CHAPTER 3

WIGNER FUNCTION DESCRIPTION OF
QUANTUM MECHANICAL NONLOCALITY

3.1. Introduction

In the field of quantum optics, contrary to popular belief, it is possible to derive correct results for most observed phenomena involving light and its interaction with atoms from a classical theory of electromagnetic waves. However, experiments performed on a restricted class of optical fields may show exclusively quantum effects that cannot be envisaged in the classical electromagnetic theory. Characteristics of antibunching (Teich and Saleh 1988, Walls 1979), sub-Poissonian photon statistics (Mandel 1979), squeezing (Walls 1983), quantum interference (Mandel 1983, Ghosh et al 1986, Ghosh and Mandel 1987), and spontaneous emission are fundamentally nonclassical. Nonlocal correlations demonstrated by the violation of classical inequalities in optical correlation experiments (Kocher and Commins 1967, Aspect et al 1981, 1982, Ou and Mandel 1988, Shih and Alley 1988) of the E-P-R kind are also specifically quantum in nature. In order to explain these effects one has to necessarily evoke the principles of quantum mechanics. Quantization introduces characteristic quantum features into the properties of the radiation field.

The electromagnetic field is quantized by the association of a quantum mechanical harmonic oscillator with each mode of the radiation field (Louisell 1973). A single mode of the electromagnetic field is described in terms of the
complex annihilation and creation operators \( \hat{a} \) and \( \hat{a}^\dagger \), which satisfy the commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\). The familiar quantization of the oscillator's total energy corresponds to having only an integral number of photons in the radiation mode (Louisell 1973). \( \hat{a} \) and \( \hat{a}^\dagger \) are related to the hermitian operators corresponding to the position \( (q) \) and momentum \( (p) \) observables of the oscillator. Most operators for the quantum field are often expressed in terms of \( \hat{a} \) and \( \hat{a}^\dagger \) (Cahill and Glauber 1968, 1969). A convenient tool in the quantum description of a light field is its photon probability distribution, or more generally its density operator, \( \hat{\rho} \). The usual way of calculating the statistical average of an observable \( \hat{A} \) is to evaluate the trace of the product of the operators \( \hat{\rho} \) and \( \hat{A} \): \( \langle \hat{A} \rangle = \text{Tr}[\hat{\rho} \hat{A}] \). It is often desirable to express these quantum mechanical ensemble averages in forms that offer the simplicity of classical ensemble averages. It is possible to express the density operator, \( \hat{\rho} \), in several representations, and for quantized fields the statistical averages for a physically important class of operators (e.g., electric field, intensity) can be written as integrals similar to the phase-space integrals of classical probability theory. The phase-space \( (q, p) \) of a system cannot have the same meaning in classical and in quantum mechanics. In the quantum case, one cannot represent a pure state of the system by a point in phase-space, since from Heisenberg's uncertainty principle, \( q \) and \( p \) cannot be known simultaneously with arbitrary precision. But the statistical representation of a many-body system is possible via a distribution function, in terms of which the formulation of quantum mechanics is identical to the classical one, but the physical interpretation of the distribution function cannot be the same as in classical mechanics. Classical probability distributions are positive definite and non-singular. However, for quantum
systems the functions which bear some resemblance to phase-space distributions, the 'quasiprobability' distribution functions, can be negative. They are useful calculational tools and they provide important insights into the connection between the classical and the quantum. Due to the noncommutativity of the operators \( \hat{a} \) and \( \hat{a}^\dagger \), the various quasiprobability distributions in quantum optics are used to calculate expectation values of products of operators depending on their rule of ordering (Cahill and Glauber 1968, 1969). One such representation of the density operator that is particularly suited for the averaging of operators written in symmetric order is in terms of the Wigner distribution function (Wigner 1932, Wigner and Moyal 1948). Other prominent quasiprobability distributions are the Q-function (Kano 1964, Mehta and Sudarshan 1965) and the Glauber-Sudarshan P-function (Glauber 1963, Sudarshan 1963).

Nonlinear optical processes have been known to generate fields with strong quantum features such as squeezing (Walls 1983) and antibunching (Mandel 1979, Teich and Saleh 1988). The strong correlation between photon pairs produced in nonlinear processes like parametric down-conversion and four-wave mixing has been used in several optical correlation experiments (Kocher and Commins 1967, Aspect et al 1981, 1982, Ou and Mandel 1988, Shih and Alley, 1988) to investigate the quantum mechanical paradox relating to Einstein locality. Each of these quantum phenomena is normally expressed by the violation of a particular inequality associated with the quantum state under consideration (viz., Fock state, squeezed state, superposition state etc.). Therefore, a natural question arises: Is it possible to obtain a general description, may be in the form of a generalized inequality, which
would encompass all the known nonclassical and nonlocal effects and
distinguish between the classical and the quantum fields in general? Fields
generated in many of these nonlinear processes (including those with losses)
have a Wigner distribution function which is Gaussian centered around the mean
value of the field (Agarwal and Adam 1988). In the following (Venugopalan and
Ghosh 1991, 1993) we show that the Bell inequality for an optical correlation
experiment with two coupled modes generated in a nonlinear process can be
expressed as an inequality relating to the parameters of the underlying Wigner
distribution function. The examples of parametric down-conversion (Agarwal and
Dutta Gupta 1989) and a two-photon squeezed laser (Ghosh and Agarwal 1989;
Agarwal and Ghosh 1994), which can be used in experiments to demonstrate the
quantum nonlocality, are considered. Similar inequalities involving the Wigner
parameters can also be written down for various other nonclassical effects,
such as squeezing, sub-Poissonian statistics, etc. We look for a possibility
of distinguishing a quantum system from a classical one in terms of the
underlying distribution function parameters for any general system.

The organization of this chapter is as follows. In Section II we
introduce the concept of distribution functions to describe the quantum state
of an electromagnetic field. In Section III various nonclassical and nonlocal
features of the electromagnetic field are discussed. In Section IV we discuss
how nonclassical light is generated by nonlinear optical processes. In Section
V we briefly introduce the master equation in quantum optics. In Section VI we
consider a Gaussian Wigner distribution function and show how various
inequalities corresponding to nonclassical effects can be interpreted in terms
of the Wigner parameters for specific systems. In Section VII, specific
examples of parametric down-conversion in a cavity and a two-photon squeezed laser are discussed. In each case, dynamical evolution of the system is given in terms of the density-matrix operator and the corresponding equation for the Wigner distribution function is shown to have the form of a linearized Fokker-Planck equation, the solution of which is Gaussian. In Section VIII the Bell inequality for a polarization correlation experiment is considered and transformed into an inequality involving the parameters of the Gaussian Wigner distribution function for the system. Finally, the results of this chapter are summarized in section IX.

3.II Distribution functions and quasiprobabilities.

