CHAPTER-IV

SEMI-RIEMANNIAN QUASI EINSTEIN MANIFOLDS
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Introduction:

This chapter is devoted to the study of semi-Riemannian Quasi Einstein manifolds. Such an $n$-dimensional manifold is denoted by the symbol $\Psi(QE)_n$. It is known that in a semi-Riemannian manifold a vector field may be timelike, spacelike or lightlike. This chapter is divided into four sections of which the first deals with $\Psi(QE)_n$ with timelike generator and unequal associated scalars. In section 2 $\Psi(QE)_n$ with unequal associated scalars and timelike generator satisfying $\text{div} R = 0$ is studied. Section 3 deals with $\Psi(QE)_n$ with timelike generator in which the relation $R(X,Y)\cdot S = 0$ holds. In section 4 we study a type of $\Psi(QE)_4$ with timelike generator in which Einstein’s equation without cosmological constant is satisfied.

Preliminaries:

Let $\Psi(QE)_n$ be a semi-Riemannian $(QE)_n$ with associated scalars $a,b$, associated 1-form $A$ and generator $U$.

Then

$$(IV.p.1) \quad S(X,Y) = ag(X,Y) + bA(X)A(Y)$$

where

$$(IV.p.2) \quad g(X,U) = A(X) \quad \forall X$$

If $U$ is timelike, then

$$(IV.p.3) \quad g(U,U) = -1 \quad \text{or} \quad A(U) = -1$$

Hence

$$(IV.p.4) \quad S(X,U) = aA(X) - bA(X)$$

$$= (a-b)A(X)$$
We have

\((\nabla_x g)(U,U) = \nabla_x g(U,U) - g(\nabla_x U,U) - g(\nabla_x U,U)\)

Hence

(IV.p.5) \(g(\nabla_x U,U) = 0\)

These formulas will be used in the sequel.

\section{ SECTION-1 }

\(\Psi(QE)_n \text{ with timelike generator and unequal associated scalars.}\)

In this section we consider a \(\Psi(QE)_n\) with unequal associated scalars \(a,b\) and timelike generator \(U\).

Then

(IV.1.1) \(S(Y,Z) = a g(Y,Z) + b A(Y) A(Z)\)

Differentiating (IV.1.1) covariantly with respect to \(X\) we get

(IV.1.2) \((\nabla_X S)(Y,Z) = d a(X) g(Y,Z) + d b(X) A(Y) A(Z) + b [ (\nabla_X A)(Y) A(Z) + (\nabla_X A)(Z) A(Y) ]\)

Hence

(IV.1.3) \((\nabla_X S)(Y,Z) - (\nabla_X S)(Y,X) = d a(X) g(Y,Z) - d a(Z) g(Y,X) + d b(X) A(Y) A(Z) - d b(Z) A(Y) A(X) + b [ (\nabla_X A)(Y) A(Z) - (\nabla_X A)(Z) A(Y) ] - (\nabla_Z A)(Y) A(X) + (\nabla_Z A)(X) A(Y) ] = d a(X) g(Y,Z) - d a(Z) g(Y,X) + A(V) [ d b(X) A(Z) - d b(Z) A(X) ] + b [ (\nabla_X A)(Y) A(Z) - (\nabla_X A)(Y) A(X) ] + b A(Y) [ (\nabla_X A)(Z) - (\nabla_X A)(X) ]\)
From (IV.1.3) it follows that divR is not, in general, zero. This leads to the following result:

**Theorem 14:** In a $Ψ(QE)_n$ with unequal associated scalars and timelike generator, the divergence of $R$ is not, in general, zero.

**Section-2**

$Ψ(QE)_n$ with unequal associated scalars and timelike generator in which $\text{div} R = 0$.

In this section we suppose that $\text{div} R = 0$.

