical' via environment induced decoherence is characterized by the emergence of a ‘preferred basis’ of states in which the density matrix of the system becomes diagonal and hence classically interpretable. An interesting example is the existence of a macroscopic (classical) harmonic oscillator in a special class of states called the coherent states (Zurek et al 1993). However, the question as to which basis would emerge as the preferred basis for a given system is still not completely answered. In this Chapter we analyse the solutions for a free particle coupled to a dissipative environment and establish that the momentum basis is the emergent ‘preferred basis’ selected by the environment (Venugopalan 1994), which is contrary to the general expectation that position should emerge as the preferred basis, since the coupling with the environment is through the position coordinate.

As has been already discussed above, quantum mechanics is characterised by distinct nonclassical features which arise from its basic formalism. A unique quantum feature is that of nonlocal correlations which is best illustrated by the E-P-R paradox (Einstein et al 1935) which can be understood by a gedankenexperiment involving spatially separated spin-1/2 particles in an initial singlet state. The experiment demonstrates that if reality and locality are assumed to be natural requirements of any physical
theory, the quantum mechanical description of systems is incomplete (Einstein et al 1935). The experiment also brings forth the counterintuitive concept of the nonlocal nature of quantum mechanics which tempted many authors to look for a 'hidden-variable theory' that agreed with all the predictions of quantum theory and yet conformed to locality and realism. In 1965 John Bell proved that any hidden-variable theory that satisfied a physically reasonable condition of locality and the nonexistence of action-at-a-distance made statistical predictions about restricted correlations in some phenomena (Bell 1965). These restrictions have been derived for various situations and have various forms which are together referred to as Bell inequalities. Bell also showed that under appropriate conditions, quantum mechanics makes statistical predictions which violate these restrictions (inequalities). Advances in technology have made it possible to perform highly sophisticated experiments that demonstrate the violation of Bell inequalities (Aspect et al 1982). These experiments are delicate and carefully controlled to explore the quantum regime that shows nonlocal correlations. How do quantum systems that show nonlocality lead to classical systems that conform to realism and locality? How is this transition reflected in Bell inequalities? It is interesting to consider the effect of an environment on Bell inequalities for an E-P-R setup. In this Chapter we also analyse the effect of an environment consisting of a fluctuating magnetic field on the quantum correlations of an E-P-R pair, i.e., two spin-1/2 particles in an initial singlet state. We show that decoherence due to the environment causes quantum correlations to disappear and the system goes from the usual 'violation' of Bell inequalities to a 'nonviolation' over a characteristic time scale.
Our main approach in this thesis is based on the environment induced decoherence theory discussed above. We would like to mention here that there has been a resurgence of many other new approaches to tackle the measurement problem in recent years. We would like to briefly mention some of these theories. Ghirardi, Rimini and Weber (1986) have proposed a modification of the Schrödinger equation (the continuous spontaneous localization theory) to resolve the measurement problem. This modified linear Schrödinger equation possesses a new term that depends upon a randomly fluctuating force. The state vector is, thus, more fundamental to their treatment and the dynamics lets microscopic systems spread out and interfere while it prevents macroscopic systems from doing so, thus explaining the emergent 'classicality'. The implied merit of this treatment is that it does away with the need to break up a composite quantum system into system, apparatus and environment. In recent years there has also been a considerable interest in the subject of quantum cosmology in which the notion of an external classical domain is rather ambiguous. The 'environment-induced decoherence theory' explains the emergence of classicality as a consequence of the 'openness' of quantum systems. The quantum Universe is perhaps the only system that is effectively closed and does not possess any external environment. How does a classical Universe emerge from an underlying quantum one? Griffiths (1984), Gell-Mann and Hartle (1990), and Omnés (1990) have developed the decoherent (or 'consistent') histories approach suitable for quantum cosmology through several important papers. We will not go into the conceptual structure of these theories and refer the interested reader to the above mentioned papers. The 'Many Hilbert Space' approach to the measurement problem was proposed by Machida and Namiki (1980). The philosophy of their approach is similar to the
environment induced decoherence theory where the 'detector' is a macroscopic object which is not isolated from its environment. The main feature here is that such a system is treated as having a 'many Hilbert space structure'. Another approach to tackle the measurement problem is the AgBr model developed by Hepp (1972) where an interaction is considered between a particle and a one-dimensional N-spin array (like a linear 'photographic emulsion' of AgBr molecules). Thus several important theories with apparently different approaches have emerged to find answers to the queries of quantum measurement. All these theories and, indeed, the 'environment-induced decoherence theory', though not completely free of criticisms (Zurek 1993a), signify an important development in the attempts to resolve the measurement problem. In their aim to explain the emergent classicality of systems within the quantum mechanical framework, these theories may well share a relationship that is closer than what is apparent.

The remaining Chapter is organized as follows. In Section II, we present details of the calculations for the density matrix for the Zurek-type apparatus discussed above. Section III contains the calculations and analysis of the density matrix for the Stern-Gerlach apparatus. In Section IV we look at the Stern-Gerlach recombination experiment and the effect of decoherence by environmental influence. In Section V we examine the role of the preferred basis in a measurement process by analysing the effect of decoherence for a free particle interacting with an environment via a dissipative coupling. In Section VI we analyse the effect of an environment on the correlations manifested by the violation of Bell inequalities in an E-P-R setup. Finally, in Section VII we summarize the results of this Chapter.
2.2 Density matrix for the Zurek-type apparatus.

Consider Zurek's treatment of the quantum measurement problem (Zurek 1983). A pair of two-level quantum systems is taken: the system (spin) and the apparatus (atom) with \(|\uparrow\rangle\), \(|\downarrow\rangle\), and \(|\pm\rangle\), \(|\mp\rangle\) denoting their basis states, respectively. There is also an environment which is a large collection of \(n\) quantum two-level systems with eigenstates \{|\pm\}, \{|\mp\}\} for each pair. Proceeding along the lines of von Neumann (1932), the first stage of Zurek's two-stage process involves establishing a correlation between the system and the apparatus. The Hamiltonian for the system - apparatus interaction is

\[
H_{SA} = i\mathcal{A} (|\pm\rangle\langle\mp| - |\mp\rangle\langle\pm|) \otimes (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|),
\]  

(2.2.1)

where \(\mathcal{A}\) is the strength of the interaction. The Hamiltonian evolution transforms an initial direct-product state

\[
|\psi(0)\rangle = (a|\uparrow\rangle + b|\downarrow\rangle) (c|\pm\rangle + d|\mp\rangle)
\]  

(2.2.2)

to

\[
|\psi(t)\rangle = a|\uparrow\rangle (c|\pm\rangle e^{-i\mathcal{A}t} + d|\mp\rangle) + b|\downarrow\rangle (c|\mp\rangle e^{-i\mathcal{A}t} + d|\pm\rangle e^{i\mathcal{A}t}).
\]  

(2.2.3)

Zurek considered the special case in which \(c = d = 1/\sqrt{2}\). At \(t = t_f = \pi/4\mathcal{A}\),

\[
|\psi(t_f)\rangle = (a|\uparrow\rangle |\pm\rangle + b|\downarrow\rangle |\mp\rangle)/\sqrt{2},
\]  

(2.2.4)
in which the correlation between the system and the apparatus states has been established. In general, the combined density matrix is more complicated, containing overlaps between the apparatus states. The first stage of Zurek's treatment terminates at the special state (2.2.4), after which the system-apparatus interaction is switched off. Now the apparatus-environment interaction is switched on via the interaction Hamiltonian:

\[ H^{AE} = \sum_{j=1}^{n} i g_k (|\pm\rangle (|r\rangle - |r\rangle)| (|\pm\rangle) \otimes (|\pm\rangle |\phi\rangle |\phi\rangle) \prod_{j \neq k} |I\rangle, \] (2.2.5)

where \( I_j \) denotes a unit operator. The initial state is taken as the correlated state (2.2.4) of stage 1 and a direct product with a general 'environment state'. At any arbitrary time \( t \), the density matrix describing the system-apparatus combination after taking partial trace over the states of the environment is:

\[ \rho = |a|^2 |\uparrow\rangle\langle\downarrow| |\pm\rangle |\phi\rangle |\phi\rangle + |b|^2 |\downarrow\rangle\langle\uparrow| |\pm\rangle |\phi\rangle |\phi\rangle + Z(t) a b^* |\uparrow\rangle\langle\downarrow| |\pm\rangle |\phi\rangle |\phi\rangle + \bar{Z}(t) a^* b |\downarrow\rangle\langle\uparrow| |\pm\rangle |\phi\rangle |\phi\rangle, \] (2.2.6)

where

\[ Z(t) = \prod_{k=1}^{n} \left( \cos(2g_k t) + i \left( |\alpha_k|^2 - |\beta_k|^2 \right) \sin(2g_k t) \right), \] (2.2.7)

\( \alpha_k \) and \( \beta_k \) are related to the initial state of the environment. It is shown numerically that for a smooth distribution of \( g_k \)'s, \( Z(t) \) being a product of a large number of oscillatory factors of magnitude less than unity, gives rise
to apparent decays for large times \( t \), (but \( t < \ll T \), where \( T \) is the characteristic 'recurrence' time for the periodic function, as explained in the Introduction). The correlations between the apparatus and the measured system are left intact in the preferred basis of the apparatus (Zurek 1983).

In the following treatment we choose the same initial state as in Zurek's treatment but look at the combined evolution of the system, apparatus and environment. This treatment is more realistic as it does not require termination of any interaction at a specific time. The complete interaction Hamiltonian for looking at the combined evolution of the system, apparatus and environment is

\[
H_{\text{SAE}} = g \sigma_z L_y + \sum_{k=1}^{n} g_k L_z J_{ky} \bigotimes_{j \neq k} I_j. 
\]  

(2.2.8)

For convenience we use the Pauli spin operators \( \sigma, L, J \) for the system, apparatus and environment, respectively. The eigenvalues and the eigenfunctions of \( H_{\text{SAE}} \) can be easily found by working in a representation in which \( \sigma_z \) and \( \{J_{ky}\} \) are diagonal. Denoting the eigenvalues of \( \sigma_z \) by \( s ( = \pm 1 ) \) and the eigenvalues of \( J_{ky} \) by \( \mu_k ( = \pm 1 ) \), the Hamiltonian in the subspace \((s, \{\mu_k\})\) is

\[
H' = g s L_y + f L_z.
\]  

(2.2.9)

where

\[
f = \sum_k g_k \mu_k.
\]  

(2.2.10)
The eigenvalues $E(s, \mu_k)$ are
\begin{equation}
E(s, \mu_k) = \pm \sqrt{g^2 + f^2} = \pm E_{\mu},
\end{equation}
(2.2.11)
and the eigenvectors in an obvious notation are
\begin{equation}
|s, \pm \sqrt{g^2 + f^2}, \mu_k\rangle = |s\rangle_z |\psi_{\pm \mu}\rangle \prod_{k} |\mu_k\rangle_y, \tag{2.2.12}
\end{equation}
where
\begin{equation}
|\psi_{+ \mu}\rangle = \cos (\theta_{\mu}/2) |\pm\rangle + i \sin (\theta_{\mu}/2) |\mp\rangle, \tag{2.2.13}
\end{equation}
\begin{equation}
|\psi_{- \mu}\rangle = i \sin (\theta_{\mu}/2) |\pm\rangle + \cos (\theta_{\mu}/2) |\mp\rangle, \tag{2.2.14}
\end{equation}
\begin{equation}
\theta_{\mu} = \pm \tan^{-1}(g/f) = \pm \theta_{\mu}. \tag{2.2.15}
\end{equation}

We choose the initial state of the system to be
\begin{equation}
|\psi(0)\rangle = |a|^\pm \rangle + b|\downarrow\rangle \otimes ((|\pm\rangle + |\mp\rangle)/\sqrt{2}) \otimes \prod_{k} (\alpha_k |^{+E}\rangle_k + \beta_k |^{-E}\rangle_k), \tag{2.2.16}
\end{equation}
where $|^{+E}\rangle_k, |^{-E}\rangle_k = (|\pm\rangle \pm i |\mp\rangle)/\sqrt{2}$ are the two eigenstates $|\mu\rangle_y$ of $J_{kY}$.

The expression for the reduced density matrix $\rho_r$ of the system and the apparatus is obtained by tracing over the environment states:
\( \rho_R = \sum_{\mu} \frac{|C(\mu)|^2}{2} \left( |a|^2 |\uparrow\rangle \langle \uparrow| + |b|^2 |\downarrow\rangle \langle \downarrow| \right) (|\pm\rangle \langle \pm| + |\mp\rangle \langle \mp|) + \sin \theta_\mu \sin(2E_\mu t) \left( |a|^2 |\uparrow\rangle \langle \uparrow| - |b|^2 |\downarrow\rangle \langle \downarrow| \right) \times (|\mp\rangle \langle \mp| - |\pm\rangle \langle \pm|) + (|a|^2 |\uparrow\rangle \langle \uparrow| + |b|^2 |\downarrow\rangle \langle \downarrow|) \times \left( \cos(2E_\mu t) - i \sin(2E_\mu t) \cos \theta_\mu \right) |\pm\rangle \langle \pm| + \text{C.C.} \right) + \left\{ a \ b^* \ |\uparrow\rangle \langle \uparrow| e^{-i \theta_\mu} \left( \cos \theta_\mu + i \sin \mu \cos(2E_\mu t) \right) \times (|\pm\rangle \langle \pm| + |\mp\rangle \langle \mp|) + \left( i \sin \theta_\mu + \cos \theta_\mu \cos(2E_\mu t) \right) \times (|\pm\rangle \langle \pm| + |\mp\rangle \langle \mp|) + \sin(2E_\mu t) \left( |\mp\rangle \langle \mp| - |\pm\rangle \langle \pm| \right) + \text{C.C.} \right\} . 
(2.2.17)

The summation over \( \mu \) corresponds to the summation over all possible permutations of the various environment states, and

\[
C(\mu) = \prod_{k=1}^{n} \gamma_k(\mu_k),
\]
(2.2.18)

with

\[
\gamma_k(\pm 1) = \alpha_k, \quad \gamma_k(-1) = i \beta_k.
\]
(2.2.19)

\(|C(\mu)|^2\) is a smoothly varying function. Except for the first term on the RHS in (2.2.17), the coefficients of all the other terms are oscillatory in time and depend on \( \theta_\mu \) which, as defined in (2.2.15), is a function of \( g_k \)'s. Again
it can be assumed that due to their oscillatory nature, in the limit $n \to \infty$, and in the large time limit all these coefficients are driven to zero. The reduced density matrix thus becomes:

\[ \rho_R = \sum_{\mu} \frac{|C(\mu)|^2}{2} \left( |a|^2 |\uparrow\rangle \langle \uparrow| + |b|^2 |\downarrow\rangle \langle \downarrow| \right) (|\pm\rangle \langle \pm| + |\mp\rangle \langle \mp|). \]  

(2.2.20) clearly indicates that although the environmental influence does drive the density matrix of the apparatus and the system to become diagonal, there is no one-to-one correlation between the states of the apparatus and that of the system. From the above treatment it is clear that the reason the quantum apparatus does not correlate with the system states is that the overlap in states aligned with that of $\sigma_z$ do not vanish unless the special conditions $t = t_f$ and $c = d$ are taken.

