Chapter 4

\(C_{4p}\)-frame of Complete Multipartite Multigraphs

In this chapter, some necessary or sufficient conditions are established for the existence of a \(C_{2k}\)-frame in \((K_m \otimes I_n)(\lambda)\). In fact, the results obtained in this chapter provide a complete solution to the existence of \(C_{4p}\)-frame in \((K_m \otimes I_n)(\lambda)\), when \(p\) is a prime.

4.1 Introduction and Preliminary Results

A partial factor of \(K_m \otimes I_n\) is a factor of \((K_m \otimes I_n)\mid V_i\), for some \(i \in \{1, 2, \ldots, m\}\) and \(V_i\) is an \(i^{th}\) partite set of \(K_m \otimes I_n\). A partial \(H\)-factor of \(K_m \otimes I_n\) is a partial factor of \(K_m \otimes I_n\) such that each component of it is isomorphic to the subgraph \(H\). An \(H\)-frame of \(K_m \otimes I_n\) is a decomposition of \(K_m \otimes I_n\) into partial \(H\)-factors.

Heinrich et.al[48] proved that there exists a near \(C_m\)-factorization of \(K_{ms+1}(2)\) for all odd \(m \geq 3\) and all \(s \geq 1\). Burling and Heinrich [18] have shown that there exists a near \(C_{2m}\)-factorization of \(K_n(2)\) if and only if \(n \equiv 1 \pmod{2m}\). Stinson [81] proved that there exists a \(C_3\)-frame of \((K_m \otimes I_n)(\lambda)\) if and only
if \( m \geq 4 \), \( \lambda n \equiv 0 \pmod{2} \) and \( n(m - 1) \equiv 0 \pmod{3} \). Wang [92] established necessary and sufficient conditions for the existence of cube frames in complete multipartite multigraphs. In [22], Cao, et.al. proved that there exists a \( C_k \)-frame of \( (K_m \otimes I_n)(\lambda) \) for \( k \in \{4, 5, 6\} \) if and only if \( n(m - 1) \equiv 0 \pmod{k} \), \( \lambda n \equiv 0 \pmod{2} \), \( m \geq 3 \) when \( k \in \{4, 6\} \), \( m \geq 4 \) when \( k = 5 \) and \( (k, \lambda, n, m) = (6, 1, 6, 3) \). In [84], Tiemeyer also proved that there exists a \( C_4 \)-frame of \( (K_m \otimes I_n)(\lambda) \) if and only if \( n(m - 1) \equiv 0 \pmod{4} \), \( \lambda n \equiv 0 \pmod{2} \), \( m \geq 3 \).

To prove the main results we require the following:

**Theorem 4.1** ([54]). For \( t \geq 3 \), \( K_{t,t,t} \) has a \( C_t \)-factorization.

**Theorem 4.2.** [18] For \( n \geq 3 \), \( K_n(2) \) has a near \( C_{2k} \)-factorization if and only if \( n \equiv 1 \pmod{2k} \).

**Theorem 4.3.** [49] \( K_{m,n} \) has a \( C_k \)-factorization if and only if

1. \( m = n \equiv 0 \pmod{2} \)
2. \( k \equiv 0 \pmod{2} \geq 4 \), and
3. \( 2n \equiv 0 \pmod{k} \)

with precisely one exception, namely \( m = n = k = 6 \).

**Theorem 4.4.** [28] There exists a \( K_2 \)-frame of \( (K_m \otimes I_n)(\lambda) \) if and only if \( m \geq 3 \) and \( n(m - 1) \equiv 0 \pmod{2} \).

**Theorem 4.5** ([81]). There exists a \( C_3 \)-frame of \( (K_m \otimes I_n)(\lambda) \) if and only if \( m \geq 4 \), \( \lambda n \equiv 0 \pmod{2} \) and \( n(m - 1) \equiv 0 \pmod{3} \).

**Theorem 4.6.** [22] There exists a \( C_k \)-frame of \( (K_m \otimes I_n)(\lambda) \) for \( k \in \{4, 5, 6\} \) if and only if \( n(m - 1) \equiv 0 \pmod{k} \), \( \lambda n \equiv 0 \pmod{2} \), \( m \geq 3 \) when \( k \in \{4, 6\} \), \( m \geq 4 \) when \( k = 5 \), and \( (k, \lambda, n, m) = (6, 1, 6, 3) \).
Theorem 4.7. [60, 61] There exists a pair of mutually orthogonal latin squares of order n for every n = 2, 6.

Theorem 4.8. [50] Let G be a graph with chromatic number χ(G). Then

(i) G / G ⊗ I_m if χ(G) ≤ N(m) + 2 and

(ii) G \ G ⊗ I_m if χ(G) ≤ N(m) + 1,

where N(m) is the maximum number of pairwise orthogonal latin squares of order m.

4.2 C_{2k}-frame of K_m ⊗ I_n

This section deals with the existence of some general results on C_{2k}-frame of complete multipartite graph.

Lemma 4.9. If (K_m ⊗ I_n)(λ) has a C_{2k}-frame, then

(i) m ≥ 3

(ii) (m - 1)n ≡ 0 (mod 2k)

(iii) at least one of m and n must be even when λ is odd.

Proof. It is clear from the definition of C_{2k}-frame that m ≥ 3. As C_{2k}-frame is the edge disjoint union of partial C_{2k}-factors of (K_m ⊗ I_n)(λ), the number of vertices in (K_m ⊗ I_n)(λ) \ V_i must be divisible by 2k, so (m - 1)n ≡ 0 (mod 2k). Since each partial C_{2k}-factor of (K_m ⊗ I_n)(λ) consists of (m - 1)n edges, the number of partial C_{2k}-factors in (K_m ⊗ I_n)(λ) is

\[ \lambda \frac{m(m-1)}{2} n^2 = \frac{1}{2} \lambda m n \]

which implies that at least one of m, n must be even, if λ is odd. ∎
4.2. $C_{2k}$-frame of $K_m \otimes I_n$

Lemma 4.10. For any $s > 0$, if $K_m \otimes I_n$ has a $C_{2k}$-frame, then so does $K_m \otimes I_{ns}$.

Proof. Let $C = \{C_{2k}, C_{2k}, \ldots, C_{2k}\}$ be a $C_{2k}$-frame of $K_m \otimes I_n$, where each $C_{2k}$ is a partial $C_{2k}$-factor of $K_m \otimes I_n$.