Consider a particle in one dimension with its position denoted by \( q \) and its momentum by \( p \). Classically the particle is described statistically by a phase space distribution \( P_{c1}(q,p) \). The average of any function of the position and momentum, \( A(q,p) \) can then be expressed as

\[
\langle A \rangle_{c1} = \int dq \int dp \ A(q,p) P_{c1}(q,p).
\]  

Quantum mechanically, a particle can be described by a density matrix \( \hat{\rho} \). For a quantum electromagnetic field, complete information is contained in its density operator \( \hat{\rho} \) and information about the statistical properties of the field can be obtained from the moments of the field operators. The average of a function of the position and momentum operators \( \hat{A}(q,p) \) is
Given a classical expression $A(q,p)$, the corresponding self-adjoint operator $\hat{A}$ is not uniquely defined. However, it is possible to express the quantum mechanical average as

$$\langle \hat{A} \rangle_{\text{quant.}} = \text{Tr} [\hat{A} \hat{\rho}]. \quad (3.2.2)$$

Here the statistical information is transferred from the density operator $\hat{\rho}$ to a weight functional $P_Q(q,p)$. $P_Q(q,p)$ thus corresponds to $\hat{\rho}$ and $A(q,p)$ to the operator $\hat{A}$. The function $P_Q(q,p)$, in general, cannot be interpreted as a true probability distribution as it is not always positive definite and nonsingular, and hence is referred to as a 'quasiprobability' distribution (Wigner 1932, Cahill and Glauber 1968, 1969). The various procedures for expressing expectation values in the form (3.2.3) differ principally in the way in which the functions $P_Q(q,p)$ and $A(q,p)$ correspond to the operators $\hat{\rho}$ and $\hat{A}$. The various quasiprobability distribution functions are thus associated with specific operator ordering rules (Hillery et al 1984). Here the function $A(q,p)$, thus, can be derived from the operator $\hat{A}(p,q)$ by a well-defined mapping rule and this allows one to cast quantum mechanical results into a form in which they resemble classical ones. Thus, for quantum fields, quasiprobability distributions are important calculational tools for expressing the averages of operators (observables) in a fashion similar to the phase space integrals for classical systems. The use of quasiprobability distributions also enables us to examine dynamics in quantum optics in close analogy to classical statistical optics within the framework of pure quantum mechanics. A number of quasiprobability distributions are used in quantum
optics. Each of these distribution functions can be used to evaluate expectation values of operators ordered according to a certain rule. Since there are various possible correspondences between operators and functions, the choice of a distribution function for a particular problem is based on convenience. The most useful correspondences are based on three types of orderings: normal ordering, e.g., \( \hat{a}^+ \hat{a} \); antinormal ordering, e.g., \( \hat{a} \hat{a}^* \); and symmetric ordering, e.g. \( \frac{1}{2} (\hat{a}^+ \hat{a} + \hat{a} \hat{a}^*) \). Corresponding distribution functions are defined below.

The Glauber-Sudarshan \( P \)-function.

As mentioned above, a single mode of the quantized electromagnetic field can be described in terms of the annihilation and creation operators \( \hat{a} \) and \( \hat{a}^* \). A particularly useful complete (but non-orthogonal) set of states is of the coherent states \( \{ \alpha \} \), which are the eigenstates of the annihilation operator \( \hat{a} \) for each mode:

\[
\hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \tag{3.2.4}
\]

where \( \alpha \) is a complex number (Glauber 1963). The density operator of the electromagnetic field can be expressed in the diagonal coherent-state representation in the following way:

\[
\hat{\rho} = \int P(\alpha) |\alpha\rangle \langle \alpha| \ d^2(\alpha). \tag{3.2.5}
\]

Here \( |\alpha\rangle \) are the coherent states and \( P(|\alpha|) \) is some weight function or
phase-space density, and the integral is to be taken over all of complex $\alpha$-plane. If $P(|\alpha|)$ is not a true probability density, then the corresponding state is nonclassical. In general, $P(|\alpha|)$ can be negative and highly singular. $P(|\alpha|)$ is called the Glauber-Sudarshan $P$-function and is usually used to evaluate expectation values of operators which are normally ordered (e.g., $a^*a$) (Glauber 1963).

The $Q$-function.

When we are interested in calculating averages of operators which are antinormally ordered (e.g., $aa^*$), a distribution function analogous to the $P$-function can be obtained:

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{1} | \alpha \rangle$$

(3.2.6)

This is called the $Q$-function (Kano 1964, Mehta and Sudarshan 1965).

The Wigner function.

When we are interested in calculating the averages of operators that are symmetrically ordered (e.g. $(a^*a + aa^*)$), the phase-space distribution often used is the Wigner distribution function, which, for a single mode of an electromagnetic field is (Louisell 1973)

$$W(z,z^*) = \pi^{-4} \text{Tr} \left\{ \hat{\rho} \int d^2q \exp \left\{ -i \{ q z^* - \hat{a}^* - q^* (z - \hat{a}) \} \right\} \right\}.$$  

(3.2.7)

where the integral is over the entire complex $q$-plane.
The phase-space distributions discussed above, thus, provide a useful analogy between classical and quantum optics. It must be noted, however, that their behaviour is radically different for classical and quantum light fields. These distribution functions for classical light, e.g. coherent light or thermal light, are positive definite and nonsingular, i.e., true probability distribution functions. For a quantum field this is not necessarily the case. In general, there are two quite distinct ways to prescribe a quantum state of an electromagnetic field:

(i) If the average photon occupation number per mode is less than unity, then the field cannot be treated as classical for some purposes. This is the most familiar condition based on the correspondence principle.

(ii). If the Glauber-Sudarshan P-function is not a probability density, (i.e., it may be negative or/and highly singular), the state is nonclassical.

An optical field behaves as a classical wave field in all respects only when both conditions (i) and (ii) are violated. In the following Section we discuss some features which are manifestations of certain nonclassical states of light.

3. III. Quantum features of the electromagnetic field

Nonlocality

Realism and locality are assumed to be the basis of all classical theories and our perceptions of the world around us. Realism is a philosophical view which assumes the existence of an objective reality.
independent of whether a system is being observed or not. Locality assumes that forces or information can only travel between bodies at speeds less than or equal to that of light. When quantum theory was conceived and developed, it was tempting to expect it to be consistent with the concepts of realism and locality (Clauser and Shimony 1978). However, quantum mechanics rejects the idea that objective physical reality can be attributed to quantum systems and uses the wave function to predict only the relative probabilities of possible outcomes of a measurement on a system. This aspect has been explored in some detail in Chapter 2. The quantum description of certain correlated systems suggests that these correlations are not local. This aspect was first demonstrated by the celebrated E-P-R paradox (Einstein et al 1935) which deals with measurements on spatially separated correlated systems. According to the quantum theory measurements made on one of a pair of spin-1/2 particles (initially in a singlet state) affects the outcome of a measurement on the other even though the other particle is left undisturbed. The theory suggests the existence of 'action-at-a-distance' which is incompatible with locality. Nonlocality and the inability of quantum theory to predict with certainty the behaviour of a quantum system gave rise to a feeling of incompleteness and the need to search for alternative theories which can predict with certainty the outcome of a given experiment and are consistent with realism and locality. However, Bell's theorem (Bell 1965) ultimately proved that quantum theory makes certain predictions that are incompatible with any realistic, local, hidden variable theory. The inequalities derived by Bell and others (Bell 1965, Clauser and Shimony 1978) have been tested in several optical correlation experiments which have proven beyond doubt the existence of nonlocality - a uniquely quantum feature.
Sub-Poissonian statistics