The models considered above belong to the class of system-apparatus-environment models where $[H_{SA}, H_{AE}, \sigma_z] = 0$, i.e., the total Hamiltonian $H_{SAE}$ commutes with the measured observable. In the specific model considered by Zurek and in this Chapter, $[H_{SA}, H_{AE}] \neq 0$, i.e., the two parts of the Hamiltonian do not commute. It can be easily shown that even if we choose commuting Hamiltonians ($[H_{SA}, H_{AE}] = 0$), we are led to a situation like (2.2.20), i.e., the apparatus does not perform a 'measurement' of spin (no one-to-one correlation).

The measurement of spin is clearly not accomplished by using a two-state quantum apparatus coupled to a bath of many two-state systems (the
environment). The apparatus here has no classical limit. In the next Section we show that a 'spin measurement' is possible when the interaction with the environment is via an apparatus which has a well understood classical limit.

2.III. Density matrix for the Stern-Gerlach apparatus

In the following treatment we consider a Stern-Gerlach type set-up for investigating the measurement of spin. Since the problem of motion of a spinless particle in simple potentials and in interaction with the environment has been studied at great length in recent literature (Dekker 1981, 1977, Barchielli et al 1982, 1983, Kumar 1984, Ghirardi et al 1986, Caldeira and Leggett 1983, 1985), we draw upon this work to deal directly with a reduced density matrix equation for the particle in which environmental degrees of freedom have been traced over. Though the density matrix equation has been derived in a number of ways, the derivation which is in the spirit of the work described in this thesis was given by Caldeira and Leggett using Feynman-Vernon path integral approach (Feynman and Vernon 1963). In their study, a free particle coupled to a collection of harmonic oscillators, which constitutes the environment. The interaction between the free particle and the environment is linear via a coordinate-coordinate coupling (Caldeira and Leggett 1983, 1985). The total Hamiltonian for the composite system can be written as

\[ H = \frac{p^2}{2m} + \sum_j \left( \frac{p_j^2}{2m_j} + \frac{m_j \omega_j^2}{2} \left( x_j - \frac{c_j Q}{m \omega_j^2} \right)^2 \right) \]  

(2.3.1)
Here, $P$, $Q$ are the momentum and position coordinates of the free particle, and $p_j$ and $x_j$, those of the $j$th harmonic oscillator. $c_j$s are coupling strengths and $\omega_j$s are the frequencies of the oscillators. The Hamiltonian of (2.3.1) is known as the 'independent oscillator' model. More frequently seen in the literature is the 'linear coupling' model where the Hamiltonian is without the 'counter term' (last term in the summation) and the coupling is represented by adding a term of the form $Q \sum_j c_j x_j$. As pointed out by Ford et al (1988), the 'linear coupling' Hamiltonian is unphysical and corresponds to a 'passive' heat bath and is not invariant under translations. All information about the harmonic oscillator heat bath which is required for the description of the particle via a reduced density matrix is contained in the spectral density function (Ford et al 1988) and the initial temperature of the bath. Using the Feynman-Vernon influence functional technique (Feynman and Vernon 1963), Caldeira and Leggett have shown that for a white (ohmic) noise spectrum in the high temperature limit, one can write an equation of motion for the reduced density matrix $\rho_R$ of the free particle. In the position representation this equation can be written (Kumar 1984, Zurek 1993) as

$$\frac{\partial \rho_R(x,y,t)}{\partial t} = \left\{ -\frac{\hbar}{2im} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - \gamma(x-y) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \right\} \rho_R(x,y,t) - \frac{D}{4\hbar^2} (x-y)^2 \rho_R(x,y,t),$$

(2.3.2)

where $m$ is the mass of the particle, $\hbar$ is Planck's constant, $\gamma$ is the
Langevin friction coefficient and $D$ has the usual interpretation of the diffusion constant. $\gamma$ and $D$ are related to the parameters of the Hamiltonian (2.3.1). For a high temperature thermal bath, $D = 2m \gamma k_B T$. (2.3.2) has been used to study the dynamics of systems like the free particle and the harmonic oscillator in interaction with a heat bath. The high temperature limit is needed here as the earlier work (Kumar 1984) shows that the solution then evolves to a classical stochastic distribution. Note that the Liouvillian of the master equation (2.3.2) is not of the Lindblad-Gorini-Kossakowski-Sudarshan form (Lindblad 1976a, Gorini et al 1976) and hence its solution is not guaranteed to be positive. A term with a double commutator with $p$ is then thought to be essential (Lindblad 1976b). However, our master equation is the markovian limit of a generalized master equation and is valid in the large time domain where time $t$ is much larger than than the relaxation time of the environment. One can check our solutions explicitly (see equations (2.3.13), (2.3.18) and (2.3.21) later) for positive definiteness and they also satisfy $\text{Tr} (\rho) = 1$ for all $t$.

We consider a spin-1/2 particle of mass $m$ in an inhomogeneous magnetic field (Gerlach and Stern 1922). Here spin plays the role of the system and the positional degree of freedom of the particle that of the apparatus. The particle is further coupled to an environment via its position, and this coupling is intended to drive its translational behaviour classical to perform a measurement. The Hamiltonian of the combined system is

$$H_{\text{SEAE}} = p^2/(2m) + \lambda \sigma_z + \varepsilon \times \sigma_z + H_{\text{AE}} + H_{\text{E}}.$$  

(2.3.3)
Here \(x\) and \(p\) denote the position and momentum (taken in one dimension for convenience) of the particle, \(\lambda \sigma_z\) the Hamiltonian of the system, \(\varepsilon\) the product of the field gradient and the magnetic moment of the particle, \(H^{AE}\) the interaction of the environmental degrees of freedom with \(x\), and \(H^E\) denotes the Hamiltonian for the environmental degrees of freedom. We look at the time evolution of the density matrix in the \(|s,x\rangle\) representation, where \(|s\rangle\) refers to the eigenstates of \(\sigma_z\) and \(|x\rangle\) are the position states. Corresponding to the four elements of the spin space (\(\uparrow\uparrow\), \(\downarrow\downarrow\), \(\uparrow\downarrow\), \(\downarrow\uparrow\)), the equations for the elements of the reduced density matrix \(\rho_{ss'}(x,y,t)\) for our Hamiltonian (2.3.3) are:

\[
\frac{\partial \rho_{ss'}(x,y,t)}{\partial t} = \left\{ \frac{-\hbar}{2\lambda m} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - \gamma(x-y) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \right.
\]
\[
- \frac{D}{4\hbar^2} (x-y)^2 + \frac{i\varepsilon(x-s')}{\hbar} \left( x - y \right)^2 + \left. \frac{i\lambda(s-s')}{\hbar} \right\} \rho_{ss'}(x,y,t), \tag{2.3.4}
\]

where \(s, s' = +1\) (for \(\uparrow\)) or \(-1\) (for \(\downarrow\)). It is more convenient to work with variables: \(r = (x-y)\), \(R = (x+y)/2\). Then the spin-diagonal density matrix elements \(\rho_d\), and the spin-off-diagonal density matrix elements \(\rho_{od}\) obey the equations:

\[
\frac{\partial \rho_d(R,r,t)}{\partial t} = \frac{-\hbar}{\lambda m} \frac{\partial^2 \rho_d}{\partial R^2} - \gamma R \frac{\partial \rho_d}{\partial r} - \frac{D}{4\hbar^2} \rho_d \pm \frac{i\varepsilon r}{\hbar} \rho_d, \tag{2.3.5}
\]

and

\[
\frac{\partial \rho_{od}(R,r,t)}{\partial t} = \frac{-\hbar}{\lambda m} \frac{\partial^2 \rho_{od}}{\partial R^2} - \gamma R \frac{\partial \rho_{od}}{\partial r} - \frac{D}{4\hbar^2} \rho_{od} \pm \frac{2i\varepsilon R}{\hbar} \rho_{od} \pm \frac{2i\lambda}{\hbar} \rho_{od}. \tag{2.3.6}
\]
To solve these equations, it is convenient to take a partial Fourier transform in the variable $R$:

$$
\rho(Q,r,t) = \int_{-\infty}^{\infty} \exp(iQR) \rho(R,r,t) \, dR.
$$

(2.3.7)

The equations (2.3.5) and (2.3.6) simplify to a pair of first-order partial differential equations:

$$
\frac{\partial \rho_d(Q,r,t)}{\partial t} = \left( \frac{h}{m} - \gamma \right) \frac{\partial \rho_d}{\partial r} - \frac{D}{4 \hbar^2} \rho_d \pm \frac{i \epsilon r}{\hbar} \rho_d,
$$

(2.3.8)

$$
\frac{\partial \rho_{od}(Q,r,t)}{\partial t} = \left( \frac{h}{m} - \gamma \right) \frac{\partial \rho_{od}}{\partial r} - \frac{D}{4 \hbar^2} \rho_{od} \pm \frac{2 \epsilon}{\hbar} \frac{\partial \rho_{od}}{\partial Q} \pm \frac{2i \lambda}{\hbar} \rho_{od}.
$$

(2.3.9)

Such equations, being of first order, can be solved by the method of characteristics (Courant and Hilbert 1962). The physical significance of the solution can be clearly understood if we choose the following Gaussian wave packet of width $\sigma$ and mean momentum $\bar{p}$ for the initial condition:

$$
\psi(x,0) = \frac{1}{(\sigma \sqrt{\pi})^{1/2}} \exp \left\{ i\bar{p}x - x^2/2\sigma^2 \right\}.
$$

(2.3.10)

Then the solution of (2.3.8) is (see appendix for details)

$$
\rho_d(Q,r,t) = \exp \left\{ i\bar{p} r_0 - r_0^2/4\sigma^2 - Q^2\sigma^2/4 + i\bar{p}(r-r_0)e^{-\tau} - 1/4\sigma^2((r-r_0)^2e^{-2\tau}
$$

$$
+ 2r_0(r-r_0)e^{-\tau} + D/(4\hbar^2) \left\{ r_0^2\tau + 2r_0 (r-r_0)(1-e^{-\tau})
$$

$$
+ (r-r_0)^2(1-e^{-2\tau})/2 \right\} + i\epsilon/\hbar \gamma \left\{ r_0\tau + (r-r_0)(1-e^{-\tau}) \right\} \right\}.
$$

(2.3.11)
where \( \tau = \gamma t \) and \( r_0 = \hbar Q/m \gamma \). To understand the measurement aspect implied by this solution, we consider the solution in the momentum representation, i.e.,

\[
\rho_d(u, \gamma, t) = \int \rho_d(x, y, t) e^{i(\overline{u}x + \overline{v}y)} \, dx \, dy.
\]

This is obtained by taking a Fourier transform with respect to the variable \( r \) in (2.3.11) and identifying \( Q = \overline{u} - \overline{v} \) and \( q = (\overline{u} + \overline{v})/2 \). This solution is

\[
\rho_d(Q, q, t) = \frac{1}{\mathcal{N}(\tau)} \exp \left\{ \frac{-1}{\mathcal{N}(\tau)} \left[ q + \frac{\overline{p}}{\hbar} \, e^{-\tau} + \frac{\varepsilon}{\hbar \gamma} (1-e^{-\tau}) \right. \right.
\]
\[
+ \frac{i \hbar Q}{2 \sigma^2 m \gamma} e^{-\tau} (1-e^{-\tau}) - \frac{i Q D}{4 \hbar \gamma^2 m} (1-e^{-\tau})^2 \right. \left. \right\}^2
\]
\[
- \left[ \frac{\hbar^2}{4 \sigma^2 m^2 \gamma^2} (1-e^{-\tau})^2 + \frac{\sigma^2}{4} \right. \right.
\]
\[
+ \frac{D}{2m^2 \gamma^3} (2\tau - 3 + 4e^{-\tau} - e^{-2\tau}) \right. \left. \right\} Q^2 + \left[ \frac{i \overline{p} \hbar}{m \gamma} (1-e^{-\tau}) \right.
\]
\[
+ \frac{i \varepsilon \tau}{m \gamma^2} \pm \frac{i \varepsilon}{m \gamma^2} (1-e^{-\tau}) \right. \left. \right\} Q \bigg],
\]

where

\[
\mathcal{N}(\tau) = (D/2\hbar^2 \gamma) (1-e^{-2\tau}) + (1/\sigma^2) e^{-2\tau}.
\]

Now we see that the momentum off-diagonal components \( Q \neq 0 \) vanish with time, reducing \( \rho_d \) to the diagonal form (Caldeira and Leggett 1983, Zurek...

\[ \tau_d^{-1} = \frac{D Q^2}{m^2 \gamma^3} , \]

or,

\[ t_d \equiv \frac{\tau_d}{\gamma} = \frac{m \gamma}{2k_B T Q^2} , \]

(2.3.15)

where \( Q \) is the extent of momentum-space off-diagonality. This is a classical time scale of momentum relaxation. If one looks at the spatial transform of (2.3.13), one can see that the spatial off-diagonal parts rapidly decay on a time scale

\[ t_r = \frac{\hbar^2}{2m \gamma k_B T r^2} , \]