By definition, we write

$$K_m \otimes I_{ns} = (K_m \otimes I_n) \otimes I_s = (C_k \otimes C_{2k} \otimes \cdots \otimes C_{2k}) \otimes I_s = (C_k \otimes I_s) \otimes (C_{2k} \otimes I_s) \otimes \cdots \otimes (C_{2k} \otimes I_s).$$

By Theorem 4.8, each $C_{2k} \otimes I_s$ has a $C_{2k}$-frame. This completes the proof. □

Lemma 4.11. For $n \geq 1$, if $K_{m+1} \otimes I_{2n}$ has a $C_{2k}$-frame and $K_{mn,mn}$ has a $C_{2k}$-factorization, then $K_{4m+1} \otimes I_{2n}$ has a $C_{2k}$-frame.

Proof. Let $V(K_{4m+1}) = \bigcup_{i=0}^{S} B_i \cup \{\infty\}$, where $B_i = \{mi + j, 1 \leq j \leq m\}$, $V(I_{2n}) = \bigcup_{j=0}^{S} D_j$, where $D_j = \{nq + j, 1 \leq q \leq n\}$ and $V(K_{4m+1} \otimes I_{2n}) = \bigcup_{i=0}^{S} B_i \cup \{\infty \times I_{2n}\}$, where $B_i = B_i \times D_j$.

Find a new graph $B$ from $K_{4m} \otimes I_{2n}$ by identifying each $B_i$, $i \in Z_4, l \in Z_2$ into a single vertex $b'_l$ and join two vertices $b'_l$ and $b'_s$ if there exists a complete bipartite graph $K_{B'_l, B'_s} \cong K_{mn,mn}$ between $B'_l$ and $B'_s$ in $K_{4m} \otimes I_{2n}$.

The resulting graph $B$ is isomorphic to $K_4 \otimes I_2$. We know by Theorem 4.4, $K_4 \otimes I_2$ has a $K_2$-frame $\{rF_0, rF_1, rF_2, rF_3, 1 \leq r \leq 2\}$ where $rF_i$ is an $r$th partial $K_2$-factor which does not contain the vertices of $i$th partite set of $K_4 \otimes I_2$, $i = 0, 1, 2, 3$. While we blow up, corresponding to each $\bigcup_{r=1}^{S} F_i$, $i = 0, 1, 2, 3$, we have $6K_{mn,mn} (\cong C_6 \otimes I_{mn})$ in $K_{4m} \otimes I_{2n}$ which does not contain the vertices of $K_{4m} \otimes I_{2n}$ corresponding to the $i$th partite set of $K_4 \otimes I_2$, which is nothing but $B_i \times I_{2n}$. Now the missing vertices $B_i \times I_{2n}$ of $K_{4m} \otimes I_{2n}$ along with $\infty \otimes I_{2n}$ form a $K_{B_i \cup \infty} \otimes I_{2n} \cong K_{m+1} \otimes I_{2n}$ in $K_{4m+1} \otimes I_{2n}$ which covers the rest of the vertices of $K_{4m+1} \otimes I_{2n}$ that are not covered by the above $6K_{mn,mn}$ correspond
to the partial $K_2$-factors $\mathcal{S} = \bigcap_{r=1}^{\infty} F_r$ of $K_4 \otimes I_2$.

By the hypothesis, $6K_{mn,mn}(= C_6 \otimes I_{mn})$ has a $C_{2k}$-factorization consisting $mn$ $C_{2k}$-factors and $K_{m+1} \otimes I_{2n}$ has a $C_{2k}$-frame consisting of $nm + n$ partial $C_{2k}$-factors. Note that the $mn$ $C_{2k}$-factors of $6K_{mn,mn}$ and $mn$ partial $C_{2k}$-factors of $K_{m+1} \otimes I_{2n}$ together give $mn$ partial $C_{2k}$-factors of $K_{4m+1} \otimes I_{2n}$. The remaining $n$ partial $C_{2k}$-factors of $K_{m+1} \otimes I_{2n}$ with missing partite set $\infty \times I_{2n}$ in $K_{4m+1} \otimes I_{2n}$ are kept aside for future purpose. So corresponding to each $F_r$ and missing part $i$ of $K_4 \otimes I_2$, $0 \leq i \leq 3$, we have $mn$ partial $C_{2k}$-factors of $K_{4m+1} \otimes I_{2n}$ and $n$ partial $C_{2k}$-factors of $K_{B_i \cup \infty} \otimes I_{2n} \cong K_{m+1} \otimes I_{2n}$ with missing partite set $\infty \times I_{2n}$ in $K_{4m+1} \otimes I_{2n}$.

Taking the union of all $n$ partial $C_{2k}$-factors of $K_{B_i \cup \infty} \otimes I_{2n}$ with missing partite set $\infty \times I_{2n}$, which are kept aside for future purpose, for each $i$, $0 \leq i \leq 3$, we have another $n$ partial $C_{2k}$-factors of $K_{4m+1} \otimes I_{2n}$. Totally we get $4mn + n$ partial $C_{2k}$-factors of $K_{4m+1} \otimes I_{2n}$ and hence a $C_{2k}$-frame exists. The above construction will be more clear from Figure 4.1(a)-(d).

![Figure 4.1: Construction of $C_{2k}$-frame of $K_{4m+1} \otimes I_{2n}$](image-url)
Lemma 4.12. For \( n \geq 1 \), if \( K_{m+1} \otimes I_{2n} \) has a \( C_{2k} \)-frame and \( K_{2mn,2mn} \) has a \( C_{2k} \)-factorization, then \( K_{mt+1} \otimes I_{2n} \) has a \( C_{2k} \)-frame, \( t \geq 3 \) is an odd integer.

Proof. Let \( V(K_{mt+1}) = \bigcup_{j=0}^{S} B_j \cup \{\infty\} \), where \( B_i = \{mi + j, 1 \leq j \leq m\} \). We obtain a new graph \( B \) from \( K_{mt} \), by identifying each \( B_i \) into a single vertex \( b_i \) and join two vertices \( b_i \) and \( b_j \) if the corresponding \( B_i \) and \( B_j \) form a \( K_{B_i,B_j} \) in \( K_{mt} \). Then the new graph \( B \) is isomorphic to \( K_{t} \). By Theorem 4.4, \( K_{t} \) has a near 1-factorization \( \{F_0, F_1, \ldots, F_{t-1}\} \), where \( F_i \) is a near 1-factor of \( K_{t} \) with missing vertex \( i \). While we blow up, each \( F_i \) of \( K_{t} \) will give to raise to \( (\frac{t-1}{2})K_{B_i,B_j} \) in \( K_{mt} \). So corresponding to each \( F_i \) of \( K_{t} \), we have \( (\frac{t-1}{2})K_{m,m} \otimes I_{2n} \) in \( K_{mt} \otimes I_{2n} \). Note that \( K_{m,m} \otimes I_{2n} \cong K_2 \otimes I_{2mn} \cong K_{2mn,2mn} \), i.e. corresponding to each \( F_i \) of \( K_{t} \), we have a \( (\frac{t-1}{2})K_{2mn,2mn} \) in \( K_{mt} \otimes I_{2n} \) which is a subgraph of \( K_{mt+1} \otimes I_{2n} \). Corresponding to the missing vertex \( i \) of \( F_i \) in \( K_{t} \), we have the \( B_i \) in \( K_{mt} \) and hence \( K_{B_i,\{\infty\}} \otimes I_{2n} \cong K_{m+1} \otimes I_{2n} \) covers the rest of the vertices of \( K_{mt+1} \otimes I_{2n} \) that are not covered by \( (\frac{t-1}{2})K_{m,m} \otimes I_{2n} \).