Information about the statistical properties of a light field is often obtained by photon counting experiments. The beam of light under investigation is made to fall on a photoelectric detector connected by suitable electronics to a count meter which registers the number of photons falling on the phototube. The results are often expressed as a probability distribution $P_n(T)$ for the counting of $n$ photons during an observation time $T$. When completely coherent light falls on a photoelectric detector, the number of photoelectric counts, $n$, registered in some finite time interval obeys Poisson statistics for which the variance $\langle (\Delta n)^2 \rangle$ of $n$ equals the mean number, $n$. For classical waves, in general, $\langle (\Delta n)^2 \rangle > \langle n \rangle$, as a consequence of intensity fluctuations. However, there exist quantum states of the electromagnetic field for which the photon statistics is sub-Poissonian, i.e.,

$$\langle (\Delta n)^2 \rangle - \langle n \rangle < 0. \quad (3.3.1)$$

An example of such quantum states of the light field is that of the single-mode photon number states (Fock states). Nonclassical light fields with sub-Poissonian photon statistics have been observed experimentally (Short and Mandel 1983). These states have no classical description and one needs the quantized field approach to explain this phenomenon.

Squeezing

Classical light is often visualized as a smooth wave whose shape can be
described with absolute certainty. However, according to quantum theory, any coherent light field must be accompanied by a certain minimum amount of fluctuation in photon number which originates from the uncertainty principle. These 'vacuum fluctuations' cause ordinary light to be noisy and set a fundamental limit on the precision of optical interferometric measurements. The coherent state of the radiation field (Glauber 1963, Sudarshan 1963) is the closest counterpart to a classical electromagnetic field and is defined as that whose uncertainty product $\Delta E \Delta H$ for the electric and magnetic fields is minimum when subject to the simple harmonic potential characteristic of the field. The electric field associated with a single mode of angular frequency $\omega$ at a given position is

$$\hat{E}(t) = E_0 \left[ \hat{X}_1 \cos(\omega t) + \hat{X}_2 \sin(\omega t) \right], \quad (3.3.2)$$

where $E_0$ is a constant, and the quadrature field operators $\hat{X}_1 = (\hat{a}^* + \hat{a})$ and $\hat{X}_2 = i(\hat{a}^* - \hat{a})$ are analogous to the position and momentum operators of a simple harmonic oscillator with $[\hat{X}_1, \hat{X}_2] = 2i$. The fluctuations in $\hat{X}_1$, $\hat{X}_2$ obey the uncertainty relation:

$$\langle (\Delta \hat{X}_1)^2 \rangle \langle (\Delta \hat{X}_2)^2 \rangle \geq 1. \quad (3.3.3)$$

The equality sign holds for coherent states which are thus also known as the minimum uncertainty states for which

$$\langle (\Delta \hat{X}_1)^2 \rangle = \langle (\Delta \hat{X}_2)^2 \rangle = 1. \quad (3.3.4)$$

Squeezed states (Walls 1983) of the electromagnetic field are a unique set of
quantum states (which may or may not be minimum uncertainty states) with less fluctuation in one quadrature phase at the expense of increased fluctuations in the other phase, i.e.,

\[ \langle (\Delta \hat{X}_i)^2 \rangle < 1, \quad i = 1 \text{ or } 2. \] (3.3.5)

Since these states are generated by the redistribution of noise in the quantum mechanical vacuum, squeezing is a specifically quantum feature which has no classical counterpart. Squeezed states were first demonstrated in an experiment using degenerate four-wave mixing (Slusher et al 1985).

**Antibunching**

In a photon counting experiment photon bunching is the tendency of photons to distribute themselves preferentially in bunches with time. This means that when a light beam falls on a photon detector more photon pairs are detected close together in time than further apart. Antibunching is the opposite effect in which fewer photon pairs are detected closer together than further apart (Teich and Saleh 1988, Walls 1979, Zou and Mandel 1990). These effects are quantified via a second-order correlation function, which, for a single mode field can be defined as

\[ g^{(2)}(\tau) = \frac{\langle T: I(t)I(t+\tau): \rangle}{\langle I(t)\rangle \langle I(t+\tau) \rangle}, \] (3.3.6)

where \( I(t) \) is the intensity of the light field, \( T \) stands for time ordering and \( :: \) stands for normal-ordering of the operators. For classical light, \( g^{(2)}(\tau) \geq 1 \) (for thermal light \( g^{(2)}(0) = 2 \) and \( \lim_{\tau \to \infty} g^{(2)}(\tau) = 1 \), and for coherent light
as produced by an ideal laser, $g^{(2)}(\tau) = 1$ for all $\tau \geq 0$). For light which shows antibunching, $g^{(2)}(0) < g^{(2)}(\tau)$. This condition violates the Schwarz inequality for classical fields and thus antibunching is an effect which cannot be explained by classical electromagnetic theory. It is a quantum phenomenon and has been experimentally seen to be exhibited by nonclassical light fields produced by resonance fluorescence from a single two-level atom (Kimble et al 1977). For sub-Poissonian statistics, the condition is $g_2(0) < 1$. Under conditions of $g_2(\tau) < 1$, photon antibunching implies a sub-Poissonian distribution, but in general, one may occur without the other.

3.IV. Nonlinear optical processes for the generation of nonclassical light

Nonlinear optical processes provide versatile techniques for the generation of nonclassical light. Nonlinear effects are based on the fact that if the intensity of the incident light beam is large enough, then the optical properties of the medium (e.g., the refractive index) are modified by the incident field which in turn affect the propagation of the light beam in the medium. More specifically, for a nonlinear material subject to an electric field $E$, the susceptibility $\chi$ is field dependent and can be written as a power expansion in $E$. The induced polarization $P$ of the medium is then

$$P(E) = \varepsilon_0 \left( \chi^{(1)} E(\omega) + \chi^{(2)} E^2(\omega) + \chi^{(3)} E^3(\omega) + \ldots \ldots \right), \quad (3.4.1)$$

