Chapter 5

Semi-perfect 1-factorization and Craft’s Conjecture

In this chapter, the existence of semi-perfect 1-factorization of complete bipartite graphs and $n$-cube $Q_n$ has been established.

5.1 Introduction and Preliminaries

For $n \geq 2$, let $G$ be an $n$-regular graph. A semi-perfect 1-factorization is a decomposition of a graph $G$ into distinct 1-factors $F_1, F_2, \ldots, F_n$ such that $F_i \cup F_j$, $2 \leq i \leq n$ forms a Hamilton cycle. In general, a 1-factorization $\{F_1, F_2, \ldots, F_n\}$ of $G$ is called $k$-semi-perfect if $F_i \cup F_j$ forms a Hamilton cycle for every $1 \leq i \leq k$ and every $k + 1 \leq j \leq n$. A 1-factorization is called perfect if the union of any two 1-factors is a Hamilton cycle.

The existence of perfect or semiperfect 1-factorization for various graphs $G$ have been extensively studied by many authors. A. Kotzig [58] conjectured that the complete graph $K_{2n}$ has a perfect 1-factorization for all $n \geq 2$. This is one of the outstanding open problems in graph theory. It has many applications, particularly in designing topologies for wireless communications. Kotzig’s
conjecture has been shown to hold when \( n \) is a prime, \((2n - 1)\) is a prime, or \( 2n \in \{16, 28, 36, 40, 50, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, 6860\} \) (the references can be seen in [71]). A weaker version of Kotzig’s conjecture raised by David Craft [29] in the case of semi-perfect 1-factorization is stated as follows:

**Conjecture** (D. Craft): There is a semiperm fact. 1-factorization of \( Q_n \), \( n \geq 2 \).

Královič and Královič [59] have shown that the graphs \( K_{2n} \), \( Q_{2n+1} \) and tori \( T_{2n \times 2n} \) admit a semi-perfect 1-factorization. Further, Craft [29] and Gochev and Gochev [44] have shown that \( Q_3 \), \( Q_4 \) and \( Q_6 \) admit a semi-perfect 1-factorization. Also, Gochev and Gochev [44] proved that for \( k = 1 \) or \( k = 5 \) and \( p \geq 1 \), there exists a \( k \)-semiperfect 1-factorization of \( Q_{k+2p} \); if \( k \geq 1 \) and \( p \geq 1 \) there exists a \( 2k \)-semiperfect 1-factorization of \( Q_{2k+2p} \).

In this chapter, the existence of semi-perfect 1-factorization of \( K_{n,n} \), \( n \geq 2 \) and \( Q_{2n} \), \( n \geq 1 \) has been established. As a consequence, Craft’s Conjecture has been settled affirmatively.

To prove the main results, we require the following:

**Definition 5.1.** [59]. Let \( P \) be a 1-factor and \( H_1, H_2, \ldots, H_k \) be Hamilton cycles of \( G \). A \( P \)-cover of \( G \) is a set \( \{H_1, H_2, \ldots, H_k\} \) such that \( H_i \cap P = P \) and every edge of \( G \setminus P \) is in exactly one of the \( H_i \)'s of \( G \).

**Observation 5.2.** [59]. It is clear that finding a \( P \)-cover \( \{H_1, H_2, \ldots, H_k\} \) for a given pair \( (G, P) \) gives the required semi-perfect 1-factorization of \( G \).

**Theorem 5.3.** (Walecki’s Construction [5]) The complete graph \( K_n \) has a Hamilton cycle decomposition for all \( n \geq 3 \).

**Corollary 5.4.** The symmetric digraph \( K_n^* \) has a directed Hamilton cycle decomposition for all odd \( n \geq 3 \).

**Theorem 5.5.** [85]. For \( 2m \geq 8 \), \( K_{2m} \) has a directed Hamilton cycle decomposition.
Theorem 5.6. [82]. For $m \geq 2$, $Q_{2m+1}^*$ has a directed Hamilton cycle decomposition.

In order to construct semi-perfect 1-factorization of complete bipartite graphs and hypercubes, the graph operations $O(G)$ and $RO(G)$ are defined as follows:

Definition 5.7. Let $\overline{G}$ be a directed graph. We obtain a new graph $O(\overline{G})$, by duplicating every vertex $i \in V(\overline{G})$ by a pair $(i_0, i_1)$ with an edge $i_0i_1$, and replacing every arc $ij \in E(\overline{G})$ by an edge $i_0j_1$. i.e. $O(\overline{G}) = (V', E')$ with $V' = \{i_0, i_1 / i \in V(\overline{G})\}$ and $E' = \{i_0i_1 / i \in V(\overline{G})\} \cup \{i_0j_1 / ij \in E(\overline{G})\}$.