By the hypothesis, \( K_{2mn,2mn} \) has a \( C_{2k} \)-factorization consisting of \( mn \) \( C_{2k} \)-factors and \( K_{m+1} \otimes I_{2n} \) has a \( C_{2k} \)-frame consisting of \( mn + n \) partial \( C_{2k} \)-factors. Note that the \( mn \) \( C_{2k} \)-factors of \( (\frac{t-1}{2})K_{2mn,2mn} \) and \( mn \) partial \( C_{2k} \)-factors of \( K_{m+1} \otimes I_{2n} \) together give \( mn \) partial \( C_{2k} \)-factors of \( K_{mt+1} \otimes I_{2n} \). The remaining \( n \) partial \( C_{2k} \)-factors of \( K_{m+1} \otimes I_{2n} \) with missing partite set \( \infty \times I_{2n} \) are kept aside for future purpose. So corresponding to each \( F_i \) and the missing vertex \( i \) of \( K_{t} \), \( 0 \leq i \leq t-1 \), we have \( mn \) partial \( C_{2k} \)-factors of \( K_{mt+1} \otimes I_{2n} \) and \( n \) partial \( C_{2k} \)-factors of \( K_{B_i,\infty} \otimes I_{2n} \cong K_{m+1} \otimes I_{2n} \) with missing partite set \( \infty \times I_{2n} \) in \( K_{m+1} \otimes I_{2n} \).

By taking the union of all \( n \) partial \( C_{2k} \)-factors of \( K_{B_i,\infty} \otimes I_{2n} \) with missing partite set \( \infty \times I_{2n} \), which are kept aside for future purpose, for each \( i, 0 \leq i \leq t-1 \), we have another \( n \) partial \( C_{2k} \)-factors of \( K_{mt+1} \otimes I_{2n} \). Totally we get \( mnt + n \) partial \( C_{2k} \)-factors of \( K_{mt+1} \otimes I_{2n} \) and hence a \( C_{2k} \)-frame exists.

\[ \square \]
Lemma 4.13. For \( k \geq 2 \), the graph \( K_{2k+1} \otimes I_2 \) has a \( C_{2k} \)-frame.

Proof. Let \( V(K_{2k+1} \otimes I_2) = \{i_1, i_2; 0 \leq i \leq 2k-1\} \cup \{\infty_1, \infty_2\} \). Now we construct a \( C_{2k} \)-frame of \( K_{2k+1} \otimes I_2 \) in two cases as follows:

Case(i): \( k \) even.

For \( 1 \leq i \leq k \),

\[
C_{2k}^i = (i_1, (1+i)_1, (2k-1+i)_2, (2+i)_1, (2k-2+i)_2, (3+i)_1, \\
(2k-3+i)_2, (4+i)_1, (2k-4+i)_2, \ldots, (k-2+i)_1, (k+2+i)_2, \\
(k-1+i)_1, (k+1+i)_2, \infty_2)
\]

\[
(i_2, (1+i)_2, (2k-1+i)_1, (2+i)_2, (2k-2+i)_1, (3+i)_2, \\
(2k-3+i)_1, (4+i)_2, (2k-4+i)_1, \ldots, (k-2+i)_2, (k+2+i)_1, \\
(k-1+i)_2, (k+1+i)_1, \infty_1)
\]

For \( k+1 \leq i \leq 2k \),

\[
C_{2k}^i = (i_1, (1+i)_2, (2k-1+i)_2, (2+i)_2, (2k-2+i)_2, (3+i)_2, \\
(2k-3+i)_2, (4+i)_2, (2k-4+i)_2, \ldots, (k-2+i)_2, (k+2+i)_2, \\
(k-1+i)_2, (k+1+i)_2, \infty_2)
\]

\[
(i_2, (1+i)_1, (2k-1+i)_1, (2+i)_1, (2k-2+i)_1, (3+i)_1, \\
(2k-3+i)_1, (4+i)_1, (2k-4+i)_1, \ldots, (k-2+i)_1, (k+2+i)_1, \\
(k-1+i)_1, (k+1+i)_1, \infty_1) \text{ and }
\]

\[
C_{2k}^{2k+1} = (0_1, 1_1, 2_2, 3_1, 4_2, \ldots, k_2, (k+1)_2, (k+2)_1, \ldots, (2k-1)_1) \\
(0_2, 1_2, 2_1, 3_2, 4_1, 5_2, \ldots, k_1, (k+1)_2, (k+2)_2, \ldots, (2k-1)_2),
\]

where the additions are taken modulo \( 2k \).
4.2. $C_{2k}$-frame of $K_m \otimes I_n$

Case (ii): $k$ odd.

For $1 \leq i \leq 2k - 1$, $C_{2k}^i$ can be constructed as in case (i).

Let $C_{2k}^{2k} = (0_2, 1_2, (2k - 1)_2, 2_2, (2k - 2)_2, 3_2, (2k - 3)_2, \ldots, (k - 2)_2,$

\[(k + 2)_2, (k - 1)_2, (k + 1)_2, \infty_2) \]

\[(0_1, 1_1, (2k - 1)_1, 2_1, (2k - 2)_1, 3_1, (2k - 3)_1, \ldots, (k - 2)_1, \]

\[(k + 2)_1, (k - 1)_1, (k + 1)_1, \infty_1) \text{ and} \]

$C_{2k}^{2k+1} = (0_1, 2_1, 3_1, \ldots, (k - 1)_1, k_2, (k + 1)_2, (k + 2)_2, (k + 3)_2, \ldots,$

\[(2k - 2)_1, (2k - 1)_1) \]

\[(0_2, 1_2, 2_2, 3_2, \ldots, (k - 1)_2, k_1, (k + 1)_2, (k + 2)_2, (k + 3)_2, \ldots,

\[(2k - 2)_2, (2k - 1)_2). \]

Clearly, each $C_{2k}^i$ is a partial $C_{2k}$-factor of $K_{2k+1} \otimes I_2$. Hence $\{C_{2k}^i; 1 \leq i \leq 2k + 1\}$ gives a $C_{2k}$-frame of $K_{2k+1} \otimes I_2$ in both the cases. \qed

Lemma 4.14. For $k \equiv 1 \pmod{2}$, the graph $K_{2k+1} \otimes I_2$ has a $C_{4k}$-frame.