(2.3.16)

\( r \) being the spatial separation, even though the density matrix in the position representation does not eventually become diagonal. The momentum distribution function can be obtained by looking at the diagonal elements of the density matrix (2.3.13), i.e., for \( Q = 0 \) and \( q = \bar{u} \):

\[ \rho_{\bar{u}}(0,\bar{u},t) \equiv |\psi(\bar{u})|^2 = 2 \left[ \frac{\pi}{N(\tau)} \right] \exp \left\{ \frac{-1}{N(\tau)} \left( \bar{u} + \bar{p} e^{-\tau} \mp \frac{\varepsilon}{\hbar \gamma} (1-e^{-\tau}) \right)^2 \right\} . \]

(2.3.17)

This has the classical Ornstein–Uhlenbeck form with the spin-dependent drift caused by the field. In the large \( t \) limit, the momentum distribution is centered around \( \varepsilon/\hbar \gamma \) (\( -\varepsilon/\hbar \gamma \)) for up (down) spin. Thus we see that the measurement of momentum of the particle can determine the spin. It is also interesting to consider this solution in the space coordinates. The diagonal
distribution in momentum necessarily implies that the density matrix does not reduce to a diagonal form with respect to space coordinates. The Fourier transform in $Q$ and $q$ of $\rho_d(Q,q,t)$ gives the density matrix in position representation:

$$\rho_d(R,r,t) = \frac{1}{M(\tau)} \exp \left\{ -\left( \frac{1}{4\sigma^2} e^{-2\tau} + \frac{D}{8\hbar^2} (1-e^{-2\tau}) \right) r^2 + \left( i\hbar e^{-\tau} \frac{\partial}{\partial Q} \left( 1-e^{-\tau} \right) \right) R - \frac{\hbar}{m\gamma} \left( R - \frac{\vec{p} \cdot \hbar}{m\gamma} \right) + \frac{\hbar}{m\gamma^2} \left( 1-e^{-\tau} \right) \right\} \right\}.$$

where $\tau = \gamma t$ and

$$M(\tau) = \sigma^2 + \frac{\hbar^2}{\sigma^2 m^2 \gamma^2} (1-e^{-2\tau})^2 + \frac{D}{2m^2 \gamma^3} \left( 2\tau - 3e^{-\tau} - 4e^{-\tau} e^{-2\tau} \right).$$

As $t \to \infty$, one can see that the off-diagonal elements of the density matrix do not vanish, and the diagonal elements give the position distribution function obtained by setting $r = 0$ and $R = x$:

$$|\psi(x)|^2 = \frac{\pi}{M(\tau)} \exp \left\{ -\frac{1}{M(\tau)} \left( x - \frac{\vec{p} \cdot \hbar}{m\gamma} \right) + \frac{\hbar \gamma}{m\gamma^2} \left( \frac{\hbar \tau}{m\gamma^2} \right)^2 \right\}.$$

The results corresponding to $\pm \epsilon$ are for the up and down spins. The centres of the position distribution function shift with time and are clearly...
different for up and down spins. Though the distribution $\rho_d(R,r,t)$ implies a nonlocality through its dependence on $r$, we observe that the width $w_1$ of the distribution in $r$ is considerably smaller than the width $w_2$ of the distribution in $R$. In the large $\tau$ limit, $w_2/w_1 = h^2m^2\gamma^4/D^2$. If $\rho_d(R,r,t)$ is coarse-grained over length scales $l$, such that $l$ is larger than the de Broglie wavelength of the particle $\gamma/\epsilon$ and $w_1$, but smaller than $w_2$, we should have a local distribution in position space. This means $l > \text{Max}(\gamma/\epsilon, \gamma/D)$, and $l < Dh^2/\gamma m^2\gamma^2$, which is surely possible for large enough $\tau$.

Let us now look at the solution for the spin off-diagonal elements of the density matrix. The solution for (2.3.9) is described in the appendix. For the initial conditions of (2.3.10) the solution is:

$$
\rho_{o_d}(Q,r,t) = \exp \left( \frac{-\gamma^2 D\epsilon^2}{3m^2\gamma^5 h^2} \right) \exp \left( \pm \frac{2i\gamma t}{h} \right)
$$

$$
\times \exp \left\{ -\frac{1}{4} \left( \frac{D}{h^2\gamma} \left( 1-e^{-2\tau} \right) + \frac{1}{\sigma^2} e^{-2\tau} \right) r^2 \right\}
$$

$$
+ \left( i\sigma e^{-\tau} \mp \frac{\epsilon e^{-2\tau}}{\gamma^2 m} \mp \frac{\epsilon D}{2h^2\gamma^3} \left( \tau(1-e^{-2\tau}) - 2(1-e^{-\tau}) \right) \right)
$$

$$
\pm \frac{D\tau}{h^2\gamma} r - \frac{1}{4} \left( \frac{1}{\sigma^2} (1-e^{-\tau})^2 \mp \frac{D}{2h^2\gamma} (4\tau - 3 + 4e^{-\tau} - 2e^{-2\tau}) \right) \bar{r}_Q^2
$$

$$
+ \left( i\sigma (1-e^{-\tau}) - \frac{1}{4\sigma^2} \left( 2r \mp \frac{4\epsilon\tau}{\gamma^2 m} \right)(1-e^{-\tau}) \right).
$$

40
\[
\frac{D_r}{4\hbar^2\gamma} (1-e^{-\gamma})^2 \pm \frac{D_\varepsilon}{2\hbar^2\gamma^3 m} (1-e^{-\gamma}(1+e^{-\gamma})-2) \pm \frac{D_{\varepsilon}\gamma^2}{2\hbar^2 m\gamma^2}
\]

\[\pm \frac{D_{\varepsilon}\gamma}{\hbar^2 m\gamma^2} r_Q \exp \left(- \left(\frac{Q^2\sigma^2}{4} + \frac{\varepsilon^2\gamma^2\sigma^2}{\hbar^2\gamma^2}\right)\right), \tag{2.3.21}\]

where

\[r_Q = Q\hbar/m\gamma \pm 2\varepsilon\gamma/m\gamma^2 \mp 2\varepsilon/m\gamma^2. \tag{2.3.22}\]

The solution has a factor going as $e^{-A\tau^3}$ which drives the entire expression to zero with time, independent of all other arguments in the density matrix. This means that the density matrix is driven diagonal in the spin-space. The time scale over which this happens is given by

\[\tau_s = \left(\frac{3m^2\gamma^5\hbar^2}{e^2D}\right)^{1/3},\]

or, \[t_s = \frac{\tau_s}{\gamma} = \left(\frac{3m\gamma^2}{2e^2k_B T}\right)^{1/3}. \tag{2.3.23}\]

More explicitly, if the initial wavefunction of the system-apparatus is a product of the Gaussian wave-packet of (2.3.10) and the apparatus state $(a \mid \uparrow\rangle + b \mid \downarrow\rangle)$, i.e.,

\[\psi(x,0) = \frac{1}{(\sigma/\pi)^{1/2}} \left[ a \mid \uparrow\rangle + b \mid \downarrow\rangle \right] \exp \left(-\frac{\gamma p^2}{\sigma^2}\right), \tag{2.3.24}\]

in the model without the environment, the time evolution of the density matrix is
\[
\rho = |a|^2 |\uparrow \rangle \langle \uparrow | \psi_\uparrow(x, t) \psi_\downarrow(y, t) + |b|^2 |\downarrow \rangle \langle \downarrow | \psi_\downarrow(x, t) \psi_\uparrow(y, t) \\
+ a^* b |\uparrow \rangle \langle \downarrow | \psi_\uparrow(x, t) \psi_\downarrow(y, t) + a b^* |\downarrow \rangle \langle \uparrow | \psi_\downarrow(x, t) \psi_\uparrow(y, t),
\]

(2.3.25)

where \(\psi_\pm(x, t)\) are the wavefunctions of the particle in the potential \(\pm V(x)\). The environment causes the decay of the off-diagonal elements and the large time limit of the density matrix assumes the form

\[
\rho = |a|^2 |\uparrow \rangle \langle \uparrow | \rho_{\uparrow\uparrow} + |b|^2 |\downarrow \rangle \langle \downarrow | \rho_{\downarrow\downarrow},
\]

(2.3.26)

with \(\rho_{\uparrow\uparrow}\) and \(\rho_{\downarrow\downarrow}\) being given by (2.3.13) in the momentum representation, and by (2.3.18) in the coordinate representation. This calculation clearly establishes the measurement of spin via a momentum measurement. The spin diagonal density matrix evolves to a diagonal form in the momentum space, while the spin off-diagonal matrix goes to zero with time. Further, the probability distributions of up and down spins are also given by the initial amplitudes \(a\) and \(b\) according to the quantum prescription. Figs. 1(a), (b) and (c) show the real part of the sum of the density matrices, \(\rho = \text{Real}(\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow})\), given by (2.3.13) in the momentum representation for the diagonal spin elements for three different values of the scaled time \(\tau = \gamma t\). As \(\tau\) increases, the off-diagonal elements clearly decay, leaving a diagonal distribution which is centred around two different mean momenta, corresponding to up and down spins. One can identify these mean momenta as those with which the centres of the wave packets corresponding to up and down spins move.
Fig. 1(a): Plot of the sum of the spin-diagonal density matrices in the momentum representation, $\rho = \text{Real}(\rho_{++} + \rho_{+-})/\sigma$, given by Eq. (2.3.13), versus dimensionless momenta $u = (q + Q/2)\sigma$ and $v = (q - Q/2)\sigma$, for $\tau = \gamma t = 0$, $\epsilon/m\gamma^2 = 0.0$, with $\overline{\rho} = 0.2/\sigma$, $D/m^2\gamma^3 = 1.0\sigma^2$, $m\gamma/\hbar = 0.5/\sigma^2$. 
Fig. 1(b): Plot of the sum of the spin-diagonal density matrices in the momentum representation, \( \rho = \text{Real}(\rho_{11} + \rho_{1-1})/\sigma \), given by Eq.(2.3.13), versus dimensionless momenta \( u = (q + Q/2)/\sigma \) and \( v = (q - Q/2)/\sigma \), for \( \tau = 1 \), \( \epsilon/m\gamma^2 = 2.0 \), with \( \overline{p} = 0.2/\sigma \), \( D/m^2\gamma^3 = 1.0\sigma^2 \), \( m\gamma/\hbar = 0.5/\sigma^2 \).
Fig. 1(c): Plot of the sum of the spin-diagonal density matrices in the momentum representation, \( \rho = \text{Real}(\rho_{++} + \rho_{+-})/\sigma \), given by Eq. (2.3.13), versus dimensionless momenta \( u = (q + Q/2)\sigma \) and \( v = (q - Q/2)\sigma \), for \( \tau = 3 \), with \( \epsilon/m^2 = 2.0 \), \( \bar{p} = 0.2/\sigma \), \( D/m^2\gamma^3 = 1.0\sigma^2 \), \( m\gamma/\hbar = 0.5/\sigma^2 \).
The significant aspect of this model of measurement is that it occurs via environmental decoherence which does not happen instantaneously but rather, over a finite time. So, it is important to analyse the time scales given by (2.3.16) and (2.3.23). Both the time scales involve two macroscopic parameters, namely, the temperature $T$ of the environment and the dynamical friction coefficient $\gamma$. For a particle larger and more massive than the particles constituting the environment, $\gamma = 6\pi\eta a/m$, where $a$ is the radius of the particle and $\eta$ is the coefficient of viscosity of the medium. So, in this situation the time scales become independent of the mass of the particle. To make an estimate of the time scales for atomic scale particles, we make the following plausible assumptions: $\gamma \approx 10^{12}s^{-1}$, $m \approx 10^{-24}gms$, $T = 300K$, $Q \approx 10^5 cm^{-1}$, $\epsilon \approx 1eV/cm$ and $r \approx 100\AA$. This gives $t_r \approx 10^{-17}s$, $t_s \approx 10^{-10}s$ and $t_d \approx 10^{-9}s$.

2.IV. Spin–recombination and decoherence.

In the previous Section we have shown the effect of environment–induced decoherence on a Stern–Gerlach model of an apparatus for the measurement of spin. Decoherence, as we have seen, is a consequence of the macroscopic nature of the apparatus, viz., the particle trajectory, the particle being coupled to an environment through its positional degree of freedom. Destruction of coherence results in a diagonal density matrix where the spin states correlate with the momentum of the particle leading to a measurement of spin. The irreversible nature of the decoherence phenomena can be explored
by applying it to a Stern-Gerlach interferometer (SGI).

A Stern-Gerlach interferometer combines Stern-Gerlach deflecting magnets to reconstitute initially split beams in such a way that an arbitrary initial spin state is recovered. For example, if the initial beam has x-polarized spins, the first Stern-Gerlach apparatus (SGA) splits the beam into two beams with 'up' and 'down' z-polarizations and a second SGA subsequently recombines these to give back the original x-polarized beam. Such a device has been discussed in some detail originally by Bohm (1951) and Wigner (1963). There have also been many recent theoretical (Scully et al 1989 and references therein) and experimental (Badurek et al 1986 and references therein) studies on the SGI. In the presence of environmental influence one can question, for the above situation, whether the reconstituted beam will be in a coherent x-polarized state or will its coherence be destroyed?