Definition 5.8. Let $G$ be a simple bipartite graph with bipartition $X = \{1_0, 2_0, \ldots, n_0\}$, $Y = \{1_1, 2_1, \ldots, n_1\}$ containing the 1-factor $\{1_01_1, 2_02_1, \ldots, n_0n_1\}$ as a subset of $E(G)$. Define a new digraph $RO(G)$ as follows: By contracting the edge $i_0i_1$ for each $i = 1, 2, \ldots, n$ and replacing each edge $i_0j_1$ by an arc $ij$, we get a new digraph $RO(G)$, i.e. $RO(G) = (V', E')$ be a digraph with $V' = \{i / i_0i_1 \in E(G)\}$ and $E' = \{ij / i_0j_1 \in E(G)\}$.

![Figure 5.1: Graph G, O(G) and RO(O(G))](image)

Observation 5.9. From the Definitions 5.7 and 5.8 one can observe the following:

(i) $O(K_n) \cong K_{n,n}$ and $RO(K_{n,n}) \cong K_n^*$

(ii) $O(Q^*_{n-1}) \cong Q_n$ and $RO(Q_n) \cong Q_{n-1}^*$, $n \geq 2$
5.2 Semi-perfect 1-factorization of $K_{n,n}$ and $Q_{2n}$

Lemma 5.10. For $n \geq 3$, the graph $G^t = O(C_n)$ admits a semi-perfect 1-factorization.

Proof. Let $C_n = (1, 2, \ldots, n)$ be a directed cycle on $n$ vertices. Then $V(G^t) = \{i_0, i_1 | i \in V(C_n)\}$ and $E(G^t) = \{i_0i_1, i \in V(C_n)\} \cup \{i_0(i+1) | i \in \{1, 2, \ldots, n\}\}$, the addition is taken modulo $n$. Let $P = \{i_0i_1, i \in V(C_n)\}$ be a 1-factor of $G^t$. By Definition 5.7, $G^t$ itself is a Hamilton cycle containing the 1-factor $P$, which is our required $P$-cover of $G^t$.

Lemma 5.11. Let $C_{2n} = (1_0, 1_1, 2_0, 2_1, 3_0, 3_1 \ldots n_0, n_1)$ be a cycle on $2n$ vertices. For $n \geq 2$, the graph $G^t = RO(C_{2n})$ is a directed Hamilton cycle on $n$ vertices.

Proof. Follows from the Definition 5.8.

Theorem 5.12. For all $n = 4, 6$, the graph $K_{n,n}$ admits a semi-perfect 1-factorization.

Proof. Let $n \geq 2$ and $n = 4, 6, V(K^t_n) = \{1, 2, \ldots, n\}$. We define $K_{n,n}$ as follows: $V(K_{n,n}) = \{i_0, i_1 | i \in V(K^t_n)\}$ and $E(K_{n,n}) = \{i_0i_1, i \in V(K^t_n)\} \cup \{i_0j_1, j \in V(K^t_n)\} \in E(K^t_n)$. Let $P = \{i_0i_1, i \in \{1, 2, \ldots, n\}\}$ be a 1-factor of $K_{n,n}$. Clearly, $K_{2,2}$ has a semi-perfect 1-factorization. By the Corollary 5.4 and Theorem 5.5, $K^t_n$ has a directed Hamilton cycle decomposition $\{H^t_1, H^t_2, \ldots, H^t_{n-1}\}$ (say). We shall construct a $P$-cover $\{H^t_1, H^t_2, \ldots, H^t_{n-1}\}$ of $K_{n,n}$ as follows: By Observation 5.9, we have

$$K_{n,n} = O(K^t_n) = O(\bar{H}_1 \oplus \bar{H}_2 \oplus \cdots \oplus \bar{H}_{n-1}) \cong (O(\bar{H}_1)|P) \oplus (O(\bar{H}_2)|P) \oplus \cdots \oplus (O(\bar{H}_{n-1})|P) \oplus P$$