Proof. Let $V(K_{2k+1} \otimes I_2) = \{i_1, i_2; i \in Z_{2k}\} \cup \{\infty_1, \infty_2\}$. Now we construct a $C_{4k}$-frame of $K_{2k+1} \otimes I_2$ as follows: The construction is obvious when $k = 1$. So we consider $k \geq 3$.

For $i = 1, 3, 5, 7, \ldots, 2k - 1$,

$C_{4k}^i = (i_1, (1 + i)_1, (2k - 1 + i)_2, (2 + i)_2, (2k - 2 + i)_1, (3 + i)_1,$

\[(2k - 3 + i)_2, (4 + i)_2, (2k - 4 + i)_1, \ldots, (k - 2 + i)_1, (k + 2 + i)_2, \]

\[(k - 1 + i)_2, (k + 1 + i)_1, \infty_1, i_2, (1 + i)_2, (2k - 1 + i)_1, (2 + i)_1, \]

\[(2k - 2 + i)_2, (3 + i)_2, (2k - 3 + i)_1, (4 + i)_1, (2k - 4 + i)_2, \ldots, \]

\[(k - 2 + i)_2, (k + 2 + i)_1, (k - 1 + i)_1, (k + 1 + i)_2, \infty_2). \]
for \( i = 2, 4, 6, 8 \ldots 2k, \)

\[
C_{4k}^i = (i_2, (1 + i)_2, (2k - 1 + i)_1, (2 + i)_1, (2k - 2 + i)_2, (3 + i)_2, (2k - 3 + i)_1, (4 + i)_1, (2k - 4 + i)_2, \ldots, (k - 2 + i)_2, (k + 2 + i)_1, (k - 1 + i)_1, (k + 1 + i)_2, (2 + i)_2, (2k - 2 + i)_1, (3 + i)_1, (2k - 3 + i)_2, (4 + i)_2, (2k - 4 + i)_1, \ldots, (k - 2 + i)_1, (k + 2 + i)_2, (k - 1 + i)_2, (k + 1 + i)_1, \infty) \text{ and }
\]

\[
C_{4k}^{2k+1} = (0_1, 1_1, 2_2, 3_2, 4_1, \ldots, (k - 1)_1, k_1, (k + 1)_2, (k + 2)_2, \ldots, (2k - 1)_1, 0_2, 1_2, 2_1, 3_1, 4_2, 5_2, \ldots, (k - 1)_2, k_2, (k + 1)_1, (k + 2)_1, \ldots, (2k - 1)_2),
\]

where the additions are taken modulo 2k.

Clearly, each \( C_{4k}^i, 1 \leq j \leq 2k + 1 \) is a partial \( C_{4k} \)-factor of \( K_{2k+1} \otimes I_2 \). Hence \( \{C_{4k}^i ; 1 \leq j \leq 2k + 1 \} \) gives a \( C_{4k} \)-frame of \( K_{2k+1} \otimes I_2 \).

**Lemma 4.15.** For \( k \equiv 3 \pmod{4} \), the graph \( K_{k+1} \otimes I_4 \) has a \( C_{4k} \)-frame.

**Proof.** Let \( V(K_{4s+4} \otimes I_4) = \{X_i / i \in \mathbb{Z}_{4s+3}\} \cup \{X_\infty\}, \) where \( X_i = \{i; 1 \leq t \leq 4\} \) and \( X_\infty = \{\infty; 1 \leq t \leq 4\} \). Then \( E(K_{4s+4} \otimes I_4) = (\sum_{0 \leq i < j \leq 4s+2} F_i(X_i, X_j)) \)

\[
(\sum_{i \in \mathbb{Z}_{4s+3}^\prime} \cup_{0 \leq i < j \leq 4s+2} F_i(X_i, X_\infty)).
\]

We now construct \( C_{4k} \)-frame of \( K_{4s+4} \otimes I_4 \) as follows:

Let \( C_{4k}^{\infty, 1} = \sum_{i=0}^{4} \begin{cases} 1 \end{cases} F_1(X_i, X_{i+1}) \cup F_1(X_0, X_{4s+2}) \) and

\[
C_{4k}^{\infty, 2} = \sum_{i=0}^{4} \begin{cases} 1 \end{cases} F_3(X_i, X_{i+1}) \cup F_3(X_0, X_{4s+2}).
\]
For $i \in \mathbb{Z}_{4s+3}$,

\[ C_{4k}^{i,1} = F_0(X_\infty, X_i) \cup F_0(X_i, X_{i+1}) \cup \bigcup_{j=1}^{4s+1} F_0(X_{i+j}, X_{i-j}) \]

\[ \bigcup_{j=1}^{4s+1} (F_1(X_{i-j}, X_{i+j})) \cup F_1(X_{i+2s+1}, X_\infty) \]

\[ C_{4k}^{i,2} = F_2(X_\infty, X_i) \cup F_2(X_i, X_{i+1}) \cup \bigcup_{j=1}^{4s+1} F_2(X_{i+j}, X_{i-j}) \]

\[ \bigcup_{j=1}^{4s+1} (F_3(X_{i-j}, X_{i+2s+1})) \cup F_3(X_{i+j+1}, X_\infty), \]

where the addition in the subscripts are taken modulo $(4s+3)$. Clearly, each $C_{4k}^{i,j}$ is a partial $C_{4k}$-factor of $K_{4s+4} \otimes I_4$ and hence \{\( C_{4k}^{i,1}, C_{4k}^{i,2}, C_{4k}^{\infty,1}, C_{4k}^{\infty,2} ; i \in \mathbb{Z}_{4s+3} \}\) gives a $C_{4k}$-frame of $K_{4s+4} \otimes I_4$. \qed

**Lemma 4.16.** For $k \equiv 1 \pmod{4}$, the graph $K_{k+1} \otimes I_4$ has a $C_{4k}$-frame.

**Proof.** Let $V(K_{4s+2} \otimes I_4) = \{ X_i/i \in \mathbb{Z}_{4s+1} \} \cup X_\infty$, where $X_i = \{ i_t; 1 \leq t \leq 4 \}$ and $X_\infty = \{ \infty_t / 1 \leq t \leq 4 \}$. Then $E(K_{4s+2} \otimes I_4) = \bigcup_{i=0}^{4s} F_i(X_i, X_j)$.