In this Section we look at the effect of an environment of the type discussed in Section III on a Stern-Gerlach recombination setup. Our aim is to start with an initial state which is a coherent superposition of 'up' and 'down' z-polarized spin states with corresponding spatial parts which are Gaussian wave packets centered around two different mean values, ±x₀:

\[
|\psi\rangle = \frac{1}{(\sigma\sqrt{\pi})^{1/2}} \left( \exp \left\{ -\frac{(x - x_0)^2}{2\sigma^2} \right\} |\uparrow\rangle + \exp \left\{ -\frac{(x + x_0)^2}{2\sigma^2} \right\} |\downarrow\rangle \right), \quad (2.4.1)
\]

where σ is the width of the wave packets. Such a state can be thought to be created when a particle represented by a Gaussian wave packet and x-polarized
in spin enters the first SGA in the SGI (Bohm 1951). Before we consider environmental influence in the SGI model, let us look at the pure quantum evolution of the state (2.4.1) via the following Hamiltonian (cf. (2.3.3)):

$$H_{SA} = \frac{p^2}{2m} + \lambda \sigma_z + \epsilon x \sigma_z,$$  (2.4.2)

where the field is a 'reversed' one with respect to the one that created the state (2.4.1). We would like to point out here that a 'spin-recombination' may not merely be a consequence of 'time reversal' as one would intuitively expect for a pure quantum evolution. It must be noticed here that the spin-1/2 particle is described by a Gaussian wave packet with its spin degree of freedom coupled to an inhomogeneous magnetic field through its position coordinate. The time evolution via Hamiltonian (2.4.2) is thus, complicated, say, compared to the simple case when we are working only in the spin space and the Hamiltonian is simply proportional to $S.B$, where $S$ is the spin vector and $B$ is the external magnetic field. Earlier references to the SGI by Bohm (1951) and Wigner (1963) have acknowledged some technical difficulties with the experimental realization of the SGI. More recently (Schwinger et al 1988, Englert et al 1988, Scully et al 1989), studies show that there are some fundamental limitations on the maintenance of spin coherence and some loss of coherence is inevitable even when a pure quantum evolution is considered.

Consider the time dependent solutions for the components $\rho_{\uparrow\uparrow}$, $\rho_{\downarrow\downarrow}$ ($\rho_d$, i.e., diagonal in spin space) and $\rho_{\downarrow\uparrow}$, $\rho_{\uparrow\downarrow}$ ($\rho_{od}$, i.e., off-diagonal in spin space) of the density matrix corresponding to the state (2.4.1) in the position representation (see Section III) with the changed coordinates.
\[ r = (x-y), \quad R = (x+y)/2 : \]

\[
\rho_d(R, \tau) = 2 \left\{ \frac{\pi}{\sigma^2 + \frac{\hbar^2 \ell^2}{\sigma^2 m^2}} \exp \left\{ \frac{r^2}{4\sigma^2} \pm \frac{i\epsilon \tau t}{\hbar} - \frac{1}{\sigma^2 + \frac{\hbar^2 \ell^2}{\sigma^2 m^2}} \right\} \right. \\
\left. + \frac{i\hbar \tau}{2\sigma^2 m} \mp x_0 \pm \frac{\epsilon \ell^2}{2m} \right\}^2, (2.4.3) \]

\[
\rho_{od}(R, \tau) = 2 \left\{ \frac{\pi}{\sigma^2 + \frac{\hbar^2 \ell^2}{\sigma^2 m^2}} \exp \left\{ - \frac{1}{\sigma^2 + \frac{\hbar^2 \ell^2}{\sigma^2 m^2}} \right\} \right. \\
\left. + \frac{i\hbar \ell^3}{2m^2 \sigma^2} \pm \frac{i\epsilon \omega \hbar}{\sigma^2 m} \right\}^2 \mp 2i\lambda \gamma - \frac{r^2}{4\sigma^2} - \frac{\sigma^2 \epsilon^2 \ell^2}{\hbar^2} - \frac{\epsilon^2 \ell^4}{4\sigma^2 m^2} \\
- \frac{x_0^2}{\sigma^2} \mp \frac{x_0^2}{\sigma^2 m^2} + \frac{\epsilon \ell^2}{2m} \right\}, (2.4.4) \]

The signs + (-) correspond to \( \rho_{\uparrow\uparrow} (\rho_{\downarrow\downarrow}) \) in \( \rho_d \) and \( \rho_{\downarrow\uparrow} (\rho_{\uparrow\downarrow}) \) in \( \rho_{od} \). Note that the density matrix remains completely coherent in the sense that the off-diagonal elements of the density matrix in the spin and coordinate space are always there. The diagonal distributions in the coordinate space (corresponding to \( r = 0, R = x \)) are:

\[
\rho_d(x, 0, \tau) = 2 \left\{ \frac{\pi}{\sigma^2 + \frac{\hbar^2 \ell^2}{\sigma^2 m^2}} \exp \left\{ - \frac{1}{\sigma^2 + \frac{\hbar^2 \ell^2}{\sigma^2 m^2}} \left( x + \mp x_0 \pm \frac{\epsilon \ell^2}{2m} \right)^2 \right\}, (2.4.5) \right. \\
\left. + \frac{i\hbar \ell^3}{2m^2 \sigma^2} \pm \frac{i\epsilon \omega \hbar}{\sigma^2 m} \right\}^2 \mp 2i\lambda \gamma - \frac{r^2}{4\sigma^2} - \frac{\sigma^2 \epsilon^2 \ell^2}{\hbar^2} - \frac{\epsilon^2 \ell^4}{4\sigma^2 m^2} \right\}.
The quantities of interest for a real physical observation are the expectation values \( \langle \sigma_x \rangle \), \( \langle \sigma_y \rangle \), and \( \langle \sigma_z \rangle \) with the individual spin-components at \( x \) measured at the time of recombination. Noting \( \langle \sigma_x \rangle = \rho_{+\downarrow} + \rho_{\downarrow+} \), \( \langle \sigma_y \rangle = i(\rho_{+\downarrow} - \rho_{\downarrow+}) \), and \( \langle \sigma_z \rangle = (\rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow}) \), where the elements of the density matrix are the diagonal distributions (2.4.5) and (2.4.6) in position representation, one can see that these quantities are constantly changing with time in a complicated way which also includes the effects due to spin precession. Thus the spin has components along all three directions, i.e, no specific polarization along \( x \), \( y \) or \( z \) axes. How do we get back the initial \( x \)-polarized state where \( \langle \sigma_x \rangle \neq 0 \), and \( \langle \sigma_y \rangle \), \( \langle \sigma_z \rangle = 0 \) ? If we look at the initial state (2.4.1), a 'recombination' can be naively visualized as the merging together (in space) of the trajectories corresponding to the two beams for 'up' and 'down' \( z \)-polarized spin states. In the light of our previous discussions on the quantum measurement problem and the notion of wave function collapse, note that in the pure quantum evolution discussed above, one cannot quite talk of trajectories. Trajectories are diagonal distributions in space (i.e., the position distributions). In the absence of a measurement model like the one discussed in Section III, the density matrices in both position and spin space in the above case are highly nonlocal. If one were to apply the projection postulate in both spin and
space parts (i.e., merely set the off-diagonal elements of the density matrix to zero), the density matrix becomes:

$$\rho = \rho_{\uparrow \uparrow}(x,t) |\uparrow\rangle\langle\uparrow| + \rho_{\downarrow \downarrow}(x,t) |\downarrow\rangle\langle\downarrow|,$$

(2.4.7)

where,

$$\rho_{\uparrow \uparrow}, \rho_{\downarrow \downarrow} = 2 \left[ \frac{\pi}{\sigma^2 + \frac{\hbar^2 \epsilon^2}{\sigma^2 m^2}} \exp \left\{ \frac{-1}{\sigma^2 + \frac{\hbar^2 \epsilon^2}{\sigma^2 m^2}} \left( x \mp x_0 \pm \frac{\epsilon \ell^2}{2m} \right)^2 \right\} \right].$$

(2.4.8)

A recombination in space (as described above) would mean that the centers of the two wave packets corresponding to the 'up' and 'down' \(z\)-polarized states are made to coincide, i.e., the term \(x \mp x_0 \pm \epsilon \ell^2/2m\) in the exponential must go to zero. This happens at time \(t = t_{re} = \sqrt{2\hbar x_0/\epsilon}\). At \(t = t_{re}\) one can see that the resulting density matrix is not quite the same as that of the initial \(x\)-polarized state that entered the first SGA and a look at the expectation values for the spin components show that at \(t = t_{re}\), \(\langle \sigma_z \rangle = 0\) like it was for the original \(x\)-polarized state, but \(\langle \sigma_x \rangle, \langle \sigma_y \rangle \neq 0\). Thus by 'merging' the trajectories we have obviously not quite recovered the original \(x\)-polarized state. At all other times, \(\langle \sigma_x \rangle, \langle \sigma_y \rangle \langle \sigma_z \rangle \neq 0\), and constantly change with time. How does one reconstitute the original because of spin precession. From (2.4.3) one can see that by a careful choice of the parameters \(\epsilon, m, \sigma\) and \(t\), it may be possible to recover a situation where \(\langle \sigma_x \rangle \neq 0, \langle \sigma_y \rangle = 0, \langle \sigma_z \rangle \neq 0\). However, this again is not the same situation as of (2.4.1) as the state is not strictly \(x\)-polarized.

Let us now examine the effect of an environment described by the model discussed in Section III. We again consider the initial condition (2.4.1) and
for the total Hamiltonian ($H_{SA}$ of (2.4.2) with an environment),

$$H_{SAE} = p^2/(2m) + \lambda \sigma_z + \epsilon \times \sigma_z + H_{AE} + HE,$$  (2.4.9)

one can write down the equations for the density matrices corresponding to the diagonal and the off-diagonal spin components similar to (2.3.8) and (2.3.9) (see Section III). For our purpose in this Section we directly write down the solutions in the position representation:

$$\rho_d(R, r, \theta) = \frac{\pi}{M(\tau)} \exp \left\{ \left[ \frac{1}{4\sigma^2} e^{-2\tau} + \frac{D}{8\hbar^2 \gamma} (1-e^{-2\tau}) \right] r^2 + \frac{i\epsilon}{\hbar \gamma} (1-e^{-\tau})r \right.$$ 

$$\left. - \frac{1}{M(\tau)} \left( R - \frac{\epsilon}{m\gamma^2} (1-e^{-\tau}) - \frac{i\hbar r}{2\sigma^2 m\gamma} e^{-\tau} (1-e^{-\tau}) \right. \right.$$ 

$$\left. + \frac{iD r}{4m\gamma^2 \hbar} (1-e^{-\tau})^2 \pm \chi_0 \right\}^2, \right.$$  (2.4.10)

where $\tau = \gamma t$ and

$$M(\tau) = \sigma^2 + \frac{\hbar^2}{\sigma^2 m^2 \gamma^2} (1 - e^{-\tau})^2 + \frac{D}{2m^2 \gamma^3} (2\tau - 3 + 4e^{-\tau} - e^{-2\tau}).$$  (2.4.11)

For the spin off-diagonal elements of the density matrix, the solution in the partial transform representation is the same as the solution (2.3.21) but for a different initial condition. As we have already seen in Section III, in the $t \to \infty$ limit, the solutions in the momentum representation for $\rho_d$ become completely diagonal while that in the position representation remains nondiagonal to the extent of the de Broglie wavelength, while the solutions
for $\rho_{ad}$ decay to zero in either representation over a characteristic time scale (see (2.3.23)). The final density matrix is thus completely diagonal in the spin space in the basis $|\uparrow\rangle, |\downarrow\rangle$ of $\sigma_z$ with the 'up' and 'down' spin states correlating with a double-peaked momentum distribution function. The pure state density matrix becomes a mixed one with classical probabilities associated with the particles having 'up' or 'down' $z$-polarizations. If we are probing length scales much larger than the thermal de Broglie wavelength of the particle, the distribution in position representation can be assumed to be approximately diagonal and these diagonal distributions represent the distinct classical trajectories associated with the 'up' and 'down' spin states:

$$M(t) = \int |\psi(x,t)|^2 \, dx = \sum_{\pm} \int \left| \psi(x,t) \right|^2 \, dx = \frac{\pi}{2M(t)} \exp \left\{ -\frac{1}{M(t)} \left( x \pm x_0 \pm \frac{\epsilon}{m\gamma^2} (1-t) \right)^2 \right\}.$$ (2.4.12)

A 'recombination' in this situation would mean the merging of these classical trajectories. It is obvious that this would only result in a statistical mixture of 'up' and 'down' $z$-polarized particles so that the average value $\langle \sigma_z \rangle = 0$. From (2.4.12) one can see that this would happen at time $t = (1/\gamma + \gamma m x_0 / \epsilon)$. At all other times 'up' and 'down' $z$-polarized particles have distinct classical trajectories centered around two different mean values.

From the above discussion it is clear that there are marked differences when an SGI is analysed with and without the environment. In the case of pure quantum evolution, the expectation values of the observable physical
quantities in the state (2.4.1), \( \langle \sigma_x \rangle \), \( \langle \sigma_y \rangle \) and \( \langle \sigma_z \rangle \) are constantly changing with time. It is difficult to talk of a recombination in terms of the merging of trajectories and although a careful choice of parameters can be made to get a finite expectation value of \( \langle \sigma_x \rangle \), the original x-polarized spin state cannot be completely recovered. When environmental influences are taken into account, the density matrix corresponding to the state (2.4.1) reduces to a diagonal mixture in a time scale characteristic of the environment. A merging together of the trajectories corresponding to the two beams cannot obviously give back the original x-polarized state as environmental influences have already destroyed coherence. The implications of the two situations, i.e., a pure quantum evolution without an environment and one with the environment are quite different and can be subjected to experimental tests. It is worth noting here that even in the absence of an environment it is difficult to get back the initial spin-polarized state in an SGI and a careful and more detailed analysis is required both in theoretical and experimental attempts to 'reconstitute' an arbitrary initial spin state.