Then from (IV.1.4) we get

\begin{align*}
(\text{IV.2.1}) & \quad da(X)g(Y,Z) - da(Z)g(Y,X) + A(Y)\left[db(X)A(Z) - db(Z)A(X)\right] \\
& \quad - b[(\nabla_x A)(Y)A(Z) - (\nabla_z A)(Y)A(X)] + \\
& \quad + bA(Y)\left[(\nabla_x A)(Z) - (\nabla_z A)(X)\right] = 0
\end{align*}

Putting $Y = U$ in (IV.2.1) we get

\begin{align*}
(\text{IV.2.2}) & \quad da(X)A(Z) - da(Z)A(X) + [db(X)A(Z) - db(Z)A(X)] + \\
& \quad + b[(\nabla_x A)(U)A(Z) - (\nabla_z A)(U)A(X)] - \\
& \quad - b[(\nabla_x A)(Z) - (\nabla_z A)(X)] = 0
\end{align*}

Since $(\nabla_x A)(U) = \nabla_x A(U) - A(\nabla_x U)$

\begin{align*}
(\text{IV.2.3}) & \quad (\nabla_x A)(U) = -g(\nabla_x U, U) \\
& \quad = 0 \quad \text{[ by (IV.p.5)]}
\end{align*}

In virtue of (IV.2.3) we can express (IV.2.2) as follows:

\begin{align*}
(\text{IV.2.4}) & \quad dc(X)A(Z) - dc(Z)A(X) - b[(\nabla_x A)(Z) - (\nabla_z A)(X)] = 0
\end{align*}

where

\begin{align*}
& \quad c = a - b \neq 0 \\
& \quad \text{Now } g(\nabla_x U, Z) = (\nabla_x A)(Z)
\end{align*}
Hence (IV.2.4) can be expressed as follows:

\[(IV.2.5) \quad dc(X)A(Z) - dc(Z)A(X) = b[g(V_xU,Z) - g(V_zU,X)]\]

If \( U \) is an irrotational vector field, then

\[g(V_xU,Z) - g(V_zU,X) = 0 \quad \forall X, Z\]

Hence

\[(IV.2.6) \quad dc(X)A(Z) - dc(Z)A(X) = 0\]

Conversely if (IV.2.6) holds, then

\[g(V_xU,Z) - g(V_zU,X) = 0 \quad [\because b \neq 0]\]

So \( U \) is an irrotational vector field.

Thus we can state the following theorem.

**Theorem 15:** If in a \( \Psi(QE)_n \) with unequal associated scalars and timelike generator, the divergence of \( R \) is zero, then the generator is an irrotational vector field if and only if

\[dc(X)A(Z) - dc(Z)A(X) = 0\]

where \( c = a - b \neq 0 \).

**SECTION-3**

\( \Psi(QE)_n \) with timelike generator in which the relation \( R(X,Y) \cdot S = 0 \) holds.

We have

\[(IV.3.1) \quad -[R(X,Y) \cdot S](Z,W) = S(R(X,Y)Z,W) + S(R(X,Y)W,Z)\]

If \( R(X,Y) \cdot S = 0 \), then from (IV.3.1) it follows that

\[(IV.3.2) \quad S(R(X,Y)Z,W) + S(R(X,Y)W,Z) = 0\]

Using (IV.3.1) the above relation can be expressed as follows:

\[aS[R(X,Y)Z,W] + bS[R(X,Y)Z]A(W)]

\[+ aS[R(X,Y)W,Z] + bS[R(X,Y)W]A(Z)] = 0\]
or

\((IV.3.3)\) \quad A(R(X,Y)Z)A(W) + A(R(X,Y)W)A(Z) = 0 \quad [\therefore b \neq 0]\)

Putting \(W = U\) in \((IV.3.3)\) we get

\((IV.3.4)\) \quad -A(R(X,Y)Z) + A(R(X,Y)U)A(Z) = 0

Next, putting \(Z = U\) in \((IV.3.4)\) we have

\((IV.3.5)\) \quad A(R(X,Y)U) = 0

Hence from \((IV.3.4)\) it follows that

\((IV.3.6)\) \quad A(R(X,Y)Z) = 0

or,

\((IV.3.7)\) \quad g[R(X,Y)Z,U] = 0

Contracting the above equation over \(Y\) and \(Z\) we get

\((IV.3.8)\) \quad S(X,U) = 0

Using \((IV.p.4)\) from \((IV.3.8)\) we get

\[a - b = 0 \quad \text{i.e.,} \quad a = b\]

Therefore we can state as follows:

**Theorem 16:** If in a \(\Psi(QE)_n\) with timelike generator, the relation \(R(X,Y).S=0\) is satisfied, then the associated scalars of the \(\Psi(QE)_n\) must be equal.

**Section-4**

\(\Psi(QE)_4\) with timelike generator in which Einstein's equation without cosmological constant is satisfied.

In this section we consider a \(\Psi(QE)_4\) with timelike generator in which Einstein's equation without cosmological constant is satisfied. Then

\((IV.4.1)\) \quad S(X,Y) - \frac{1}{2}g(X,Y) = kT(X,Y) \quad [15]
where \( r \) is the scalar curvature, \( k \) is the gravitational constant and \( T \) is the energy momentum tensor.