We now construct $C_{4k}$-frame of $K_{4s+2} \otimes I_4$ as follows:

Let $C_{4k}^{\infty,1} = \bigcup_{i=0}^{4s} F_i(X_i, X_{i+1}) \cup F_1(X_0, X_{4s})$ and $C_{4k}^{\infty,2} = \bigcup_{i=0}^{4s} F_i(X_i, X_{i+1}) \cup F_3(X_0, X_{4s})$. 

For \( i \in \mathbb{Z}_{4s+1} \),

\[
C_{4k}^{4k} = F_0(X_\infty, X_i) \cup F_0(X_i, X_{i+1}) \cup \bigcup_{j=1}^\infty \left( F_1(X_{j-2j+1}, X_{j+2j}) \cup F_1(X_{j+2j-1}, X_{j-2j+1}) \cup F_2(X_{j+2j}, X_{j-2j}) \right)
\]

\[
C_{4k}^{4k} = F_1(X_\infty, X_i) \cup F_2(X_i, X_{i+1}) \cup \bigcup_{j=1}^\infty \left( F_3(X_{j-2j+1}, X_{j+2j}) \cup F_3(X_{j+2j-1}, X_{j-2j+1}) \cup F_3(X_{j+2j}, X_{j-2j}) \right)
\]

where the addition in suffixes are taken modulo \((4s + 1)\).

Clearly, each \( C_{4k}^{4k} \) is a partial \( C_{4k} \)-factor of \( K_{4s+2} \otimes I_4 \) and hence \( \{ C_{4k}^{4k}, C_{4k}^{4k} \} \)

\( C_{4k}^{4k}, C_{4k}^{4k}; i \in \mathbb{Z}_{4s+1} \) gives a \( C_{4k} \)-frame of \( K_{4s+2} \otimes I_4 \). \( \square \)

**Lemma 4.17.** The graph \( K_{25} \otimes I_2 \) has a \( C_6 \)-frame.

**Proof.** Let \( V(K_{25}) = \bigcup_{i=0}^6 B_i \cup \{ \infty \} \), where \( B_i = \{ 6i + j, 1 \leq j \leq 6 \} \), \( V(I_2) = \bigcup_{i=0}^2 \{ 2i, 2j \} \), \( D_i, \) where \( D_i = I + 1 \) and \( V(K_{25} \otimes I_2) = \bigcup_{i=0}^6 B_i \cup \{ \infty \times I_2 \} \),

where \( B_i \) and \( D_i \) are defined as above. Find a new graph \( G \) from \( K_{25} \otimes I_2 \) by identifying each \( B_i \)

\( i \in \mathbb{Z}_4, l \in \mathbb{Z}_2 \) into a single vertex \( b_i^l \) and join two vertices \( b_i^l \) and \( b_i^{l+1} \) by an edge if there exists a complete bipartite graph \( K_{|B_i^l|, B_i^{l+1}} \). \( \cong \mathbb{Z}_{6,6} \) between \( B_i^l \) and \( B_i^{l+1} \) in \( K_{25} \otimes I_2 \). The resulting graph \( B \) is isomorphic to \( K_4 \otimes I_2 \).

By Theorem 4.5, \( K_4 \otimes I_2 \) has a \( C_3 \)-frame \( \{ F_0, F_1, F_2, F_3 \} \) where \( F_i \) is a partial \( C_3 \)-factor which does not contain the vertices of \( i \)-th partite set of \( K_4 \otimes I_2 \),
4.2. $C_{2k}$-frame of $K_m \otimes I_n$

$i = 0, 1, 2, 3$. While we blow up, corresponding to each $F_i, i = 0, 1, 2, 3$, we have $2K_{6,6,6}$ in $K_{24} \otimes I_2$ which does not contain the vertices of $B_i \times I_2$ in $K_{24} \otimes I_2$.

Now the vertices of $B_i \times I_2$ along with $\infty \otimes I_2$ form a $K_{|B_i \cup \infty|} \otimes I_2 \cong K_7 \otimes I_2$ in $K_{25} \otimes I_2$ which cover the rest of the vertices of $K_{25} \otimes I_2$ that are not covered by the above $2K_{6,6,6}$. By Theorem 4.1, $2K_{6,6,6}$ has a $C_6$-factorization consisting of 6 $C_6$-factors and by Lemma 4.13, $K_7 \otimes I_2$ has a $C_6$-frame consisting of 7 partial $C_6$-factors. Note that the 6 $C_6$-factors of $2K_{6,6,6}$ and 6 partial $C_6$-factors of $K_7 \otimes I_2$ together give 6 partial $C_6$-factors of $K_{25} \otimes I_2$. The last partial $C_6$-factor of $K_7 \otimes I_2$ with missing partite set $\infty \times I_2$ in $K_{25} \otimes I_2$ is kept aside for future purpose. By continuing the above process for each $F_i, 0 \leq i \leq 3$, we have 6 partial $C_6$-factors of $K_{25} \otimes I_2$ and last partial $C_6$-factor of $K_{|B_i \cup \infty|} \otimes I_2 \cong K_7 \otimes I_2$, with missing partite set $\infty \times I_2$ in $K_{25} \otimes I_2$. By taking the union of all last partial $C_6$-factors of $K_{|B_i \cup \infty|} \otimes I_2, 0 \leq i \leq 3$ with missing partite set $\infty \times I_2$, which are kept aside for future purpose, we have the last partial $C_6$-factor of $K_{25} \otimes I_2$. Totally we get 25 partial $C_6$-factors of $K_{25} \otimes I_2$ and hence a $C_6$-frame exists.

□

**Lemma 4.18.** For $s \geq 0, k \geq 2, n \equiv 0 \pmod{2}$ and $t \equiv 1 \pmod{2}$, the graph $K_{2^{s+1}k+1} \otimes I_n$ has a $C_{2k}$-frame.

**Proof.** Case(i). $n = 2$ and $t = 1$.

We prove this theorem by induction on $s$. For $s = 0$, the graph $K_{2k+1} \otimes I_2$ has a $C_{2k}$-frame by the Lemma 4.13. For $s = 1$, the graph $K_{8k+1} \otimes I_2$ has a $C_{2k}$-frame by the Lemmas 4.11, 4.13 and 4.17 (for the case $k = 3$). Assume that, $K_{2^{n+1}k+1} \otimes I_2$ has a $C_{2k}$-frame for all $q \leq s - 1$. Now consider

$$K_{2^{s+1}k+1} \otimes I_2 = K_{2^{s}2^{s-1}k+1} \otimes I_2 = K_{(4)2^{s-1}k+1} \otimes I_2.$$

By the induction assumption, the existence of $C_{2k}$-frame of the graph on RHS is same as that of Lemma 4.11, by taking $m = 2^{2s-1}k$. 
Case (ii). $n \equiv 0 \pmod{2} > 2$ and $t \equiv 1 \pmod{2} > 1$. 