where $\varepsilon_0$ is the dielectric permittivity of vacuum and $\chi^{(1)}$ is the ith order susceptibility tensor characteristic of the medium. For usual 'linear optics', the higher order or nonlinear terms are much smaller than the linear term and
can be safely neglected whenever one is dealing with conventional light sources/materials. However, for more intense light (as generated by high power lasers) this may not be the case and the contributions from the nonlinear terms become significant. The power series expansion (3.4.1) of the polarization is essentially a perturbative approach to nonlinear optics. It is valid in the regime of weak nonlinearities only, when the electronic structure of the medium is not much perturbed by the applied field. This approach cannot be used in the regime of strong nonlinearity, which arises when field frequencies are very close to a narrow resonance and saturation of optical transitions occurs in the medium. The nonlinear terms in the interaction of light with atoms give rise to a variety of interesting phenomena. Nonlinear effects of self-focussing, self-trapping and self-induced transparency are observed when a single beam of light propagates in a nonlinear medium. Most other nonlinear phenomena require the simultaneous presence of a number of coherent beams in the medium. In the language of quantum optics the higher-order terms in these interactions correspond to the excitation of an atom by processes in which two or more photons are absorbed. In some of the nonclassical features we have discussed above, manifestations of the quantum nature of light are found in the statistical and coherence properties of the light. Many nonlinear effects are known to be phase dependent, i.e., they depend on the degree of coherence between the participating beams. Nonlinear processes, thus, can produce nonclassical effects which show up in the measurement of photon coincidences, or intensity correlations, within a single beam of light or between two light beams of different frequencies. In the following we briefly discuss some nonlinear optical processes that we shall consider later in this Chapter.
Parametric down-conversion and four-wave mixing processes

As mentioned above, most nonlinear phenomena require the simultaneous presence of a number of coherent beams in the medium. Just as the linear frequency-dependent susceptibility of a medium provides a complete description of the linear propagation of electromagnetic waves through a medium, the propagation of these waves through a medium in which nonlinear processes occur is describable in terms of the nonlinear susceptibility of the medium. The number of beams taking part in a nonlinear mixing interaction in a medium is the number of beams entering the medium plus the number of beams generated by the medium itself. The $\chi^{(2)}$ processes (i.e., involving the second-order susceptibility) are the three-wave mixing interactions of the type

$$\omega_1 + \omega_2 \rightarrow \omega$$

(3.4.2)

where $\omega$'s are the frequencies. Physically, this means that the two beams of different frequencies produce light of frequency which is the sum of the original frequencies and vice versa. For the degenerate case of $\omega_1 = \omega_2 = \omega$, this leads to the second-harmonic generation, $\omega + \omega \rightarrow 2\omega$. The reverse of second-harmonic generation is called the degenerate parametric down-conversion, in which one beam of frequency $2\omega$ splits into two beams of frequency $\omega$ each. This corresponds to the simultaneous production of two photons. These photon pairs are quantum-mechanically correlated or 'entangled' and hence can produce a number of nonlocal and nonclassical effects. Such $\chi^{(2)}$ processes occur in non-centrosymmetric media. For such interactions to take
place with appreciable probability, the phase-matching conditions (energy and momentum conservation laws) are to be satisfied. For centrosymmetric media, the second- and all even-order susceptibilities are identically zero. The terms involving the third-order susceptibility \( \chi^{(3)} \) correspond to four-wave mixing (FWM) interactions of the type

\[
\omega_3 + \omega_4 \leftrightarrow \omega_1 + \omega_2.
\]  

(3.4.3)

As before, the phase-matching conditions are to be satisfied. The photon pairs, again, are simultaneously produced and hence show nonclassical and nonlocal features.

It is now understood how different types of nonclassical states can be generated using a variety of nonlinear interactions both in resonant and nonresonant systems. Under the assumption of negligible depletion of the pump fields, so that pump beams can be treated classically, all nonlinear interactions generating two output beams (such as down-conversion and four-wave mixing) can be written in the parametric approximation as an effective bilinear interaction of the following second-quantized form

\[
H = G \hat{\alpha}_1^* \hat{\alpha}_2^* + H.c.,
\]  

(3.4.4)

where the coupling constant \( G \) is related to the nonlinear susceptibility for the process under consideration, and \( \hat{\alpha}_1^* \), \( \hat{\alpha}_2^* \) (\( \hat{\alpha}_1 \), \( \hat{\alpha}_2 \)) are the creation (annihilation) operators corresponding to the two output modes. The Hamiltonian \( H \) creates and destroys photons in pairs. The output fields are
taken to be initially in the vacuum state $|0,0\rangle$, and the state at time $t$ will then be a two-photon state $|\text{TP}\rangle$ given by

$$|\text{TP}\rangle = \exp (-iHT\hbar) |0,0\rangle.$$  \hspace{1cm} (3.4.5)

3.V The master equation in quantum optics

The master equation in classical stochastic modelling is seen as an integro-differential equation which gives the dynamics for a conditional probability. Methods of stochastic modelling and quantum statistical mechanics are often used to study interacting quantum systems. In quantum optics one is often considering a subsystem which is part of a larger closed system. Master equation methods are useful in dealing with such open systems where the system of interest is to be described by a mixture of state vectors. We have already seen their role in the context of the quantum measurement problem in Chapter 2. In Section II we have seen that it is possible to associate quasiprobability distribution functions with the density matrix $\hat{\rho}$, characterizing the state of a quantum system. These c-number distribution functions are not unique and are based on specific operator ordering rules. Given an equation for the density operator, $\hat{\rho}$, i.e., the Liouville equation, it is possible to write down the corresponding equation for the required quasiprobability distribution through specific mapping rules. For our purpose in this Chapter we will be directly dealing with the solutions of such equations.
3.VI The Wigner function

The Wigner function corresponding to the density matrix \( \hat{\rho} \) of a single mode of the radiation field (characterized by boson operators \( \hat{a} \) and \( \hat{a}^\dagger \)) is defined (Louisell 1973) as (see Eq. (3.2.7)):

\[
W(z,z^*) = \pi^{-4} \text{Tr} \left\{ \hat{\rho} \int d^2 q \exp \left\{ -[q^* x^* - \hat{a}^\dagger] - q^*(z - \hat{a}) \right\} \right\},
\]

(3.6.1)

where the integral is over the entire complex q-plane. Note that

\[
\int W(z,z^*) d^2 z = 1.
\]

(3.6.2)

The expectation value of an arbitrary operator \( \hat{O} \) can be written as

\[
\langle \hat{O} \rangle = \int W(z,z^*) O(z) d^2 z = \langle O(z) \rangle.
\]

(3.6.3)

where \( O(z) \) is obtained from (3.6.1) by replacing \( \hat{\rho} \) by \( \pi^\dagger \). Similarly for a two mode case with \( \hat{a} \) and \( \hat{b} \) as the two modes, the Wigner function is defined as:

\[
W(z_a,z_b) = \pi^{-4} \text{Tr} \left\{ \hat{\rho} \int \int d^2 p \ d^2 q \exp \left\{ -[p(z_a^* - \hat{a}^\dagger) - p^*(z_a - \hat{a})
\right.
\right.
\left.
\left. + q(z_b^* - \hat{b}^\dagger) - q^*(z_b - \hat{b})] \right\} \right\}.
\]

(3.6.4)