2. V Preferred basis in a measurement process.

From the discussions in Sections I, II, and III it is clear that the main objective of the quantum measurement process is the establishment of a one-to-one correlation between the states of the system and the apparatus and the reduction of the pure state density matrix to a diagonal mixture. We recall that the main difficulty with a nondiagonal density matrix of the form (2.1.6),
\[
\rho^e = |\phi_e\rangle\langle\phi_e| = |\alpha|^2 |\uparrow\rangle \langle \uparrow | + |\beta|^2 |\downarrow\rangle \langle \downarrow | + \alpha^* \beta |\uparrow\rangle \langle \downarrow | + \alpha \beta^* |\downarrow\rangle \langle \uparrow |, \\
\text{(2.5.1)}
\]

is that the ensemble described by it is not unique. The density matrix can well be expressed in any other basis so that the statement that the system-apparatus combine is in the state \(|\uparrow\rangle \langle \uparrow |\) with a probability \(|\alpha|^2\) and in the state \(|\downarrow\rangle \langle \downarrow |\) with a probability \(|\beta|^2\) is completely meaningless. For a classically meaningful situation one must be able to talk of the system/apparatus existing in specific states with classical probabilities. This necessarily requires a description in terms of a density matrix that is diagonal in a uniquely specified 'pointer basis' (Zurek 1981) or the 'preferred basis' (Zurek 1991). In the previous sections we have seen how the 'environment induced decoherence' approach to the quantum measurement problem explains the reduction of the pure state density matrix to a diagonal mixture. Environmental 'superselection' (Zurek 1991) results in the emergence of a 'preferred basis' in which nonlocal superpositions vanish and the density matrix is diagonal. However, even in this approach the question of the basis in which the density matrix becomes diagonal, i.e., the preferred basis is not quite understood. It seems that regardless of the initial condition, the environment always selects a special set of states in which the density matrix is diagonal and hence classically interpretable. It is important to know what these preferred bases are for specific systems since they are directly related to the emergent 'classicality' of macroscopic systems as a consequence of decoherence. It seems plausible that such a preferred basis would be decided by the system operator which is coupled to the environmental degrees of freedom. Zurek (1991) observes that when the
interaction with the environment dominates, the reduced density matrix ends up being diagonal in the eigenspace of an observable $\Lambda$ that commutes with the interaction Hamiltonian, $[\Lambda, H_{int}] = 0$. Physically, this guarantees that the pointer observable $\Lambda$ will be a constant of motion of the interaction Hamiltonian. Thus when the system is in the eigenstate of $\Lambda$, interaction with the environment will leave it unperturbed. For example, for a free particle if the position operator is involved in coupling with the environmental degrees of freedom, one expects that the density matrix would be driven diagonal in the position space. It has been shown by Zurek (1986, 1991, 1993b) that the coherence between two Gaussian wave packets separated in space by $\Delta x$ is lost on a time scale which is typically

$$\theta = \tau \left( \frac{\hbar}{\Delta x \sqrt{4mk_B T}} \right)^2$$  \hspace{1cm} (2.5.2)$$

where $m$ is the mass of the particle, $k_B$ is Boltzmann's constant, $T$ is the temperature of the heat bath and $\tau$ is the characteristic relaxation time of the system. For classical systems and standard macroscopic separations $\Delta x$, the ratio $\theta/\tau$ can be as small as $10^{-40}$ (Zurek 1986), suggesting that the density matrix becomes diagonal in position space almost instantaneously, making position the 'preferred basis'. In this Section we, however, show that for a free particle such is not the case. In Section III we have already seen that for the measurement of spin it is the momentum basis that emerges as the preferred basis. In this Section we take a closer look at the problem for a free particle coupled dissipatively to an environment via the Caldeira-Leggett kind of coordinate-coordinate coupling which has already been
considered in detail in the previous Sections. Our conclusions in this Section are that to observe a dynamical variable the system-apparatus coupling requires a careful consideration and the process of 'superselection' of a preferred basis by the environment may not be as straightforward as it seems.

In Section III we have discussed the master equation derived by Caldeira and Leggett (1983, 1985). Consider the equation (2.3.2) in the position representation:

\[
\frac{\partial \rho_R(x,y,t)}{\partial t} = \left\{ \frac{-\hbar}{2\lambda m} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - \gamma(x-y) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \right\} \rho_R(x,y,t) - \frac{D}{4\hbar^2} (x-y)^2 \rho_R(x,y,t),
\]  

(2.3.2)

where \( m \) is the mass of the particle, \( \hbar \) is Planck's constant, \( \gamma \) is the Langevin friction coefficient and \( D \) is the diffusion constant. \( \gamma \) and \( D \) are related to the parameters of the Hamiltonian (2.3.1). For a high temperature thermal bath, \( D = 2m \gamma k_B T \). Consider the exact solutions of (2.3.2) derived earlier by Kumar (1984) for an initial Gaussian wave packet

\[
\psi(x,0) = \frac{1}{(\sigma \sqrt{\pi})^{1/2}} \exp \left\{ -\frac{x^2}{2\sigma^2} \right\},
\]  

(2.5.3)

where \( \sigma \) is the width of the wave packet. The solutions (Kumar 1984, Venugopalan et al 1993, 1994) in wave-vector and position representations in terms of the changed coordinates \( Q = u-\nu \), \( q = (u + \nu)/2 \) in wave-vector space and \( R = (x + \gamma)/2 \), \( r = (x-\nu) \) in position space, are
\[
\rho_d(Q,q,t) = 2 \pi \left( \frac{1}{N(\tau)} \right) \exp \left\{ -\frac{1}{N(\tau)} \left[ q + \frac{i \hbar Q}{2 \sigma^2 m \gamma} e^{-\tau(1-e^{-\tau})} - \frac{i}{4 \hbar \gamma^2 m} D (1-e^{-\tau})^2 \right]^2 \right\}
- \left\{ \frac{\hbar^2}{4 \sigma^2 m^2 \gamma^2} (1-e^{-\tau})^2 + \frac{\sigma^2}{4} + \frac{D}{2 m^2 \gamma^3} (2\tau - 3 + 4e^{-\tau} - e^{-2\tau}) \right\} Q^2 \right\}, \tag{2.5.4}
\]

where \( \tau = \gamma t \) and

\[
N(\tau) = \left( \frac{D}{2 \hbar^2 \gamma} \right) (1-e^{-2\tau}) + (1/\sigma^2) e^{-2\tau}, \tag{2.5.5}
\]

and

\[
\rho_d(R,r,0) = 2 \pi \left( \frac{1}{M(\tau)} \right) \exp \left\{ -\left( \frac{1}{4 \sigma^2} e^{-2\tau} + \frac{D}{8 \hbar^2 \gamma} (1-e^{-2\tau}) \right) R^2 - \frac{1}{M(\tau)} \left\{ -\frac{i \hbar}{2 \sigma^2 m \gamma} e^{-\tau(1-e^{-\tau})} + \frac{i D}{4 \hbar^2 \gamma^2 m} (1-e^{-\tau})^2 \right\}^2 \right\}, \tag{2.5.6}
\]

where

\[
M(\tau) = \sigma^2 + \frac{\hbar^2}{\sigma^2 m^2 \gamma^2} (1-e^{-\tau})^2 + \frac{D}{2 m^2 \gamma^3} (2\tau - 3 + 4e^{-\tau} - e^{-2\tau}). \tag{2.5.7}
\]

The time dependence of the above two solutions can be studied for two regimes: \( \tau >> 1 \), i.e., for times much larger than the characteristic relaxation time \( \gamma^{-1} \), and \( \tau << 1 \), i.e., for times much smaller than \( \gamma^{-1} \). The first regime (\( \tau >> 1 \)) is of greater importance in all real life systems.

(i) \( \tau >> 1 \)

It is clear from the form of the solution (2.5.4) that as \( \tau \to \infty \), the off-diagonal elements of the density matrix in the momentum representation vanish. This decay is exponential and occurs in a time

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The density matrix (2.5.4) becomes completely diagonal at long times and assumes the form

$$\rho_d(0,u,t) = 2 \frac{\pi}{N(\tau)} \exp \left\{ -\frac{u^2}{N(\tau)} \right\}. \quad (2.5.9)$$

making momentum the obvious choice for the preferred basis. (2.5.9) is the classical Maxwell distribution that one would obtain from a classical Fokker-Planck equation. We would like to point out here that the expression (2.5.8) for $t_d$ is valid for the regime $\tau \gg 1$ which implies a time scale much greater than the relaxation time of the system $\gamma^{-1}$. Moreover (2.3.2) is valid for a high temperature heat bath, where $\hbar \gamma/k_B T \ll 1$ [9]. From the expression (2.5.8) one can see that the decoherence time $t_d$ decreases with increasing $T$ and $\gamma$. On the other hand the distribution function (2.5.6) in the position representation does not become completely diagonal as $\tau \to \infty$, but assumes the form

$$\rho_d(R,r,\hat{n}) = 2 \frac{\pi}{M(\tau)} \exp \left\{ -\frac{D}{8\hbar^2 \gamma} r^2 - \frac{1}{M(\tau)} \left\{ R + i\frac{D}{4m\gamma^2 \hbar} (1-e^{-\gamma})^2 \right\}^2 \right\}. \quad (2.5.10)$$

with

$$M(\tau) \approx \frac{D\tau}{m^2 \gamma^3}. \quad (2.5.11)$$

The density matrix is obviously non-diagonal in the position space representation which is a consequence of the fact that (2.5.4) and (2.5.6) are related by Fourier transforms. The asymptotic width for the distribution
in variable \( r \) is \( D/8\hbar^2\gamma \), which for a thermal bath is \( \pi/2\lambda_d^2 \), since

\[
\hbar^2\gamma/D = \frac{\lambda_d^2}{4\pi} \tag{2.5.12}
\]

where \( \lambda_d \) is the thermal de Broglie wavelength of the particle ( \( \hbar/\sqrt{2nmk_BT} \)). One can see that if the extent of 'off-diagonality' is much greater than \( \lambda_d \), the magnitude of the off-diagonal elements, which is weighted by \( e^{-r^2\pi/2\lambda_d^2} \), is very small and the density matrix in position space can be considered nearly diagonal. In principle, however, it remains nonlocal to the extent of the de Broglie wavelength, \( \lambda_d \). It is interesting to see that for the initial condition considered by Zurek (1986, 1991, 1993) where the initial position-space density matrix contains four well-separated peaks (\( \Delta x \gg \sigma \)), an exact solution of (2.3.2) shows that for the peaks which are along the diagonal for which \( R \approx \pm \Delta x; r \approx 0 \), the major contribution to the density matrix is from the diagonal elements, which are peaked around \( \pm \Delta x \). However, for the peaks along the off-diagonal, for which \( R \approx 0; r \approx \pm \Delta x \), the factor \( e^{-r^2\pi/2\lambda_d^2} \) (see (2.5.10)) is now \( e^{-\Delta x^2\pi/2\lambda_d^2} \). One can see that for \( \Delta x \gg \lambda_d \), this factor is very small and hence the elements of the density matrix corresponding to these two off-diagonal peaks are negligible in magnitude. Thus, position seems to emerge as an approximate preferred basis, which works well only when one is probing length scales which are much larger than \( \lambda_d \).

(ii) \( \tau \ll 1 \)

In this regime, one is probing the system at time scales which are much smaller than the characteristic relaxation time \( \gamma^{-1} \). This regime is not very realistic since one is usually interested in the state of the system long
after it has been left in an environment. If we retain terms only upto first order in \( \tau \) in (2.5.4) and (2.5.6), the momentum and position space density matrices assume the forms

\[
\rho_d(Q,q,\theta) = 2 \left[ \frac{\pi}{N(\tau)} \right] \exp \left\{ \frac{-1}{N(\tau)} \left[ q + \frac{i\hbar Q}{2\sigma^2 m \gamma} \tau (1-\tau) - i \frac{Q D}{4h^2 \gamma m} \tau^2 \right] \right\}^2
\]

\[
- \left\{ \frac{\hbar^2}{4\sigma^2 m^2 \gamma^2} \tau^2 + \frac{\sigma^2}{4} \right\} Q^2 \right\}, \tag{2.5.13}
\]

where

\[
N(\tau) \propto (D\tau/h^2 \gamma) + (1/\sigma^2) (1-2\tau), \tag{2.5.14}
\]

and

\[
\rho_d(Rr,\theta) = 2 \left[ \frac{\pi}{M(\tau)} \right] \exp \left\{ \left\{ \frac{1}{4\sigma^2} (1-2\tau) + \frac{D\tau}{4h^2 \gamma} \right\} r^2 - \frac{1}{M(\tau)} \left\{ R \right. \right.
\]

\[
- \frac{i\hbar r}{2\sigma^2 m \gamma} \tau (1-\tau) + \frac{iD r}{4m^2 \gamma h} \tau^2 \left. \right\}^2 \right\}, \tag{2.5.15}
\]

where

\[
M(\tau) \propto \sigma^2 + \frac{\hbar^2 \tau^2}{\sigma^2 m^2 \gamma^2} \tag{2.5.16}
\]

It is clear from (2.5.13) and (2.5.15) that the density matrices in both representations remain nondiagonal and neither show any exponential decay with time for the off-diagonal elements. For the position density matrix (2.5.15), the leading order decaying term is \( \sim \exp \left\{ - \frac{1}{2} \left\{ \frac{D}{2h^2 \gamma} - \frac{1}{\sigma^2} \right\} r^2 \tau \right\} \). Note that \( D\tau^2/h^2 \gamma \sim 4\pi r^2 \tau/\lambda_d^2 \) and the 'decay' time is similar to (2.5.2) obtained by Zurek (1986, 1991, 1993b). However, since this is valid only for \( \tau \ll 1 \), it cannot be interpreted as a complete exponential decay of the off-
diagonal elements in position space. In this time regime, thus, both the position and the momentum space density matrices remain highly nonlocal and one cannot talk in terms of any emergent preferred basis. From the above analysis it is clear that in the $\tau \to \infty$ regime, which is a significant regime for realistic systems, momentum emerges as the basis selected by the environment. The position space density matrix remains nonlocal, the extent of nonlocality being $\lambda_d$.

Thus we see that the role of the preferred basis is extremely crucial to the study of any transition from 'quantum' to 'classical'. Although quantum mechanics allows systems to exist in a wide variety of states, macroscopic objects are always seen to exist only in a specific 'classical subset' of these states (Zurek 1993b). These are the states of the preferred basis. An interesting example considered by Zurek (1993b) is that of a harmonic oscillator interacting with an environment in thermal equilibrium. The decoherence process resulting from this interaction selects a 'preferred basis' for the harmonic oscillator. Interestingly, these states are the coherent states (Glauber 1963), which have been known for long as quantum counterparts of classical points in phase space. Although the complete dynamics is not worked out, Zurek (1993b) characterizes the 'preferred basis' in terms of the increase of entropy of the density matrices resulting from various initial states. On the basis of this 'predictability sieve' (Zurek 1993b) Zurek shows that the states which yield the least entropy increase (i.e., those that are most 'stable') for a harmonic oscillator are the coherent states. Bhandari (1987) pointed out that in the case of the coherent states for a harmonic oscillator, macroscopic superpositions are nearly zero.
He also points out the similarity of the mathematical structure of the spin coherent states and the coherent states of the harmonic oscillator suggesting their role as the preferred basis for macroscopic spins. One can, similarly, look for the preferred basis for other interesting systems.