Proof follows from Lemmas 4.10, 4.12 and Case (i). \qed

Lemma 4.19. For $s \geq 0$, $n \equiv 0 \pmod{2}$ and $k, t \equiv 1 \pmod{2}$, the graph $K_{2^{2s+1}k+1} \otimes I_n$ has a $C_{4k}$-frame.

Proof. Case (i). $n = 2$, and $t = 1$.

We prove this theorem by induction on $s$. For $s = 0$, the graph $K_{2k+1} \otimes I_2$ has a $C_{4k}$-frame by the Lemma 4.14. For $s = 1$, the graph $K_{8k+1} \otimes I_2$ has a $C_{4k}$-frame by the Lemmas 4.11 and 4.14. Assume that, $K_{2^{2q+1}k+1} \otimes I_2$ has a $C_{4k}$-frame for all $q \leq s - 1$. Now consider

$$K_{2^{2s+1}k+1} \otimes I_2 = K_{2^{2s-1}k+1} \otimes I_2$$

By the induction assumption, the existence of $C_{4k}$-frame of the graph on RHS is same as that of Lemma 4.11, by taking $m = 2^{2s-1}k$.

Case (ii). $n \equiv 0 \pmod{2} > 2$ and $t \equiv 1 \pmod{2} > 1$.

Proof of the case follows from Lemmas 4.10, 4.12 and Case (i). \qed

Lemma 4.20. For $s \geq 0$, $n \equiv 0 \pmod{4}$ and $k, t \equiv 1 \pmod{2}$, the graph $K_{2^{2s}k+1} \otimes I_n$ has a $C_{4k}$-frame.

Proof. Case (i). $n = 4$ and $t = 1$.

We prove this theorem by induction on $s$. For $s = 0$, the graph $K_{k+1} \otimes I_4$ has a $C_{4k}$-frame by the Lemma 4.15 or 4.16 according as $k \equiv 3 \pmod{4}$ or $k \equiv 1 \pmod{4}$. For $s = 1$, the graph $K_{4k+1} \otimes I_4$ has a $C_{4k}$-frame by the Lemmas 4.11, 4.15 and 4.16. Assume that, $K_{2^{2q}k+1} \otimes I_4$ has a $C_{4k}$-frame for all $q \leq s - 1$. Now consider

$$K_{2^{2s}k+1} \otimes I_4 = K_{2^{2s-2}k+1} \otimes I_4$$

By the induction assumption, the existence of $C_{4k}$-frame of the graph on RHS is same as that of Lemma 4.11, by taking $m = 2^{2s-2}k$. \qed
By the induction assumption, the existence of $C_{4k}$-frame of the graph on RHS is same as that of Lemma 4.11, by taking $m = 2^{2s-2k}$.

Case(ii). $n \equiv 0 \pmod{4} > 4$ and $t \equiv 1 \pmod{2} > 1$.

Proof of the case follows from Lemmas 4.10, 4.12 and Case(i). \hfill \Box

**Theorem 4.21.** If $m \equiv 1 \pmod{2}$ and $n \equiv 0 \pmod{2k}$, then there exists a $C_{4k}$-frame of $K_m \otimes I_n$.

**Proof.** Let $m = 2s + 1$, $n = 2kt$. Let $F = \{F_0, F_1, \ldots, F_{2s}\}$ be a near 1-factorization of $K_{2s+1}$ such that the near 1-factor $F_i$ does not contain the vertex $i$. Then $K_{2s+1} \otimes I_n = (F_0 \otimes F_1 \otimes F_2 \otimes \cdots \otimes F_{2s}) \otimes I_n = \bigotimes_{j=0}^{2s} F_j \otimes I_n$. Clearly, each $F_i \otimes I_n$ is an $n$-regular subgraph of $K_{2s+1} \otimes I_n$ containing all vertices except the vertices of $j$th partite set. Also $F_i \otimes I_n \cong sK_{n,n}$. By Theorem 4.4, each $F_i \otimes I_n$ has $k$ partial $C_{4k}$-factors of $K_{2s+1} \otimes I_n$ and hence the theorem. \hfill \Box

**Theorem 4.22.** If $m \equiv 0 \pmod{2}$ and $n \equiv 0 \pmod{4k}$, then there exists a $C_{4k}$-frame of $K_m \otimes I_n$.

**Proof.** Let $m = 2s$ and $n = 4kt$. By Theorem 4.6, let $\{C^i_4, C^2_4, \ldots, C^s_4\}$ be a $C_4$-frame of $K_{2s} \otimes I_4$, where $C^i_4$ is a partial $C_4$-factor of $K_{2s} \otimes I_4$. Then

\[
K_m \otimes I_n \cong K_{2s} \otimes I_{4kt} \cong (K_{2s} \otimes I_4) \otimes I_{kt} \\
\cong (C^i_4 \otimes C^2_4 \otimes \cdots \otimes C^s_4) \otimes I_{kt} \\
\cong (C^i_4 \otimes I_{kt}) \otimes (C^2_4 \otimes I_{kt}) \otimes \cdots \otimes (C^s_4 \otimes I_{kt})
\]

By Theorem 4.3, each $C^i_4 \otimes I_{kt}$ gives $kt$ partial $C_{4k}$-factor of $K_m \otimes I_n$, since $C_4 \otimes I_{kt} \cong K_{2kt,2kt}$ and hence the theorem. \hfill \Box

### 4.3 $C_{4p}$-frame of $(K_m \otimes I_n)(\lambda)$

This section deals with the existence of $C_{4p}$-frame of complete multipartite graph.
Lemma 4.23. The graph $K_5 \otimes I_2$ has a $C_8$-frame.