It has been shown (Agarwal and Adam 1988) that for fields generated in a large class of nonlinear optical processes (even in the presence of dissipation) the quantum state of the radiation field corresponds to a Gaussian Wigner function.
which is centered around the mean value of the field. Consider a field with Gaussian Wigner function:

\[
W(z,z^*) = \left(\frac{\pi}{\sigma^2 - 4|\mu|^2} \right)^{1/2} \exp \left\{ - \left( \frac{\mu(z-z_0)^2 + \mu^*(z^*-z_0)^2 + \sigma |z-z_0|^2}{\sigma^2 - 4|\mu|^2} \right) \right\},
\]

where

\[
\langle \hat{a} \rangle = z_0,
\]

\[
\langle \hat{a}^2 \rangle = -2\mu^* + z_0^2,
\]

\[
\langle \hat{a}^* \hat{a} \rangle = -2\mu + z_0^2,
\]

\[
\langle \hat{a} \hat{a} \rangle = \sigma - 1/2 + |z_0|^2,
\]

and

\[
\sigma > \mu + \mu^*. 
\]

The positive definiteness of the density matrix can be ensured through the following parameterization of \(\mu\) and \(\sigma\):

\[
\mu = \left(\frac{Q}{4}\right) \sinh x \exp(-i\theta),
\]

\[
\sigma = \left(\frac{Q}{2}\right) \cosh x,
\]

with the restriction that

\[
Q \geq 1.
\]

Fluctuations in the photon-number \(\hat{n} = \hat{a}^+ \hat{a}\) can be written in terms of the
Wigner parameters as

\[ (\Delta \hat{\Omega})^2 = \langle \hat{\Omega}^2 \rangle - \langle \hat{\Omega} \rangle^2 = \sigma^2 + 2 |z_0|^2 \sigma - 1/4 - 2z_0^* \mu^* - 2z_0 \mu + 4 |\mu|^2. \]  \hspace{1cm} (3.6.14)

For sub-Poissonian photon statistics,

\[ \langle (\Delta \hat{\Omega})^2 \rangle - \langle \hat{\Omega} \rangle < 0, \]  \hspace{1cm} (3.6.15)

which gives (Agarwal and Adam 1989)

\[ \sigma^2 + 2 |z_0|^2 \sigma + 1/4 - 2z_0^* \mu^* - 2z_0 \mu + 4 |\mu|^2 - 1 |z_0|^2 - \sigma < 0. \]  \hspace{1cm} (3.6.16)

Similarly, for super-Poissonian photon statistics, we get

\[ \sigma^2 + 2 |z_0|^2 \sigma + 1/4 - 2z_0^* \mu^* - 2z_0 \mu + 4 |\mu|^2 - 1 |z_0|^2 - \sigma > 0. \]  \hspace{1cm} (3.6.17)

Note that for \( z_0 = 0 \) (below threshold),

\[ \langle (\Delta \hat{\Omega})^2 \rangle - \langle \hat{\Omega} \rangle = (\sigma - 1/2)^2 + 4 |\mu|^2 \geq 0, \]  \hspace{1cm} (3.6.18)

and so the photon statistics cannot be sub-Poissonian. The condition for squeezing can also be expressed as an inequality involving the Wigner parameters. The component \( \hat{X}_1 = (\hat{a}^* + \hat{a}) \) will be squeezed if the inequality (3.3.5) holds with \( i = 1 \), i.e., if

\[ \langle \Delta (\hat{a}^* + \hat{a})^2 \rangle < 1. \]  \hspace{1cm} (3.6.19)
For a field described by a Gaussian Wigner function, this leads to the following condition (Agarwal 1987) on the parameters $\mu$, $\mu^*$ and $\sigma$ for the distribution (3.6.5):

$$0 < \sigma - \mu - \mu^* < 1/2 .$$

(3.6.20)

The above considerations can be easily generalized to the case of a multimode radiation field. Let $\psi$ be a 2N-component column matrix with

$$\psi_{2j} = z_j^*, \psi_{2j-1} = z_j, \ j = 1,2,...,N .$$

(3.6.21)

Then the joint (Gaussian) Wigner distribution function can be expressed as

$$W ( \{z\} ) = \frac{1}{\pi^N |\text{det}X|^{1/2}} \exp \left\{ -1/2 (\psi^* - \langle \psi \rangle^*) X^{-1} (\psi - \langle \psi \rangle) \right\} ,$$

(3.6.22)

where

$$X_{\alpha\beta} = \langle (\psi - \langle \psi \rangle)_{\alpha} (\psi - \langle \psi \rangle)_{\beta}^* \rangle .$$

(3.6.23)

It can be shown that

$$\langle \hat{a} \rangle = \langle |z_a|^2 \rangle - 1/2 .$$

(3.6.24a)

$$\langle \hat{a}^2 \rangle = \langle |z_a|^4 \rangle - \langle |z_a|^2 \rangle .$$

(3.6.24b)

3.VII. Specific examples of Gaussian Wigner function.

In the case of two-photon squeezed laser (Ghosh and Agarwal 1989, Agarwal and Ghosh 1994) and degenerate parametric down-conversion in a cavity
(Agarwal and Dutta Gupta 1989), the dynamical equation for the Wigner function is known to have the form of a linearized Fokker-Planck equation. The resulting solution is a Gaussian, given by (3.6.5). We quote below the results for these processes.

(a) Two-photon squeezed laser (Ghosh and Agarwal 1989). Let \( \omega \) be the frequency of the pump laser causing two-photon excitations in an active nonlinear medium placed inside a cavity, and \( \omega_a \) and \( \omega_b \) be the frequencies of the two radiation fields generated due to four-wave mixing. Let \( \hat{a} \) (\( \hat{a}^+ \)) and \( \hat{b} \) (\( \hat{b}^+ \)) be the annihilation (creation) operators for the fields at \( \omega_a \) and \( \omega_b \), respectively. Under the resonant condition \( 2\omega = \omega_a + \omega_b \), the dynamical evolution of the generated fields at frequencies \( \omega_a \) and \( \omega_b \) is described by the master equation for the field density matrix \( \rho \):

\[
\frac{\partial \rho}{\partial t} = iG(\hat{a}^+ \hat{b}^+ \rho - \hat{b}^+ \hat{a} \rho) - iG(\hat{a} \hat{b}^+ \rho - \hat{b} \hat{a}^+ \rho) - \frac{1}{2} K (\hat{a}^+ \hat{b}^+ \hat{a} \rho - \hat{b} \hat{a}^+ \hat{a} \rho - 2\hat{a}^+ \hat{b}^+ \hat{a} \rho) - \gamma_a (\hat{a}^+ \hat{a} \rho - \hat{a} \hat{a}^+ \rho) - \gamma_b (\hat{b}^+ \hat{b} \rho - \hat{b} \hat{b}^+ \rho) - \gamma_a (\hat{b} \hat{b}^+ \rho - \hat{b}^+ \hat{b} \rho).
\]

(3.7.1)