To summarize, in this Section we have clarified the roles of position and momentum for a free particle which is dissipatively coupled to a heat bath with an ohmic noise spectrum. The nature of the emergent preferred basis is expected to depend on the form of the system-environment coupling. We use a Hamiltonian with a coordinate-coordinate coupling between the system and the environment, but it is unitarily equivalent to other types of coupling (coordinate-momentum etc.) (Ford et al 1988). The preferred basis need not necessarily be that of the system variable which couples to the environmental degrees of freedom. For the coordinate-coordinate coupling, we have shown that the momentum basis clearly is the emergent preferred basis. This is contrary to the general expectation that position should emerge as the preferred basis.

2.VI. Effect of decoherence on Bell's inequalities for an EPR pair.

The basic formalism of quantum theory which involves concepts of probability amplitudes and the superposition principle has been the source of many paradoxes and strange implications. A celebrated paradox that illustrates the counterintuitive conceptual framework of quantum mechanics is the 'E-P-R paradox' which is based on a gedankenexperiment proposed by Einstein, Podolsky and Rosen in 1935. This paradox has a historical
significance since it was first proposed as an argument against the completeness of quantum mechanics and later paved the way for fascinating developments led by the Bell’s theorem (Bell 1965) and its stunning experimental tests (Freedman and Clauser 1972, Aspect et al 1982).

The E-P-R paradox is based on the following two premises: (i) Realism is a philosophical view according to which the world is made up of objects whose existence is independent of whether or not they are being observed. All physical attributes which can be measured without disturbing the system at hand are regarded as 'elements of reality'. It is intuitive to expect all physical theories to conform to realism. (ii) In nature there is no 'action-at-a-distance', i.e., no influence of any kind that can propagate faster than light. For two spatially separated parts of a composite system, the measurement on one part is not causally affected by what one chooses to measure on the other part. This requirement is called 'locality' and is another natural demand on any physical theory. E-P-R showed that if one requires quantum theory to conform to locality and realism then the quantum mechanical description of systems is incomplete. In the following we describe David Bohm’s version of the E-P-R gedankenexperiment (Clauser and Shimony 1978).

Consider two spatially separated spin-1/2 particles which were initially in a singlet state. These can be thought of as being produced in the process of the dissociation of a spin-0 particle. They can be described by the following wave function,
\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow_{1n}\rangle \otimes |\downarrow_{2n}\rangle - |\downarrow_{1n}\rangle \otimes |\uparrow_{2n}\rangle \right), \quad (2.6.1) \]

where \( |\uparrow_{1n}\rangle \) and \( |\downarrow_{1n}\rangle \), \( i = 1, 2 \) describe the spin states in which particle \( i \) has spin 'up' and 'down' respectively, along the direction \( n \). The expectation value:

\[ \langle \psi | \sigma_{1n} \sigma_{2,n} | \psi \rangle = -1, \quad (2.6.2) \]

implies a perfect negative correlation between the spins of 1 and 2. According to E-P-R, various spin components of each of these particles may be measured at the option of the experimenter in the following way. Suppose one measures the spin of particle 1 along the x-axis. The outcome is not pre-determined by the description \( |\psi\rangle \), but from it one can predict that if 1 is found to have its spin parallel to the x-axis, then 2 will have its spin antiparallel to the x-axis in accordance with (2.6.2). Thus the experimenter can arrange his or her apparatus in such a way that the value of the x-component of the spin of particle 2 can be predicted presumably without interacting with it (assuming no action-at-a-distance). Likewise, the apparatus can be arranged to predict the value of any other component of the spin of 2. Thus, all the spin components of the particle 2 can be predicted completely. This conclusion is contradictory to the quantum mechanical description of systems where only relative probabilities can be predicted and no two components of the spin can be completely specified simultaneously. In the quantum description there are no preset 'elements of reality', i.e., in this case, no fixed values of the spin components, prior to a measurement. In
contrast, the E-P-R experiment arguably showed that it is possible to determine all three components of the spin of 2 without disturbing it and hence these three values must be pre-existing 'elements of reality' of the system. E-P-R thus concluded (Einstein et al. 1935) that the quantum mechanical description of systems is incomplete. The 'shortcomings' of quantum mechanics were thought to be a consequence of its incompatibility with the concepts of realism and locality. Excellent agreements between quantum mechanical predictions and real experiments had proved quantum theory's success beyond any doubt. A natural requirement, thus, was to look for a theory that agreed with all the predictions of quantum mechanics and yet did not share its conflicts with realism and locality. Many attempts were made to find such a complete 'hidden variable theory' (Clauser and Shimony 1978, Pipkin 1978). In 1965 Bell provided a proof that assuming there is no action-at-a-distance, no hidden-variable theory that satisfied a physically reasonable condition of locality can agree with all of the statistical predictions of quantum mechanics concerning the spins of a pair of spin-1/2 particles in the E-P-R singlet state; in other words, quantum mechanics is nonlocal. This fact can be represented by the violation of the Bell inequalities which put an upper bound on the strength of the correlations between spatially separated particles with local interactions (Clauser and Shimony 1978, Pipkin 1978). This formulation allows for an experimental test of quantum mechanics against any local hidden-variable theory. It is now firmly established by highly successful experiments (Aspect et al 1982) that there exist nonseparable quantum correlations of the E-P-R kind that violate Bell inequalities, thus supporting the validity of quantum mechanics.
In this Section we restrict ourselves to one particular form of the Bell inequality applicable to the E-P-R experiment with a slight modification introduced by Bohm (Clauser and Shimony 1978, Bohm 1951). Consider the E-P-R single state (2.6.1). Two detectors are arranged to measure the spin component of particle 1 along a direction \( a \) and that of particle 2 along a direction \( b \). \( a \) and \( b \) are at an angle \( \theta \) with respect to each other. According to quantum formalism, the expectation value can be written as

\[
E(a,b) = \langle \psi | \sigma_1.a \sigma_2.b | \psi \rangle = -\cos \theta. \tag{2.6.3}
\]

One can see that for \( \theta = 0 \), \( E(a,b) = -1 \) (as in (2.6.2)). This implies the existence of a perfect negative correlation between the spins of 1 and 2, i.e., if the value of the spin component of 1 along the direction \( n \) is +1, then that of 2 is certainly -1. For our purpose in this Section we look at the following form of the Bell inequality (Clauser and Shimony 1978):

\[
|E(a,b)-E(a,c)| \leq 1 + E(b,c), \tag{2.6.4}
\]

where \( a, b \) and \( c \) are three directions along which the detectors measure the spin components of 1 and 2. One can easily see that for the condition \( \theta(a,b) = \theta(b,c) = \pi/3 \) and \( \theta(a,c) = 2\pi/3 \), quantum mechanics clearly violates this inequality (2.6.4). As mentioned above, (2.6.4) and a whole class of Bell inequalities provide an excellent means of testing experimentally the predictions of quantum mechanics against any local hidden variable theory and many experiments have shown the existence of nonlocal quantum correlations through a violation of Bell inequalities (Freedman and Clauser 1972, Aspect
et al 1982). These results establish nonlocality as an inescapable ‘fact of life’. There are many exciting implications of quantum nonlocality like quantum cryptography and quantum computers. Particularly fascinating is the idea of ‘teleportation’ of a quantum state using E-P-R pairs (Bennett et al 1993). However, we know from everyday experience that nonlocal correlation of the E-P-R kind are not seen in the macroscopic physical world. In the classical world particles well separated in space cannot affect each other in a nonlocal way. Inspite of the fact that Bell’s theorem and its subsequent experimental verifications support quantum theory, thus demolishing common sense notions of realism and locality, classical systems are known to conform to realism and locality. How does this transition take place? How do nonlocal correlations disappear? How does classicality emerge?

In this Section we study the effect of an environment on an E-P-R setup. It is interesting to note that this is of direct significance to problems like the practical implementation of the ‘teleportation’ idea which exploits the quantum correlations of an E-P-R pair (Bennett et al 1993). Our model is the following. Consider the E-P-R singlet state (2.6.1). To simulate an E-P-R situation we assume that the particles 1 and 2 are acted upon by forces which push them in opposite directions so that the two particles start moving apart at $t > 0$. The influence of the environment is two-fold. Firstly, the two particles are coupled to a bath of harmonic oscillators via a Caldeira-Leggett type of dissipative coupling through their positional degrees of freedom. This aspect of the model has been discussed in detail in Section III. Secondly, the spin degrees of freedom for the two particles are coupled to an environment modelled by a fluctuating external magnetic field. The
Hamiltonian for this model is:

\[
H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \epsilon X_1 + \epsilon X_2 + \sum \frac{p_{1k}^2}{2m_{1k}} + \frac{m_{1k}\omega_{1k}^2}{2} \left( x_{1k} - \frac{c_{1k}X_1}{m_{1k}\omega_{1k}^2} \right)^2 + \\
\sum \frac{p_{2k}^2}{2m_{2k}} + \frac{m_{2k}\omega_{2k}^2}{2} \left( x_{2k} - \frac{c_{2k}X_2}{m_{2k}\omega_{2k}^2} \right)^2 - \frac{\xi_1\mu\hbar}{2} B_1 \cdot \sigma_1 - \frac{\xi_2\mu\hbar}{2} B_2 \cdot \sigma_2 ,
\]

(2.6.5)

where \( P_1, X_1 \) and \( P_2, X_2 \) are the momentum and position coordinates of particles 1 and 2 respectively, \( \epsilon \) is the strength of the applied force (positive for 1 and negative for 2), \( p_{1k}, x_{1k} \) and \( p_{2k}, x_{2k} \) are the momentum and position coordinates of the \( k \)th harmonic oscillator of the baths which couple to 1 and 2, \( c_{1k} \) and \( c_{2k} \) are the respective coupling strengths and \( \omega_{1k} \) and \( \omega_{2k} \) are the frequencies of the oscillators. \( B_1 \) and \( B_2 \) are the external magnetic fields which couple to the spins of particles 1 and 2, \( \xi_1 \) and \( \xi_2 \) are the strength of the coupling for 1 and 2, and \( \mu = qg/2mc \), where \( q \) is the charge of the particles, \( g \) is the gyromagnetic ratio and \( c \) is the speed of light. We would like to point out here that the spin and the space (position/momentum) degrees of freedom for the two particles are not coupled directly. However, it is possible to simultaneously study the time evolution of the space and the spin parts of the problem.

As before, we can write down the equation for the density matrix for the composite system as:

\[
\frac{\partial \rho}{\partial t} = -\hbar \frac{\partial^2 \rho}{\partial R_1 \partial r_1} - \gamma R_1 \frac{\partial \rho}{\partial r_1} - \frac{D R_1^2}{4 \hbar^2} \rho + \frac{i\epsilon R_1}{\hbar} \rho - \frac{i}{\hbar} \left[ \xi_1 \mu B_1 \cdot \sigma_1, \rho \right]
\]
Here we are using the variables $r_i = x_i - y_i$ and $R_i = (x_i + y_i)/2$, $i = 1, 2$ (see Section III). It can be easily seen that the interactions of the position and spin degrees of freedom of particles 1 and 2 with their respective environments are completely independent of each other and we can separate the equations for the space and spin parts for the two particles:

$$\frac{\partial \rho_i(\text{space})}{\partial t} = -\frac{\hbar}{m} \frac{\partial^2 \rho_i(\text{space})}{\partial r_i \partial r_i} - \gamma \frac{D}{4 \hbar^2} \rho_i(\text{space}) + \frac{\hbar}{i} \rho_i(\text{space}), \quad i = 1, 2,$$

and

$$\frac{\partial \rho_i(\text{spin})}{\partial t} = -\frac{i}{\hbar} \left[ \xi_i \mu B_z \sigma_i, \rho_i(\text{spin}) \right], \quad i = 1, 2. \quad (2.6.8)$$

For an initial condition:

$$\psi = \phi(1) \phi(2) \otimes \frac{1}{\sqrt{2}} \left( |\uparrow_{1n}\rangle \otimes |\downarrow_{2n}\rangle - |\downarrow_{1n}\rangle \otimes |\uparrow_{2n}\rangle \right), \quad (2.6.9)$$

where $\phi(1), \phi(2)$ are initial Gaussian wave packets with zero initial momenta, the solutions for the space part of the problem have been worked out in detail in Section III. Here we directly write the results in position representation (see Eq. (2.3.18)).
\[
\rho_i(R_i,t) = \frac{\pi}{M(\tau)} \exp \left\{ - \left[ \frac{1}{4\sigma_i^2} e^{-2\tau} + \frac{D}{8\hbar^2} (1-e^{-2\tau}) \right] R_i^2 \pm \frac{i\epsilon}{\hbar \gamma} (1-e^{-\tau}) R_i 
\right\} 
\]

\[
- \frac{1}{M(\tau)} \left\{ R_i \pm \frac{\epsilon}{m\gamma^2} (1-e^{-\tau}) - \frac{i\hbar}{2\sigma_i^2 m\gamma} e^{-\tau}(1-e^{-\tau}) 
\right\} 
\]

\[
- \frac{iD}{4m\gamma^2 \hbar} (1-e^{-\tau})^2 \right\}^2, \quad i = 1,2 , \quad (2.6.10)
\]

where \( \tau = \gamma t \) and

\[
M(\tau) = \sigma_i^2 + \frac{\hbar^2}{\sigma_i^2 m^2 \gamma^2} (1 - e^{-\tau})^2 + \frac{D}{2m^2 \gamma^3} (2\tau - 3 + 4e^{-\tau} - e^{-2\tau}). \quad (2.6.11)
\]

If we look at the diagonal parts of (2.6.10), i.e., the position distribution functions in the \( t \to \infty \) limit, then these solutions represent two Gaussian wave packets centered around \( \pm \frac{\epsilon \tau}{m\gamma^2} \) and \( \mp \frac{\epsilon \tau}{m\gamma^2} \) which are moving away from each other with time. In the following we concentrate on solving the equations (2.6.8) for the spin degrees of freedom. It is important to note that the spins of 1 and 2 interact independently with their environments and are coupled only through their initial condition (the singlet state) as expected in an E-P-R situation.