Proof. Let $V(K_5 \otimes I_2) = \{i_1, i_2; i \in Z_4\} \cup \{\infty_1, \infty_2\}$. We now construct $C_8$-frame of $K_5 \otimes I_2$ as follows:

$$C_8^1 = (1, 2, 3, 2, 1, \infty_1, 3, 1, \infty_2, 2, 1)$$
$$C_8^2 = (0, 1, 2, \infty_1, 3, 2, 0, \infty_2, 2, 3)$$
$$C_8^3 = (0, 1, 2, \infty_2, 1, 3, 1, 1, 2, 0, \infty_1)$$
$$C_8^4 = (0, 1, 2, \infty_1, 1, 1, 0, 2, 1, 2, \infty_2)$$
$$C_8^\infty = (0, 1, 2, 2, 0, 2, 3, 1, 2, 3, 1, 1)$$

Clearly, each $C_8^i, 1 \leq j \leq 4, j = \infty$, is a partial $C_8$-factor of $K_5 \otimes I_2$ and hence a $C_8$-frame exists. 

Lemma 4.24. The graph $K_{17} \otimes I_2$ has a $C_8$-frame.

Proof. Let $V(K_{17} \otimes I_2) = \{i_1, i_2; i \in Z_{16}\} \cup \{\infty_1, \infty_2\}$. We now construct a $C_8$-frame of $K_{17} \otimes I_2$ as follows:

$$C_8^0 = (11, 8, 12, 1, 7, 2, 13, 6, 14, 1, 10, 1)(11, 8, 12, 7, 1, 13, 6, 2, 14, 2, 10)$$
$$C_8^2 = (\infty, 5, 2, 15, 4, 2, 1, 3, 2, 1, 9)$$
$$C_8^1 = (12, 9, 13, 8, 14, 7, 15, 11)(12, 9, 13, 8, 14, 7, 2, 15, 11)$$
$$C_8^3 = (\infty, 6, 2, 1, 5, 2, 1, 4, 3, 1, 10)$$
$$C_8^2 = (13, 9, 14, 1, 9, 8, 1, 12, 1)(13, 9, 14, 1, 9, 8, 2, 12)$$
$$C_8^3 = (\infty, 7, 1, 6, 2, 3, 5, 4, 1, 11)$$
$$C_8^4 = (14, 11, 15, 10, 0, 9, 1, 13)(14, 11, 15, 10, 0, 9, 2, 13)$$
$$C_8^4 = (\infty, 8, 2, 1, 7, 2, 4, 6, 2, 12)$$
$$C_8^5 = (15, 12, 0, 11, 1, 1, 10, 2, 14)(15, 12, 0, 11, 1, 1, 10, 2, 14)$$
$$C_8^5 = (\infty, 9, 2, 3, 8, 5, 7, 6, 13)$$
$$C_8^0 = (0, 1, 3, 2, 1, 12, 2, 1, 11, 3, 1, 15)(0, 1, 3, 2, 1, 12, 2, 1, 11, 3, 15)$$
$$C_8^0 = (\infty, 10, 4, 9, 2, 6, 8, 1, 7, 14)$$
Clearly, each $C_\ell$, $0 \leq \ell \leq 15$, $\ell = \infty$ is a partial $C_8$-factor of $K_{17} \otimes I_2$ and hence a $C_8$-frame exists.
Lemma 4.25. For \( s \geq 0, \ t \equiv 1 \pmod{2} \) and \( n \equiv 0 \pmod{2} \), the graph \( K_{2^{2s+2}t+1} \otimes I_n \) has a \( C_8 \)-frame.

Proof. Case(i). \( n = 2 \) and \( t = 1 \).

The proof is by induction on \( s \). For \( s = 0 \), the graph \( K_5 \otimes I_2 \) has a \( C_8 \)-frame by the Lemma 4.23. For \( s = 1 \), the graph \( K_{17} \otimes I_2 \) has a \( C_8 \)-frame by Lemma 4.24. For \( s = 2 \), the graph \( K_{65} \otimes I_2 \) has a \( C_8 \)-frame by Lemmas 4.24 and 4.11. Assume that, \( K_{2^{2s+2}t+1} \otimes I_2 \) has a \( C_8 \)-frame for all \( q \leq s - 1 \). Now consider

\[
K_{2^{2s+2}t+1} \otimes I_2 = K_{2^{2s+1}} \otimes I_2
= K_{4(2^{2s})+1} \otimes I_2.
\]

By the induction assumption and Lemma 4.11 the graph on RHS has a \( C_8 \)-frame.

Case(ii). \( n \equiv 0 \pmod{2} > 2 \) and \( t \equiv 1 \pmod{2} > 1 \).

Proof of the case follows from Lemmas 4.10, 4.12 and Case (i). □

Theorem 4.26. If \( m \equiv 1 \pmod{4} \geq 5 \) and \( n \equiv 2 \pmod{4} \), then there exists a \( C_8 \)-frame of \( K_m \otimes I_n \).

Proof. Let \( m = 4q + 1 \) for some \( q \geq 1 \). If \( q \) is odd, proof follows from Lemma 4.18. If \( q \) is even; say \( q = 2r \), then \( m = 8r + 1 \). The proof follows from Lemmas 4.18, 4.25, since any integer \( 8r + 1, r \geq 1 \) can be written as either \( 2^{2s+3}t + 1 \) or \( 2^{2s+2}t + 1 \), where \( t \) is odd and \( s \geq 0 \). □

Theorem 4.27. If \( p \geq 1 \) is odd integer, \( m \equiv 1 \pmod{4} \) and \( n \equiv p \pmod{2p} \), then there exists a \( C_{4p} \)-frame of \( (K_m \otimes I_n)(2) \).

Proof. Let \( m = 4s + 1, s \geq 1 \) and \( n = tp \), where \( t \geq 1 \) is an odd integer. By Theorem 4.2, let \( C = \{C_4^1, C_4^2, \ldots, C_4^{4s+1} \} \) be a near \( C_4 \)-factorization of \( K_{4s+1}(2) \),
4.3. $C_{4p}$-frame of $(K_m \otimes I_n)(\lambda)$

where each $C_i$ is a near $C_4$-factor of $K_{4s+1}(2)$. Then

$$(K_m \otimes I_n)(2) = (K_{4s+1} \otimes I_{lp})(2) = (C_1 \oplus C_2 \oplus \ldots \oplus C_{4s+1}^4) \otimes I_{lp}$$

By Theorem 4.3, each $C_i \otimes I_{lp}$ gives $pt$ partial $C_{4p}$-factor of $K_m \otimes I_n$, since $C_4 \otimes I_{lp} \cong K_{2pt}$. This completes the proof. 

Theorem 4.28. If $m \equiv 1 \pmod{4k}$, then there exists a $C_{4k}$-frame of $(K_m \otimes I_n)(2)$, $n \geq 1$ is an odd integer.