\( G \) is the nonlinear gain proportional to \( \chi^{(3)} \) [a redefinition of \( \hat{a} \) and \( \hat{b} \) eliminates the phase of \( G \) in (3.7.1)], \( K \) is the nonlinear absorption parameter, \( \gamma_a \) and \( \gamma_b \) are the linear loss coefficients arising because of possible leakage from the end mirrors of the cavity. Equation (3.7.1) is linearized in the vicinity of the steady-state solutions \( \alpha \) and \( \beta \) (real and positive) (set \( \hat{a} = \alpha + \hat{A} \), \( \hat{b} = \beta + \hat{B} \), \( \hat{A}/\alpha = \hat{B}/\beta \ll 1 \)) and converted into a differential equation for the Wigner function (Louisell 1973) \( W(z_a, z_B) \) by
using the rules of mapping (Agarwal 1986) associated with Weyl ordering of 
operators. The equation for \( W(z_A, z_B) \) becomes (\( \gamma \equiv \{ y_a y_b \} \))

\[
\frac{\partial W}{\partial t} = -\gamma \left\{ \frac{\partial}{\partial z_A} (z_B W) + z_B^* \frac{\partial W}{\partial z_A} \right\} + |G| (\frac{y_A}{y_B})^{1/2} \left\{ \frac{\partial}{\partial z_B} (z_A W) \right\} \\
+ |G| (\frac{y_A}{y_B})^{1/2} \left\{ \frac{\partial}{\partial z_A} (z_B W) \right\} + |G| - \gamma \left\{ \frac{\partial}{\partial z_A} (z_B W) + \frac{\partial}{\partial z_B} (z_A W) \right\} \\
+ 1/2 |G| (\frac{y_A}{y_B})^{1/2} \frac{\partial^2 W}{\partial z_A \partial z_B} + 1/2 |G| (\frac{y_A}{y_B})^{1/2} \frac{\partial^2 W}{\partial z_B \partial z_A} \\
+ 1/2 (|G| - \gamma) \left\{ \frac{\partial^2 W}{\partial z_A \partial z_B} + \frac{\partial^2 W}{\partial z_B \partial z_A} \right\} + \text{c.c.} \quad (3.7.2)
\]

Eq.(3.7.2) has the form of a linearized Fokker–Planck equation (Wang and 
Uhlenbeck 1945) in four variables whose solution is a Gaussian given by 
(3.6.22) with \( N=2 \).

(b) Degenerate parametric down-conversion in a cavity (Agarwal and Dutta Gupta
1989). The process of parametric down-conversion has been used in a number of 
recent correlation experiments demonstrating nonclassical (Ghosh and Mandel
1987) and nonlocal (Ou and Mandel 1988, Shih and Alley 1988) interference. Let 
the pump mode of frequency \( \omega_p = 2\omega_a \) generate two modes of frequencies \( \omega_a \)
each. Let \( \hat{\mathbf{p}} (\hat{\mathbf{p}}^+) \) and \( \hat{\mathbf{a}} (\hat{\mathbf{a}}^+) \) be the annihilation (creation) operators 
associated with the pump and generated modes, respectively. The Hamiltonian 
describing the down-conversion process can be written as :

\[
\hat{H} = \hbar \omega_a \hat{\mathbf{a}}^+ \hat{\mathbf{a}} + 2\hbar \omega_a \hat{\mathbf{p}}^+ \hat{\mathbf{p}} + \frac{1}{2} i \hbar (G \hat{\mathbf{a}}^+ \hat{\mathbf{p}} - \text{H.c.}) + i \hbar (\epsilon_p \hat{\mathbf{p}}^+ e^{-2i\omega_a t} - \text{H.c.}). \quad (3.7.3)
\]

\( G \) gives the magnitude of the parametric coupling and is proportional to \( \chi^{(2)} \);
\( \varepsilon_p \) denotes the coherent field driving the mode \( \hat{\beta} \). Let \( \gamma_a \) and \( \gamma_p \) be the decay rates associated with the two modes. The density matrix equation for the system can be written as

\[
\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}_p, \hat{\rho} \right] - \gamma_a (\hat{\alpha}^+ \hat{\alpha}^+ \hat{\rho} - 2\hat{\alpha}^+ \hat{\alpha}^+ \hat{\rho} \hat{\alpha}^+ \hat{\alpha}) - \gamma_p (\hat{\beta}^+ \hat{\beta}^+ \hat{\rho} - 2\hat{\beta}^+ \hat{\beta}^+ \hat{\rho} \hat{\beta}^+ \hat{\beta}).
\] (3.7.4)

After adiabatically eliminating the pump mode \( \hat{\beta} \) with the assumption that \( \gamma_p \gg \gamma_a \), and then linearizing the master equation around the steady-state value \( \langle \hat{\alpha} \rangle \) (setting \( \hat{\alpha} = \hat{\alpha} \langle \hat{\alpha} \rangle + \hat{\alpha} \langle \hat{\alpha} \rangle^\dagger \langle \hat{\alpha} \rangle^\dagger \approx 1 \)), the reduced density matrix equation for the mode \( \hat{\alpha} \) becomes

\[
\frac{\partial \hat{\rho}_{\alpha}}{\partial t} = -K (\hat{\alpha}^+ \hat{\alpha}^+ \hat{\rho}_{\alpha} - 2\hat{\alpha}^+ \hat{\alpha}^+ \hat{\rho}_{\alpha} \hat{\alpha}^+ \hat{\alpha}^+ + \hat{\alpha}^+ \hat{\alpha}^+ \hat{\rho}_{\alpha} \hat{\alpha}^+ \hat{\alpha}^+ \hat{\alpha}^+ \hat{\alpha}^+ ) - \frac{i}{\hbar} \left( \frac{1}{2} \hat{\alpha} \langle \hat{\alpha} \rangle \hat{\rho}_{\alpha} \hat{\rho}^\dagger + \text{H.c.}, \right), \hat{\rho}_{\alpha} \right),
\] (3.7.5)

where

\[
K = \gamma_a + |G|^2 \langle \hat{\alpha} \rangle^2 / \gamma_p,
\] (3.7.6)

\[
\gamma_p \langle \hat{\alpha} \rangle = \varepsilon_p - G^\dagger \langle \hat{\alpha} \rangle^2 / 2,
\] (3.7.7)

\[
\gamma_a \langle \hat{\alpha} \rangle = G \langle \hat{\alpha} \rangle^\dagger \langle \hat{\alpha} \rangle
\] (3.7.8)