Let us first consider the problem of a single spin-1/2 particle of mass \( m \) and charge \( q \) interacting with a fluctuating external magnetic field \( B \) through the Hamiltonian:
where $\xi$ is the strength of the coupling to the external field, $\mu = gq/2mc$, where $g$ is the gyromagnetic ratio. We note that the pure state density matrix for a spin-1/2 system (2 x 2 Hilbert space) can be written in the following way:

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \frac{1}{2} \left( 1 + P \sigma \right),$$  \hspace{1cm} (2.6.13)

where $1$ is a unit matrix and $\sigma$ is the Pauli spin operator. The vector $P$ has the following components:

$$P_x = \rho_{12} + \rho_{21},$$

$$P_y = i(\rho_{12} - \rho_{21}),$$

$$P_z = \rho_{11} - \rho_{22}.$$  \hspace{1cm} (2.6.14)

The Liouville equation for $\rho$ is:

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho].$$  \hspace{1cm} (2.6.15)

Substituting for $H$ and $\rho$ from (2.6.12) and (2.6.13) one can easily see that the equation (2.6.15) for $\rho$ reduces to the following equation for the vector $P$:
\[
\frac{\partial P}{\partial t} = -\xi \mu (B \times P) = -W \times P,
\]

(2.6.16)

where \( W = \xi \mu B \). Note that (2.6.16) is exactly the equation for a classical spin in an external magnetic field. In our model we assume that the external fluctuating field \( B(t) \) is a random process that is (i) stationary, i.e., there is no preferred origin of time for the statistical description of \( B(t) \), (ii) ergodic, i.e., time averages are equal to ensemble averages, and (iii) Gaussian, i.e., its marginal distribution \( B(t_1), B(t_2), \ldots, B(t_n) \) for \( n \) arbitrarily chosen time points is Gaussian. Since the field itself is stochastic, \( P(t) \) will also be a stochastic variable. The quantity of interest corresponding to a real physical observation would, therefore, be the expectation value of \( P(t) \), averaged over the ensemble of the random process \( B(t) \). One has to, thus, find the solution to a stochastic Liouville equation, i.e., an equation for the probability distribution \( f(P,t) \) for the stochastic variable \( P(t) \) (Kubo 1963). Such an equation would contain characteristic parameters of the stochastic field. One can write a Liouville equation:

\[
\frac{\partial f(P,t)}{\partial t} = -\left( \frac{\partial (\hat{P}_x f)}{\partial P_x} + \frac{\partial (\hat{P}_y f)}{\partial P_y} + \frac{\partial (\hat{P}_z f)}{\partial P_z} \right).
\]

(2.6.17)

Using (2.6.16), (2.6.17) can be written as

\[
\frac{\partial f(P,t)}{\partial t} = \left( \{ P \times \frac{\partial}{\partial P} \} : W \right) f = \dot{i} \mathcal{L} f.
\]

(2.6.18)

where the Liouville operator \( \mathcal{L} = L \cdot W \), where \( L \) is the angular momentum operator.
\[ L = -i \, P \times \frac{\partial}{\partial P} . \]  \hspace{1cm} (2.6.19)

One can write a formal solution for (2.6.18):

\[
  f(P,t) = \exp \sum_{i=0}^{t} \int_{0}^{t'} \frac{L(t')}{i} \, dt' \, f(P,0).
\]  \hspace{1cm} (2.6.20)

Here \( \exp_T(...) \) refers to the time-ordered series (Kubo 1963):

\[
  \sum_{n=1}^{\infty} \frac{1}{n!} \int_{0}^{t} \cdots \int_{0}^{t_n} \, T \{ \, \frac{L(t_1) \cdots L(t_n)}{n} \, \}.
\]  \hspace{1cm} (2.6.21)

If one chooses an initial distribution for \( f(P,0) \):

\[
  f(P,0) = \delta (P-P'),
\]  \hspace{1cm} (2.5.22)

then

\[
  f(P,t) = \exp_T \left\{ i \int_{0}^{t} \frac{L(t')}{i} \, dt' \right\} \delta (P-P').
\]  \hspace{1cm} (2.6.23)

(2.6.23) is just the solution for one particular sample of the random process \( B(t) \). For the complete solution representing the probability of being at \( P \) at
time \( t \) after having been at \( P' \) at \( t = 0 \), one has to average over the entire ensemble of \( \mathbf{B}(t) \), i.e.,

\[
\mathcal{f}(P,t/P',0) = \langle \mathcal{f}(P,t) \rangle = \langle \exp \left[ \int_0^t f(t', t) \, dt' \right] \rangle \delta(P-P').
\] (2.6.24)

If \( \mathbf{B}(t) \) (and hence \( \mathbf{W}(t) \)) is a Gaussian process then it can be shown (Kubo 1959) that

\[
\langle \exp \left[ \int_0^t f(t', t) \, dt' \right] \rangle = \exp \left[ \int_0^t \langle \mathbf{L}(t') \mathbf{L}(t) \rangle \, dt \right].
\] (2.6.25)

For the stochastic process \( \mathbf{W}(t) \), one can define the following correlation functions:

\[
\langle \mathbf{W}_\alpha(t_1) \mathbf{W}_\beta(t_2) \rangle = 0 \text{ if } \alpha \neq \beta,
\] (2.6.26)

and

\[
\langle \mathbf{W}_x(t_1) \mathbf{W}_x(t_2) \rangle = \langle \mathbf{W}_y(t_1) \mathbf{W}_y(t_2) \rangle = \psi_\perp(t_1 - t_2),
\] (2.6.27)

\[
\langle \mathbf{W}_z(t_1) \mathbf{W}_z(t_2) \rangle = \psi_{\parallel}(t_1 - t_2).
\] (2.6.28)

Using (2.6.27) and (2.6.28) it can be easily seen that

\[
\exp \left[ \int_0^t \langle \mathbf{L}(t') \mathbf{L}(t) \rangle \, dt' \right] = \exp \left[ \int_0^t \left\{ \psi_\perp(L_x^2 + L_y^2) + \psi_{\parallel}L_z^2 \right\} \, dt' \right].
\] (2.6.29)
where $L_x$, $L_y$ and $L_z$ are components of the angular momentum operator $-i \mathbf{P} \times \frac{\partial}{\partial \mathbf{P}}$. We now assume that the correlation functions $\psi_\perp$ and $\psi_\parallel$ are zero beyond a certain correlation time (the narrowing condition), i.e., we define

\[
\frac{1}{2\tau_0} = \int_0^\infty \psi_\parallel(\tau) \, d\tau ,
\]

and

\[
\frac{1}{2\tau_1} = \int_0^\infty \psi_\perp(\tau) \, d\tau .
\]

Thus, for times $t \gg \tau_0, \tau_1$, one can write the average value (2.6.24) as:

\[
\langle f(\mathbf{P},t) \rangle = \exp \left\{ t \left\{ \frac{L^2}{4\tau_1} + \left( \frac{1}{4\tau_0} - \frac{1}{4\tau_1} \right) L_z^2 \right\} \right\} f(\mathbf{P},0),
\]

where $L^2 = L_x^2 + L_y^2 + L_z^2$. (2.6.32) can be written in the polar coordinates:

\[
\frac{\partial f}{\partial t} = -G f,
\]

where

\[
G = \frac{1}{4\tau_1} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] + \left( \frac{1}{4\tau_0} - \frac{1}{4\tau_1} \right) \frac{\partial^2}{\partial \phi^2}.
\]

From (2.6.33) one can write down the equation for the average value of $\mathbf{P}$.
\[ \langle P \rangle = \int P f(P,t/P',0) f(P',0) \, dP \, dP', \quad \text{(2.6.35)} \]

as
\[ \frac{\partial \langle P \rangle}{\partial t} = -\langle G^+ P \rangle. \quad \text{(2.6.36)} \]

If the components of the vector \( P \) in the polar coordinates are \( P_z = P \cos \theta \), and \( P_{\pm} = P \sin \theta \, e^{\pm i \phi} \), then the equations for the average values of these components are:

\[ \frac{d\langle P_z \rangle}{dt} = -\frac{1}{2\tau_1} \langle P_z \rangle, \quad \text{(2.6.37a)} \]
\[ \frac{d\langle P_{\pm} \rangle}{dt} = - \left( \frac{1}{4\tau_1} + \frac{1}{4\tau_0} \right) \langle P_{\pm} \rangle. \quad \text{(2.6.37b)} \]

The solutions to the Bloch equations (2.6.37) are

\[ \langle P_z(t) \rangle = \langle P_z(0) \rangle \exp(-t/2\tau_1). \quad \text{(2.6.38a)} \]
\[ \langle P_{\pm}(t) \rangle = \langle P_{\pm}(0) \rangle \exp(-t/4\tau_1-t/4\tau_0). \quad \text{(2.6.38b)} \]

Equations (2.6.38) represent the decay of the off-diagonal elements of the density matrix \( \rho_{12}, \rho_{21} \) which correspond to \( P_{\pm} \) and the difference between the diagonal elements \( (\rho_{11} - \rho_{22}) \) corresponding to \( P_z \) over time scales that are characteristic of the fluctuating field.

Let us now get back to the problem of the E-P-R singlet state (2.6.9) where the two spins are separately interacting with the external fluctuating magnetic fields. We note that the pure state density matrix corresponding to the initial singlet state (2.6.1)
\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left\{ |\uparrow_{1n}\rangle \otimes |\downarrow_{2n}\rangle - |\downarrow_{1n}\rangle \otimes |\uparrow_{2n}\rangle \right\}. \tag{2.6.39}
\]

can be written in the following manner:

\[
\rho(0) = |\psi\rangle \langle \psi | = \frac{1}{2} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_1 \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_2 - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_1 \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_2 \right).
\tag{2.6.40}
\]

In terms of the Pauli spin operators \(\sigma^x, \sigma^y, \sigma^z\) and \(\sigma^\pm = \sigma^x \pm i\sigma^y\) for particles 1 and 2, (2.6.40) can be written as

\[
\rho(0) = \frac{1}{2} \left\{ \frac{(1 + \sigma^z_1)}{2} \frac{(1 - \sigma^z_2)}{2} + \frac{(1 - \sigma^z_1)}{2} \frac{(1 + \sigma^z_2)}{2} - \frac{\sigma^+_1 \sigma^-_2}{4} - \frac{\sigma^-_1 \sigma^+_2}{4} \right\} = \frac{1}{4} \left\{ 1 \otimes 1 - \sigma^+_1 \sigma^-_2 - \sigma^-_1 \sigma^+_2 \right\} = \frac{1}{4} \left\{ 1 \otimes 1 - \sigma_1 \cdot \sigma_2 \right\}. \tag{2.6.41}
\]

If we compare (2.6.41) with (2.6.13) for a single spin, (2.6.41) can be regarded as a superposition of three terms with three different initial values of the components of \(P\): (i) \(P^x_1 = 1, \, P^y_1 = P^z_1 = 0; \, P^x_2 = -1, \, P^y_2 = P^z_2 = 0\); (ii) \(P^y_1 = 1, \, P^x_1 = P^z_1 = 0; \, P^y_2 = -1, \, P^x_2 = P^z_2 = 0\); (iii) \(P^z_1 = 1, \, P^x_1 = P^y_1 = 0; \, P^z_2 = -1, \, P^x_2 = P^y_2 = 0\). Thus, solving for the time evolution of \(\rho\) involves solving for the various components of \(P\) for each of the spins with these initial conditions separately and substituting into (2.6.41). In this way the singlet
initial condition is incorporated with the interactions of the individual spins with their respective environments. For the calculation of Bell inequalities one is required to look at joint probability distributions for the spin components of 1 and 2 oriented along different directions. Note that (2.6.41) is a specific case where both the spin components are being looked at along the z direction. For a general case where the component of 1 is being looked at along a direction \( \mathbf{n}_1 \) and that of 2 along \( \mathbf{n}_2 \) the pure state density matrix is:

\[
\rho(0) = \frac{1}{4} \left\{ 1 - \left\{ (\mathbf{n}_1 \cdot \mathbf{z}) \mathbf{n}_1 \cdot \sigma_1 - \mathbf{n}_1 \times (\mathbf{n}_1 \times \mathbf{z}) \cdot \sigma_1 \right\} \cdot \left\{ (\mathbf{n}_2 \cdot \mathbf{z}) \mathbf{n}_2 \cdot \sigma_2 \\
- \mathbf{n}_2 \times (\mathbf{n}_2 \times \mathbf{z}) \cdot \sigma_2 \right\} - \left\{ (\mathbf{n}_1 \cdot \mathbf{x}) \mathbf{n}_1 \cdot \sigma_1 - \mathbf{n}_1 \times (\mathbf{n}_1 \times \mathbf{x}) \cdot \sigma_1 \right\} \cdot \left\{ (\mathbf{n}_2 \cdot \mathbf{x}) \mathbf{n}_2 \cdot \sigma_2 - \mathbf{n}_2 \times (\mathbf{n}_2 \times \mathbf{x}) \cdot \sigma_2 \right\} \\
- \mathbf{n}_1 \times (\mathbf{n}_1 \times \mathbf{y}) \cdot \sigma_1 \right\} \cdot \left\{ (\mathbf{n}_2 \cdot \mathbf{y}) \mathbf{n}_2 \cdot \sigma_2 - \mathbf{n}_2 \times (\mathbf{n}_2 \times \mathbf{y}) \cdot \sigma_2 \right\} \right\}. \tag{2.6.42}
\]