Let $H^t_i = O(\bar{H}_i)$. By Lemma 5.10, each $H^t_i, i = 1, 2, \ldots, n-1$ is a Hamilton cycle containing the 1-factor $P$. Hence $\{H^t_1, H^t_2, \ldots, H^t_{n-1}\}$ is a $P$-cover of $K_{n,n}$, $n = 4, 6$. \hfill \Box
5.2. Semi-perfect 1-factorization of $K_{n,n}$ and $Q_{2n}$

**Note 5.13.** The complete bipartite graphs $K_{4,4}$ and $K_{6,6}$ does not have a semi-perfect 1-factorization. For, let $V(K_{4,4}) = \{i_0 i_1 / i \in \{1, 2, 3, 4\}\}$. Let $P = \{i_0 1, 2, 3, 4, 0 1\}$ be a 1-factor of $K_{4,4}$ and $\{H_1, H_2, H_3\}$ be a $P$-cover of $K_{4,4}$. By the Observation 5.9 and Lemma 5.11, $K^*_{4} \cong RO(K_{4,4})$ and $K^*_{4} \cong RO(H_1) \oplus RO(H_2) \oplus RO(H_3)$. That is, $K^*_{4}$ has a directed Hamilton cycle decomposition, which is a contradiction to Theorem 5.5. Similarly, one can prove that $K_{6,6}$ does not have a semi-perfect 1-factorization.

**Theorem 5.14.** The graph $Q_{2n}$, $n \geq 1$, admits a semi-perfect 1-factorization.

**Proof.** Let $V(Q_{2n}) = \{i_0 i_1 / i \in \{1, 2, 3, \ldots, 2^{2n-1}\}\}$ and $P = \{i_0 i_1 / i \in \{1, 2, \ldots, 2^{2n-1}\}\}$ be a 1-factor of $Q_{2n}$. Clearly, $Q_2$ and $Q_4$ have semi-perfect 1-factorization [44]. By Theorem 5.14, let $\{ \overrightarrow{H_1}, \overrightarrow{H_2}, \ldots, \overrightarrow{H_{2n-1}}\}$ be the directed Hamilton cycle decomposition of $Q_{2n-1}$, $n \geq 3$. Now we construct a $P$-cover $\{H'_1, H'_2, \ldots, H'_{2n-1}\}$ of $Q_{2n}$ as follows: By Observation 5.9, we have

$$Q_{2n} \cong O(Q'_{2n-1})$$

$$= O(\overrightarrow{H_1} \oplus \overrightarrow{H_2} \oplus \overrightarrow{H_3} \oplus \cdots \oplus \overrightarrow{H_{2n-1}})$$

$$\cong (O(\overrightarrow{H_1}) \mid P) \oplus (O(\overrightarrow{H_2}) \mid P) \oplus \cdots \oplus (O(\overrightarrow{H_{2n-1}}) \mid P) \oplus P.$$ 

Let $H'_i = O(\overrightarrow{H_i})$. By Lemma 5.10, each $H'_i, i = 1, 2, \ldots, 2n - 1$ is a Hamilton cycle containing the 1-factor $P$. Hence $\{H'_1, H'_2, \ldots, H'_{2n-1}\}$ is a $P$-cover of $Q_{2n}$. \hfill $\Box$

Hence Theorem 5.14 together with the results of Královič and Královič [59] completely settled Craft’s Conjecture affirmatively.

Two euler tours $T_1$ and $T_2$ are said to be compatible if no path of length 2 in $T_1$ is a path of length 2 in $T_2$ and vice versa.

Existence of perfect 1-factorization will provide a platform to find compatible euler tours in a given graph $G$ which in turn provides edge disjoint Hamilton cycles in $L(G)$, the line graph of $G$. This leads to the raise of the following.
Question: Does there exist a perfect 1-factorization in $K_m \otimes I_n$ when at least one of $m - 1$ and $n$ is even?

5.3 Conclusion

It is shown that $K_{n,n}$, $n \geq 2$ and $Q_{2n}$, $n \geq 3$ admits a semi-perfect 1-factorization. Královič and Královič [59] have shown that the graphs $K_{2n}$, $Q_{2n+1}$ and tori $T_{2n \times 2n}$ admit a semi-perfect 1-factorization. Further, Craft [29] and Gochev and Gochev [44] have shown that $Q_3$, $Q_4$ and $Q_6$ admit a semi-perfect 1-factorization. Hence the results obtained in this chapter together with the results of Královič and Královič [59], and Gochev and Gochev [44] completely settled Craft’s Conjecture affirmatively.