Using the mapping for Weyl ordering (Agarwal 1986), (3.7.5) is transformed into the equation for the Wigner distribution function \( W_{\alpha}(z, z^*) \):

\[
\frac{\partial W_{\alpha}}{\partial t} = \frac{\partial}{\partial z^*} \left( Kz - \langle \hat{\alpha} \rangle G z^* W_{\alpha} \right) + \frac{\partial}{\partial z} \left( \frac{1}{2} K W_{\alpha} \right) + \text{c.c.}
\] (3.7.9)
The solution for $W_A(z,z')$ is a Gaussian Wigner function given by Eq.(3.6.5). The steady-state values of the fluctuation parameters are given as

$$\langle \hat{\mathcal{A}}^2 \rangle + 1/2 = \langle \hat{\mathcal{A}}^+ \hat{\mathcal{A}} \rangle - |\langle \hat{\mathcal{A}} \rangle|^2 + 1/2 = 1/2 \left( 1 - |\langle \hat{\mathcal{A}} \rangle / \langle \hat{\mathcal{A}} \rangle |^2 \right)^{-1} \quad (3.7.10)$$

$$\langle \hat{\mathcal{A}} \hat{\mathcal{A}} \rangle = \langle \hat{\mathcal{A}}^2 \rangle - \langle \hat{\mathcal{A}} \rangle^2 = G \langle \hat{\mathcal{A}} \rangle (1 - |\langle \hat{\mathcal{A}} \rangle / \langle \hat{\mathcal{A}} \rangle |^2)^{-1/2} \langle \hat{\mathcal{A}} \rangle \quad (3.7.11)$$

The values of the Wigner parameters in (3.6.5) are thus found to be

$$\sigma = (1 - |\langle \hat{\mathcal{A}} \rangle / \langle \hat{\mathcal{A}} \rangle |^2)^{-1/2} \quad (3.7.12a)$$

$$\mu = -(\langle \hat{\mathcal{A}} \rangle)^* (1 - |\langle \hat{\mathcal{A}} \rangle / \langle \hat{\mathcal{A}} \rangle |^2)^{-1/4} \langle \hat{\mathcal{A}} \rangle \quad (3.7.12b)$$

$$z_0 = \langle \hat{\mathcal{A}} \rangle \quad (3.7.12c)$$

$$\gamma_A \gamma_p \langle \hat{\mathcal{A}} \rangle = G \langle \hat{\mathcal{A}} \rangle (\varepsilon_p - 1/2 G^* \langle \hat{\mathcal{A}} \rangle^2) \quad (3.7.12d)$$

If one defines a parameter $\varepsilon = G \varepsilon_p / \gamma_A \gamma_p$, then for operation above threshold ($\varepsilon > 1$), one obtains from (3.7.12d)

$$|z_0|^2 = 2 |\gamma_A \gamma_p / G^2| (\varepsilon - 1) \quad (3.7.13a)$$

$$|G \langle \hat{\mathcal{A}} \rangle / \langle \hat{\mathcal{A}} \rangle | = 1 / (2 \varepsilon - 1) \quad (3.7.13b)$$

The phase of $z_0$ is given by
arg[Ge(z_0)^2] = 0. \hspace{1cm} (3.7.14)

Of course \( z_0 \) is zero below threshold \( (\epsilon < 1) \).

3.VIII. Bell's inequality for photon correlation measurements

It is now well established from several different photon polarization-correlation experiments (Ou and Mandel 1988, Shih and Alley 1988) that Einstein locality, as expressed by Bell's inequality (Bell 1965, Clauser and Shimony 1978) does not exist in nature. Consider a typical set up for such a polarization correlation experiment (see Fig.1). Let \( \hat{\alpha} \) and \( \hat{\beta} \) be two correlated modes with wave-vectors \( k_1 \) and \( k_2 \) coming out of a nonlinear material. These are made to fall from opposite sides on a beam-splitter (BS). \( \hat{\alpha} \) and \( \hat{\beta} \) are the mixed beams which arrive at the detectors \( (D_1 \) and \( D_2 ) \) placed at points \( r_1 \) and \( r_2 \) with two polarizers \( (P_1 \) and \( P_2 \) ) set at variable angles \( \theta_1 \) and \( \theta_2 \) in front of them, respectively. The Bell inequality in this case has the following well known form (Bell 1965, Clauser and Shimony 1978):

\[
S = R(\theta_1,\theta_2) - R(\theta_1,\theta_2') + R(\theta_1',\theta_2) + R(\theta_1',\theta_2') - R(\theta_1',-) - R(-,\theta_2) \leq 0.\hspace{1cm} (3.8.1)
\]

Here \( R(\theta_1,\theta_2) \) is the joint probability density of detecting two photons for polarizer settings of \( \theta_1 \) and \( \theta_2 \) measured by the coincidence counter. \( R(\theta_1,-) \) stands for the probability when the second polarizer is removed. Now the joint probability density of detection of two photons is given as:
Fig. 1. Schematic diagram of a photon correlation experiment for testing Bell's inequality.
\[ R(\theta_1, \theta_2) = K \langle \hat{\alpha}^* \hat{\alpha} \rangle, \]

where \( K \) is a constant characterizing the detectors. One may write the fields at the detectors as

\[ \hat{\alpha}(r_1, \theta_1) = X_a \hat{\alpha} + X_b \hat{\beta}, \]

\[ \hat{\beta}(r_2, \theta_2) = Y_a \hat{\alpha} + Y_b \hat{\beta}. \]

where

\[ |X_a|^2 + |Y_a|^2 = |X_b|^2 + |Y_b|^2 = 1. \]

For the correlation experiments (Ou and Mandel 1988, Shih and Alley 1988) with the down-converted signal and idler beams, where \( \hat{\alpha} \) and \( \hat{\beta} \) are \( x \)- and \( y \)-polarized, respectively,

\[ X_a = i \cos \theta_1 \sqrt{R_x} \exp(i k_{1x} r_1), \]

\[ X_b = \sin \theta_1 \sqrt{T_y} \exp(i k_{2y} r_1), \]

\[ Y_a = \cos \theta_2 \sqrt{T_x} \exp(i k_{1x} r_2), \]

\[ Y_b = -i \sin \theta_2 \sqrt{R_y} \exp(i k_{2y} r_2), \]
\[ R_x + T_x = R_y + T_y = 1. \] (3.8.6b)

Without explicitly specifying the state of the incident field at this stage, we assume that expectations of unpaired operators (e.g., of the form \( \langle \hat{a}^+ \hat{a}^+ \hat{b} \rangle \), \( \langle \hat{a}^+ \hat{b} \rangle \)) vanish. This assumption is justified when the phases of \( \hat{a} \) and \( \hat{b} \) are random and uncorrelated, and these modes therefore do not exhibit conventional (second-order) interference. For \( k_1 \) and \( k_2 \) parallel to \( k_1' \) and \( k_2' \), respectively, from (3.8.2) we get

\[
R(\theta_1, \theta_2) = K \left[ R_x T_x \cos^2 \theta_1 \cos^2 \theta_2 \langle \hat{a}^+ (\hat{a} - 1) \rangle + R_y T_y \sin^2 \theta_1 \sin^2 \theta_2 \langle \hat{b}^+ (\hat{b} - 1) \rangle \right. \\
+ \left. \left( \sqrt{T_x} \sqrt{T_y} \sin \theta_1 \cos \theta_2 + \sqrt{R_x} \sqrt{R_y} \cos \theta_1 \sin \theta_2 \right)^2 \langle \hat{a}^+ \hat{b} \rangle \right],
\] (3.8.7)

where \( \hat{a} \equiv \hat{a}^+ \hat{a} \) and \( \hat{b} \equiv \hat{b}^+ \hat{b} \) are the photon-number operators for the two beams incident on the beam-splitter. The probability density when the second polarizer is removed is calculated using unitarity:

\[
R(\theta_1, -) = R(\theta_1, \theta_2) + R(\theta_1, \theta_2 + \pi/2) = K_i R_x T_x \cos^2 \theta_1 \langle \hat{a} (\hat{a} - 1) \rangle \\
+ R_y T_y \sin^2 \theta_1 \langle \hat{b} (\hat{b} - 1) \rangle + (T_x T_y \sin \theta_1^2 + R_x R_y \cos \theta_1^2) \langle \hat{a} \hat{b} \rangle.
\] (3.8.8)