One can write the expectation value:

\[
\langle \sigma_1 \cdot \mathbf{n}_1 \; \sigma_2 \cdot \mathbf{n}_2 \rangle = \text{Trace} \left( \rho(0) \sigma_1 \cdot \mathbf{n}_1 \; \sigma_2 \cdot \mathbf{n}_2 \right) = -\cos \theta, \tag{2.6.43}
\]

where \( \theta \) is the angle between \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \). If \( \mathbf{n}_1 = \mathbf{n}_2 \), \( \langle \sigma_1 \cdot \mathbf{n}_1 \; \sigma_2 \cdot \mathbf{n}_2 \rangle = -1 \), as expected. Substituting the solutions for the Bloch equations for each term of (2.6.41) with the initial conditions described above, one can easily see that the time dependent solution is:

\[
\rho(t) = \frac{1}{4} \left\{ 1 - \left\{ (\mathbf{n}_1 \cdot \mathbf{z}) \mathbf{n}_1 \cdot \sigma_1 - \mathbf{n}_1 \times (\mathbf{n}_1 \times \mathbf{z}) \cdot \sigma_1 \right\} \cdot \left\{ (\mathbf{n}_2 \cdot \mathbf{z}) \mathbf{n}_2 \cdot \sigma_2 \\
- \mathbf{n}_2 \times (\mathbf{n}_2 \times \mathbf{z}) \cdot \sigma_2 \right\} \cdot \exp \left( -Ut_0^{(1)} - Ut_1^{(1)} - Ut_0^{(2)} - Ut_1^{(2)} \right) - \right\}.
\]
\[
\begin{aligned}
\{ (n_1 \cdot x) n_1 \sigma_1 - n_1 \times (n_1 \times x) \sigma_1 \} \ast \{ (n_2 \cdot x) n_2 \sigma_2 - \\
n_2 \times (n_2 \times x) \sigma_2 \} \ast \exp \left( - \frac{t}{4 \tau_0^{(1)}} - \frac{t}{4 \tau_1^{(1)}} - \frac{t}{4 \tau_2^{(1)}} - \frac{t}{4 \tau_1^{(2)}} \right) - \\
\{ (n_1 \cdot y) n_1 \sigma_1 - n_1 \times (n_1 \times y) \sigma_1 \} \ast (n_2 \cdot y) n_2 \sigma_2 \\
- n_2 \times (n_2 \times y) \sigma_2 \} \ast \exp \left( - \frac{t}{2 \tau_0^{(1)}} - \frac{t}{2 \tau_0^{(2)}} \right),
\end{aligned}
\]

where \( \tau_0^{(1)}, \tau_0^{(2)}, \tau_1^{(1)}, \tau_1^{(2)} \) are the characteristic time scales (2.6.30) and (2.6.31) for the particles 1 and 2. For simplicity, if we assume that all the time scales are the same, i.e., \( \tau_0^{(1)} = \tau_0^{(2)} = \tau_1^{(1)} = \tau_1^{(2)} = \tau_s \), then one can easily see that the expectation value (2.6.43) becomes:

\[
\left< \sigma_1 n_1 \sigma_2 n_2 \right> = \text{Trace} \left( \rho(t) \sigma_1 n_1 \sigma_2 n_2 \right) = -\cos \theta \exp(-t/\tau_s),
\]

transforming the Bell inequality (2.6.4) to

\[
|E(a,b) - E(a,c)| \exp(-t/\tau_s) \leq 1 + E(b,c) \exp(-t/\tau_s).
\]

As mentioned before, at \( t = 0 \) the inequality is violated for the choice of angles \( \theta(a,b) = \theta(b,c) = \pi/3 \) and \( \theta(a,c) = 2\pi/3 \). At \( t \to \infty \) one can see that the classical inequality is restored irrespective of the choice of angles. The system goes from a violation of the inequality to a nonviolation at time

\[
t = t_{c1} = -\tau_s \ln(2/3).
\]

One can find the corresponding spatial separation between the wave packets describing the two particles. Although the space and spin parts of the problem are not coupled, one can see that there is a direct correlation between the loss of quantum correlations (nonviolation of Bell inequalities) and the spatial separation of the spin carrying particles. At time \( t = t_{c1} \), when the classical inequality is restored, the separation
between the two particles (i.e., between the centres of the wave packets) is

\[ \frac{2\epsilon}{m\gamma^2} (1 - \gamma t_{c1}) \]

where \( t_{c1} \) is characteristic of the fluctuating field.

To summarize, in this Section we have studied the effect of the environment on an E-P-R setup. To simulate an E-P-R setup, we assume that the two particles of the E-P-R singlet are acted upon by forces which move them in opposite directions. We find that environmental influence causes decoherence in the quantum correlations of the initial singlet state. We have also tried to correlate this loss of coherence to the spatial separation between the two particles which are additionally coupled to an environment through the Caldeira-Leggett type of dissipative coupling. The time scales over which coherences are lost are characteristic of the environment.

2.VII. Summary

In this Chapter, we started our investigation with the query about the nature of collapse of the wavefunction of a quantum mechanical system when it interacts with an apparatus to yield a certain measurement and the subsequent emergence of classicality. Our analysis in Sections II and III of two specific apparatus models reveals some important aspects of the measurement process. A full quantum mechanical treatment of Zurek's model in Section II shows that although there is dephasing of the apparatus states due to dissipative coupling with the environment, there is no one-to-one correlation between the states of the system and that of the apparatus, and hence no measurement. This result is interesting because it suggests that for a measurement to take place it is not enough to have an apparatus coupled to a
large environment as it does not ensure correlations between the system and the apparatus states in the reduced system-apparatus density matrix. The reason for the absence of a one-to-one correlation can be traced back to time-dependent state vector of the combined system-apparatus, where the system-apparatus correlations are oscillatory in nature. We feel that the reason for this feature is the absence of a classical limit for the apparatus in the sense of the correspondence principle.

In Section III we analyse a model of a Stern-Gerlach apparatus, where a spin-1/2 particle is in an inhomogeneous magnetic field, the whole set up being in contact with a large environment. Here we find that the density matrix of the combined system and apparatus becomes diagonal and the momentum of the particle becomes correlated with a spin operator, selected by the choice of the system-apparatus interaction. This allows for a measurement of spin via a momentum measurement on the particle with associated probabilities in accordance with quantum principles. The analysis also brings out the interesting feature of the emergence of momentum as the preferred basis which is contrary to general expectations since the coupling to the environment for the particle is via the position coordinate. When compared with the apparatus model in Section II, the measuring variable, in this case the particle momentum, has a well defined classical distribution and hence justifies the success of this model. Our calculations thus show that a measurement is effected when a quantum system interacts with another macroscopic quantum system which is interacting dissipatively with an environment. This conclusion is much on the lines of the Copenhagen Interpretation which requires the measuring apparatus to be necessarily classical (macroscopic).
In this sense we can justify the concept of Bohr and von Neumann that a measurement requires the interaction of a quantum system with a classical system.

In Section IV we have looked at the effect of decoherence on a Stern-Gerlach recombination setup. We see that the environmental influence destroys coherence so that the spin polarization of a beam of particles in an initial x-polarized spin state when split and recombined in a Stern-Gerlach recombination setup cannot be recovered. This can be experimentally tested in a real Stern-Gerlach experiment and the loss of coherence can be studied with respect to the various parameters of the external environment.

In Section V we have examined the role of the preferred basis in a quantum measurement process by looking at the problem of a free particle coupled to a heat bath through the Caldeira-Leggett type of dissipative coupling which is via its position coordinates. It is intuitive to expect the position basis to emerge as the preferred basis for such a situation and it has also been shown by Zurek (1986) that environmental influence singles out position as the special basis. Our analysis, however, clearly shows that it is the momentum basis that emerges as the preferred basis in the sense that the density matrix of the system is driven completely diagonal in this basis. We show that the position space density matrix continues to remain nonlocal to the extent of the thermal de-Broglie wavelength of the particle. This result is interesting in the sense that it suggests that in the emergence of a preferred basis in a measurement process the form of the interaction with the environment requires careful considerations and that the emergent basis
need not necessarily be that of the variable that couples to the environment. In this case, as we have seen, it is actually the conjugate variable, i.e., the particle momentum.

In Section VI we have looked at the problem of environmental influence on the quantum correlations manifested in a singlet state in an E-P-R type experiment. In our model the spin-1/2 particles are coupled to a fluctuating magnetic field through their spin degrees of freedom and to a bath of harmonic oscillators through their positional degrees of freedom. Our calculations show that environmental influence destroys these quantum correlations over a time scale which is characteristic of the environment. We have correlated this loss of quantum correlations of the E-P-R singlet state to a spatial separation between the two spin-1/2 particles.

To summarize, in this Chapter we have applied the environment-induced decoherence approach to study the transition from 'quantum' to 'classical' behavior in some specific systems. The environment plays a crucial role in bringing about the transition from quantum to classical as required in a quantum measurement process and also in explaining the absence of nonlocal correlations in the everyday world around us. There is little doubt that the real world is very much a classical world that emerges from an underlying quantum substrate. The validity of quantum mechanics too is beyond any doubt as proven by many spectacular observations of quantum effects in carefully designed experiments. These effects survive the essentially classical act of measurement and necessarily require a quantum mechanical analysis. The next Chapter in this thesis deals with the description of such quantum features of
the electromagnetic radiation through phase-space distribution functions and, in that context, the 'quantum-classical' connection.
Appendix

Equation (2.3.8) is equivalent to the following set of ordinary differential equations:

\[ \frac{dt}{ds} = 1, \quad (A1) \]

\[ \frac{dr}{ds} = \gamma (2r - r_Q), \quad (A2) \]

\[ \frac{d\rho_d}{ds} = -\rho_d \left\{ \frac{D \, r^2}{4h^2} \mp \frac{i\varepsilon r}{h} \right\}. \quad (A3) \]

The invariants of these orbits with respect to \( s \) are easily found to be

\[ I_1 = (r-r_Q)e^{-\gamma t}, \quad (A4) \]

and

\[ I_2 = \rho_d \exp \left\{ -\frac{D}{4h^2} \left[ r_Q^2 \, t + \frac{2r_Q}{\gamma} (r-r_Q) + \frac{(r-r_Q)^2}{2\gamma} \right] \pm \frac{i\varepsilon}{h} \left[ r_Q t + \frac{r-r_Q}{\gamma} \right] \right\}. \quad (A5) \]

Clearly, \( I_2 = w(I_1) \), where \( w \) is an arbitrary function. This enables us to write

\[ \rho_d(Q,r,t) = w(I_1) \exp \left\{ -\frac{D}{4h^2} \left[ r_Q^2 \, t + \frac{2r_Q}{\gamma} (r-r_Q) + \frac{(r-r_Q)^2}{2\gamma} \right] \right\} \]
\[ w(I_1) \text{ is now determined from the initial condition (2.3.10) for } \rho_d(Q,r,0). \]

One can easily see that

\[ w(I_1) = w((r-r_Q)e^{-\gamma t}) = \exp \left( i \frac{\gamma}{2} (r-r_Q)e^{-\gamma t} - \frac{1}{4\sigma^2} (r-r_Q)^2 e^{-2\gamma t} \right), \]

\[ + \frac{(r-r_Q)^2 e^{-2\gamma t}}{2\gamma} \pm i \frac{\epsilon}{h} (r-r_Q)e^{-\gamma t}. \]

Substituting this in (A6) gives the result (2.3.11) for \( \rho_d(Q,r,\theta) \). To solve (2.3.9) we first make the transformation

\[ \rho_{od} = W \exp (\mp 2i\lambda \theta) \]

The equation for \( \omega(Q,r,\theta) \) is now equivalent to the following set of differential equations:

\[ \frac{dt}{ds} = 1, \]

\[ \frac{dr}{ds} = \gamma (r-r_Q), \]

\[ \frac{dQ}{ds} = \pm 2\epsilon / h. \]
\[ \frac{dW}{ds} = \frac{-D r^2}{4\hbar^2} W. \] (A12)

The invariants for this set of equations are:

\[ I_1 = \frac{h}{m\gamma} \left( Q \pm \frac{2\epsilon t}{\hbar} \right) \] (A13)

\[ I_2 = (r - \frac{hQ/\gamma m}{2\epsilon/m\gamma^2}) e^{-\gamma t} \] (A14)

The third invariant from (A11) is obviously a function of \( I_1 \) and \( I_2 \), hence,

\[
W(Q,r,\theta) = f(I_1, I_2) \exp \left\{ -\frac{D}{4\hbar^2} \left( (I_1 \mp \frac{2\epsilon}{m\gamma})^2 t + 2(I_1 \mp \frac{2\epsilon}{m\gamma^2})(\frac{1}{\gamma} I_2 e^{\gamma t} \right) \right. \\
\left. + \frac{\epsilon t^2}{m\gamma} + \frac{I_2^2 e^{2\gamma t}}{2\gamma} + \frac{4I_2}{m\gamma} e^{\gamma t} (\gamma t - 1) + \frac{4\epsilon^2 t^3}{3m^2\gamma^2} \right\} 
\] (A15)

\( f(I_1, I_2) \) can now be easily determined from the initial condition (2.3.10) for \( \rho(Q, r, 0) \):

\[
\rho(I_1, I_2) = \exp \left\{ -\frac{(Q \pm \frac{2\epsilon t}{\hbar})^2}{4\sigma^2} - \frac{1}{4\sigma^2} \left( (\frac{hQ}{m\gamma} \mp \frac{2\epsilon}{m\gamma^2}) (1 - e^{-\gamma t}) + e^{-\gamma t} \right) \right. \\
\left. \pm \frac{2\epsilon t}{\gamma} \right\}^2 + i \sqrt{\rho} \left( (\frac{hQ}{m\gamma} \mp \frac{2\epsilon}{m\gamma^2}) (1 - e^{-\gamma t}) + e^{-\gamma t} \right) 
\]

\[
x \exp \left\{ \frac{D}{2\hbar^2 \gamma} \left( (\frac{hQ}{m\gamma} \mp \frac{2\epsilon t}{m\gamma} \mp \frac{2\epsilon}{m\gamma^2}) (r - \frac{hQ}{m\gamma} \pm \frac{2\epsilon}{m\gamma^2}) e^{-\gamma t} \right) \right. \\
\left. \right. 
\]
Substituting for $f(1,1_2)$ in (A15) gives the result of (2.3.21).
References


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