Proof. Let $m = 4ks + 1$. By Theorem 4.2, let $C = \{C_4, C_4^2, \ldots, C_4^{4s+1}\}$ be a near $C_{4k}$-factorization of $K_{4ks+1}(2)$, where each $C_{4k}$ is a near $C_{4k}$-factor of $K_{4ks+1}(2)$. Then

$$(K_m \otimes I_n)(2) = (K_{4ks+1} \otimes I_n)(2) = (C_{4k} \oplus C_{4k}^2 \oplus \ldots \oplus C_{4k}^{4s+1}) \otimes I_n$$

By Theorem 4.8, each $C_{4k} \otimes I_n$ has a $C_{4k}$-frame. This completes the proof. 

Theorem 4.29. Let $m \geq 3$. There exists a $C_8$-frame of $K_m \otimes I_n$ if and only if

(i) $(m - 1)n \equiv 0 \pmod{8}$ and

(ii) at least one of $m, n$ is even.

Proof. Necessity follows from Lemma 4.9. Sufficiency can be proved in the following three cases.

(i) $m \equiv 1 \pmod{2}, n \equiv 0 \pmod{4}$;

(ii) $m \equiv 0 \pmod{2}, n \equiv 0 \pmod{8}$;
(iii) \( m \equiv 1 \pmod{4}, n \equiv 2 \pmod{4} \).

The existence of \( C_8 \)-frame of \( K_m \otimes I_n \) follows from Theorems 4.21, 4.22 and 4.26 respectively. \( \square \)

**Theorem 4.30.** Let \( m \geq 3 \) and \( p \) be a prime. There exists a \( C_{4p} \)-frame of \((K_m \otimes I_n)(\lambda)\) if and only if

1. \((m - 1)n \equiv 0 \pmod{4p}\) and
2. at least one of \( m, n \) must be even, when \( \lambda \) is odd.

**Proof.** Necessity follows from Lemma 4.9. We prove the sufficiency in two cases.

Case 1: \( \lambda = 1 \). The values of \( m \) and \( n \) fall in one of the following

(a) \( m \equiv 1 \pmod{2}, n \equiv 0 \pmod{2p}; \)

(b) \( m \equiv 0 \pmod{2}, n \equiv 0 \pmod{4p}; \)

(c) \( m \equiv 1 \pmod{2p}, n \equiv 0 \pmod{2}; \)

(d) \( m \equiv 1 \pmod{p}, n \equiv 0 \pmod{4}. \)

If \( p = 2 \), proof follows from Theorem 4.29. If \( p \) is an odd prime, proof for 1(a) and 1(b) follows from Theorems 4.21 and 4.22 respectively.

Proof of 1(c) follows from Lemma 4.19 or Lemma 4.18 according as any integer \( m = 2ps + 1, p \geq 3 \) can be written as either \( 2^{2q+1}pt + 1 \) or \( 2^{2q}pt + 1 \), where \( t > 0 \) is an odd integer and \( q \geq 0 \) is an integer.

Proof for 1(d) follows from Lemma 4.20 or Lemma 4.19 according as any integer \( m = ps + 1, p \geq 3 \) can be written as either \( 2^{2q}pt + 1 \) or \( 2^{2q-1}pt + 1 \), where \( t > 0 \) is an odd integer and \( q \geq 0 \) is an integer.

Case 2: \( \lambda = 2 \). Then the values of \( m \) and \( n \) fall in one of the following, in addition to the choices of Case 1.

(e) \( m \equiv 1 \pmod{4}, n \equiv p \pmod{2p}; \)
(f) \( m \equiv 1 \pmod{4p} \), \( n \geq 1 \) is an odd integer.

The existence of \( C_{4p} \)-frame of \( (K_m \otimes I_n)(2) \) for 2(e) and 2(f) follow from Theorems 4.27 and 4.28 respectively. From the Case 1, \( C_{4p} \)-frame of \( K_m \otimes I_n \) and hence for \( (K_m \otimes I_n)(2) \) exist from Cases 1 and 2, \( C_{4p} \)-frame of \( (K_m \otimes I_n)(2) \) and hence for \( (K_m \otimes I_n)(2s) \) exists since \( \lambda = 2s \) is even.

\[
\begin{align*}
(f) & \quad m \equiv 1 \pmod{4p}, n \geq 1 \text{ is an odd integer.} \\
& \text{The existence of } C_{4p}\text{-frame of } (K_m \otimes I_n)(2) \text{ for } 2(e) \text{ and } 2(f) \text{ follow from Theorems 4.27 and 4.28 respectively. From the Case 1, } C_{4p}\text{-frame of } K_m \otimes I_n \text{ and hence for } (K_m \otimes I_n)(2) \text{ exist from Cases 1 and 2, } C_{4p}\text{-frame of } (K_m \otimes I_n)(2) \text{ and hence for } (K_m \otimes I_n)(2s) \text{ exists since } \lambda = 2s \text{ is even.} \quad \Box
\end{align*}
\]

### 4.4 Conclusion

The results of this chapter provide a complete solution to the existence of \( C_{4p} \)-frame of complete multipartite multigraph \( (K_m \otimes I_n)(\lambda) \), when \( p \) is a prime. Also we establish the following results.

(i). the necessary conditions given in Lemma 4.9 for the existence of \( C_{2k} \)-frame of \( K_m \otimes I_n \) are sufficient when \( m = 2^{2s+1}kt + 1; n \equiv 0 \pmod{2} \) where \( s \geq 0, k \geq 2 \) and \( t \equiv 1 \pmod{2} \);

(ii). the necessary conditions \( m \geq 3, (m - 1)n \equiv 0 \pmod{4k} \) and at least one of \( m, n \) must be even for the existence of \( C_{4k} \)-frame of \( K_m \otimes I_n \) stated in Lemma 4.9 are sufficient when

\[
\begin{align*}
(i) & \quad m = 2^{2s+1}kt; n \equiv 0 \pmod{2} \text{ where } s \geq 0, k, t \equiv 1 \pmod{2}; \\
(ii) & \quad m = 2^{2s}kt; n \equiv 0 \pmod{4} \text{ where } s \geq 0, k, t \equiv 1 \pmod{2}; \\
(iii) & \quad m \equiv 1 \pmod{2}; n \equiv 0 \pmod{2k}; \\
(iv) & \quad m \equiv 0 \pmod{2}; n \equiv 0 \pmod{4k}.
\end{align*}
\]

In fact, the results presented here seems to be the first general result in this direction and also generalizes the one given by Cao. et.al [22]. Further the problem of decomposing the complete multipartite graph into cycles of length \( k \) is still open in general. Numerous partial solutions of this problem are known [6, 10, 11, 23, 24, 52, 66, 74, 77–80]. In line with them, the results of this chapter
also give a partial solution to the existence of $C_{2k}$-decomposition of complete multipartite graphs.