Let \( a, b \) be the associated scalars and \( U \) be the timelike generator of \( \Psi(QE)_4 \). Then

\[ (IV.4.2) \quad S(X,Y) = ag(X,Y) + bA(X)A(Y) \]

Using (IV.4.2) we can express (IV.4.1) as follows:

\[ (IV.4.3) \quad T(X,Y) = \frac{1}{k} [ag(X,Y) + bA(X)A(Y) - \frac{r}{2}g(X,Y)] \]

\[ = \frac{1}{k} [(a - \frac{r}{2})g(X,Y) + bA(X)A(Y)] \]

\[ = \frac{1}{k} [(-a - \frac{b}{2})g(X,Y) + bA(X)A(Y)] \]

\[ \therefore r = 4a - b \]

At first we shall determine the number of distinct eigenvalues of the energy momentum tensor.

Putting \( Y = U \) in (IV.4.3) we get

\[ (IV.4.4) \quad T(X,U) = \frac{1}{k} (-a - \frac{b}{2})A(X) \]

From (IV.4.4) it follows that \( \frac{1}{k} (-a - \frac{b}{2}) \) is an eigenvalue of the tensor \( T \) and \( U \) is an eigenvector corresponding to this eigenvalue.

Let \( V \) be any vector orthogonal to \( U \).

Then

\[ g(U,V) = 0 \]

or,

\[ \quad (IV.4.5) \quad A(V) = 0 \]

Putting \( Y = V \) in (IV.4.3) we get

\[ (IV.4.6) \quad T(X,V) = \frac{1}{k} \left[-a + \frac{b}{2}\right]g(X,V) \]
From (IV.4.6) it follows that \( \frac{1}{k} \left( -a + \frac{b}{2} \right) \) is an eigenvalue of \( T \) and \( V \) is an eigenvector corresponding to this eigenvalue. Since \( V \) is any vector orthogonal to \( U \) and the dimension of \( \Psi(QE)_4 \) is 4 the eigenvalue \( \frac{1}{k} \left( -a + \frac{b}{2} \right) \) is of multiplicity \( 4-1=3 \). [18]

Hence the multiplicity of the eigenvalue \( \frac{1}{k} \left( -a - \frac{b}{2} \right) \) is 1.

This leads to the following result:

**Theorem 17:** If a \( \Psi(QE)_4 \) with timelike generator \( U \) and associated scalars \( a, b \) satisfies Einstein's equation without cosmological constant, then the energy momentum tensor has only two distinct eigenvalues \( \frac{1}{k} \left( -a - \frac{b}{2} \right) \) and \( \frac{1}{k} \left( -a + \frac{b}{2} \right) \) of which the former is simple and the latter is of multiplicity 3. Further, in the former case \( U \) is an eigenvector corresponding to the eigenvalue \( \frac{1}{k} \left( -a - \frac{b}{2} \right) \).

Next we enquire whether a \( \Psi(QE)_4 \) of the type under consideration can admit a covariant constant energy momentum tensor.

To give an answer to this enquiry we suppose that

(IV.4.7) \( \nabla T = 0 \)

From (IV.4.1) we get

(IV.4.8) \( (\nabla_2 S)(X,Y) - \frac{1}{2} \mathrm{d}r(Z)g(X,Y) = k(\nabla_2 T)(X,Y) \)

In virtue of (IV.4.7), the above relation can be written as

(IV.4.9) \( (\nabla_2 S)(X,Y) - \frac{1}{2} \mathrm{d}r(Z)g(X,Y) = 0 \)
Contracting (IV.4.9) over $X$ and $Y$ we get
\[ dr(Z) - \frac{1}{2} dr(Z)4 = 0 \]
or, \[ dr(Z) - 2dr(Z) = 0 \]
Hence
\[ (IV.4.10) \ dr(Z) = 0 \quad \forall \ Z \]
From (IV.4.10) it follows that the scalar curvature $r$ is constant.
Hence we can state the following theorem:

**Theorem 18:** If a $\Psi(QE)_4$ with timelike generator satisfies Einstein's equation without cosmological constant, then it can admit a covariant-constant energy momentum tensor and when it does so, the scalar curvature of the $\Psi(QE)_4$ must be constant.

This is the concluding theorem of this chapter.