A similar expression can be obtained for \( R(-, \theta_2) \). For comparison of the different probabilities, all of them should be scaled by the joint probability density when both polarizers are removed:
Let \( R_x = T_x = R_y = T_y = 1/2 \), and choose angles \( \theta_1 = \pi/8, \theta_1' = 3\pi/8, \theta_2 = \pi/4, \theta_2' = 0 \).

Then from (3.8.1), we get

\[
4S/K = -0.85(\langle \hat{a}^\dagger \hat{a}^{-1} \rangle + \langle \hat{b}^\dagger \hat{b}^{-1} \rangle) + 0.41 \langle \hat{a} \hat{b} \rangle
\]  

(3.8.10)

The Bell inequality is violated whenever \( 4S/K > 0 \), i.e.,

\[
\frac{\langle \hat{a}^\dagger \hat{a}^{-1} \rangle + \langle \hat{b}^\dagger \hat{b}^{-1} \rangle}{\langle \hat{a} \hat{b} \rangle} < 0.48.
\]  

(3.8.11)

For optimum choice of angles, the right-hand-side of (3.8.11) can be made equal to 0.5. The inequality (3.8.11) can be expressed in terms of the joint Wigner function (3.6.22) of the fields at \( \hat{a} \) and \( \hat{b} \) \((N = 2)\). Making use of Eqs. (3.6.24), one gets

\[
\frac{\langle |z_a|^4 + |z_b|^4 - 2( |z_a|^2 + |z_b|^2 ) \rangle + 1}{(\langle |z_a|^2 \rangle -1/2)(\langle |z_b|^2 \rangle -1/2)} < 0.5.
\]  

(3.8.12)

It was shown by Chubarov and Nikolayev (CN) (1985) that the Bell inequality could be violated in polarization correlation experiments provided the photons of each polarization component are sufficiently sub-Poissonian in their statistics (condition (3.6.15) above). In the case treated by CN,
Hence the condition (3.8.11) for violation of the Bell inequality gives

\[
\left( \frac{\langle \Delta \hat{f} \rangle^2 - \bar{\hat{f}}}{\bar{\hat{f}}^2} \right) < -0.76, \tag{3.8.14}
\]

implying that when the Bell inequality is violated, the photon statistics is already sufficiently sub-Poissonian. This result is in agreement with CN. For the parametric down-conversion process, the photon statistics is nearly Poissonian with mean \( \bar{\hat{f}} \):

\[
\langle \hat{a} (\hat{a}^\dagger - 1) \rangle = \langle \hat{b} (\hat{b}^\dagger - 1) \rangle = \bar{\hat{f}}^2. \tag{3.8.15a}
\]

\[
\langle \hat{a}^\dagger \hat{b} \rangle = \langle \hat{f} \rangle + \langle \hat{f} \rangle^2. \tag{3.8.15b}
\]

Hence from (3.8.14) we get

\[
\frac{\langle \hat{f} \rangle^2}{\langle \hat{f} \rangle + \langle \hat{f} \rangle^2} < 0.24, \tag{3.8.16a}
\]

or

\[
\langle \hat{f} \rangle < 0.32. \tag{3.8.16b}
\]

Hence from (3.6.9), in terms of the Wigner parameters of the field given by
(3.7.13), the condition for violation of the Bell inequality in this case can be expressed as

$$\sigma + 1z_0^2 < 0.82.$$  

(3.8.17)

Once this condition holds, the fields are surely nonlocal. The beauty of this treatment is that one does not have to analyse the system concerned in great detail in order to see whether it leads to nonlocality or not – an examination of the underlying Wigner distribution function suffices.

3.IX. Summary

In line with our study of the ‘quantum-classical’ connection, in this Chapter we have looked at the quantum features of electromagnetic fields. In Chapter 2 our analysis of the quantum measurement problem dealt with explaining the emergence of classicality from an underlying quantum substrate. Although the real physical world is largely classical, quantum concepts may manifest themselves as nonclassical effects in carefully designed experiments. Till date, the most striking experiments that have probed the quantum nature of systems have involved quantum fields of light. Optical processes of resonance fluorescence and nonlinear processes like parametric down-conversion and four-wave mixing have been used in experimentally successful generation of nonclassical light. These phenomena are theoretically well-understood through field quantization and by application of quantum mechanical principles.

Several optical correlation experiments investigating the quantum
mechanical paradox relating to Einstein locality exhibit an explicit violation of the Bell inequality. It is our contention that instead of having different inequalities to describe different quantum features of the electromagnetic field, one can identify a generalized description pertaining to the basic definition of a quantum field, namely in terms of the nonclassical distribution function of the field. We have used the results of Agarwal and Adam (1988) regarding the Wigner distribution function generated in a large class of nonlinear processes producing correlated emissions, and have given a description of the quantum nonlocality (expressed by the violation of the Bell inequality) in terms of the corresponding Wigner parameters (see (3.8.12)). Other known nonclassical features of the electromagnetic field can also be described in terms of these parameters, (see (3.6.16), (3.6.20)) and thus such a description may be useful in providing a quantitative measure of the degree of nonclassicality of the radiation field. This brings out the 'quantum-classical' connections for these fields through the parameters of the phase-space distribution function.

It is interesting to note that the systems considered here may have losses, i.e., they are open interacting systems. The quantum effects, however, survive the act of measurement. The 'environment' for a typical system in quantum optics is the collection of vacuum modes. It is this coupling that is responsible for spontaneous emission processes in such systems. The role of spontaneous emissions in bringing about a measurement in an optical system is crucial and in the situations described here, measurements are shown to retain certain quantum correlations. In the absence of any losses, the 'twin' modes generated in a nonlinear process (such as parametric down-conversion or four-
wave mixing) are maximally correlated and lead to the maximum violation of the Bell inequality. However, a pure quantum state with perfect or maximum quantum correlation is not essential to disprove the local realistic theories. In the next Chapter we look at the quantum-classical connection through another interesting angle. We study a model of a classical optical experiment which displays features which are usually associated with a discrete two-level quantum system.
REFERENCES


