CHAPTER-III

ALMOST PSEUDO RICCI SYMMETRIC MANIFOLDS
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Introduction:

This chapter deals with a type of Almost Pseudo Ricci symmetric manifolds defined by equation (4) in the introduction, where A, B are two non-zero 1 forms and V has the meaning already mentioned. In such a case A and B are called the associated 1-forms of the manifold and such an n-dimensional manifold is denoted by the symbol A(PRS)_n.

If the non-zero 1-forms A and B be such that

\[(\text{III.i}) \quad g(X,U) = A(X) \text{ and } g(X,V) = B(X) \quad \forall X\]

then U and V are called the basic vector fields of the A(PRS)_n corresponding to the 1-forms A and B respectively. It is to be noted that A(X) + B(X) ≠ 0. In this chapter we study such A(PRS)_n as are Quasi-Einstein with a, b as associated scalars of which b ≠ 0 and the associated 1-form A of the A(PRS)_n as its generator. Such an n-dimensional manifold shall be denoted by the symbol A(QE)_n. This chapter is divided into three sections of which the first deals with A(QE)_n the associated scalars of which satisfy some conditions. In section 2 conformally flat A(QE)_n (n>3) is studied. Section 3 is concerned with A(QE)_n (n>3) having divergence-free conformal curvature tensor.

Preliminaries:

Let g be the metric tensor of an A(QE)_n, a, b be its associated scalars, A and B its associated 1-forms and U be its generator. Then

\[(\text{III.p.1}) \quad (V_xS)(Y,Z) = [A(X) + B(X)] S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X)\]

\[(\text{III.p.2}) \quad S(X,Y) = ag(X,Y) + B(X)A(Y)\]

and \[(\text{III.p.3}) \quad g(U,U) = 1, \quad g(X,V) = B(X)\]
Contracting (III.p.2) over X and Y we get

(III.p.4)  \( r = na + b \)

where \( r \) is the scalar curvature.

From (III.p.1) it follows that

(III.p.5)  \((\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = B(X) S(Y,Z) - B(Z) S(Y,X)\)

Contracting (III.p.5) over Y and Z we get

\[ \frac{1}{2} dr(X) = rB(X) - S(V,X) \]

or

(III.p.6)  \( dr(X) = 2rB(X) - 2S(V,X) \)

Next contracting (III.p.1) over Y and Z we get

(III.p.7)  \( dr(X) = [A(X) + B(X)]r + 2S(X,U) \)

These formulas will be used in the sequel.

**SECTION-1**

*A type of A(PRS)\(_n\) denoted by A(QE)\(_n\) whose associated scalars satisfy some conditions*

In this section the nature of a A(QE)\(_n\) is determined in the following two cases

1)  \( a + b = 0, \quad r \neq 0 \)

and 2)  \( a + b \neq 0, \quad A(V) \neq 0 \) but \( \text{div} \ R = 0 \)

*(Case - 1)*

From (III.p.2) it follows that

(III.1.1)  \( S(X,U) = (a+b)A(X) \)

Since in this case \( a + b = 0 \), from (III.1.1) we get

(III.1.2)  \( S(X,U) = 0 \)

Hence from (III.p.7) we get
(III.1.3) \[ \text{dr}(X) = [A(X) + B(X)]r \]

If \( r \) is constant, then from (III.1.3) it follows that \( r = 0 \), because \( A(X) + B(X) \neq 0 \). Therefore if \( r \neq 0 \), then it cannot be constant.

This leads to the following result.

**Theorem 10:** If \( \text{in a } A(QE) \) \( a + b = 0, \) \( r \neq 0, \) then \( r \) cannot be constant.

**(Case - 2)**

Since in this case \( \text{div}R = 0 \), from (III.p.5) we get

(III.1.4) \[ B(X)S(Y,Z) - B(Z)S(Y,X) = 0 \]

Putting \( Y = U \) in (III.1.4) we have

(III.1.5) \[ B(X)S(U,Z) - B(Z)S(U,X) = 0 \]

Since \( a + b \neq 0 \) it follows from (III.1.1) that \( S(X, U) = (a + b)A(X) \).

Hence we can write (III.1.5) as follows:

\[ B(X)(a+b)A(Z) - B(Z)(a+b)A(X) = 0 \]

or,

(III.1.6) \[ B(X)A(Z) - B(Z)A(X) = 0 \quad [\therefore a + b \neq 0] \]

Now putting \( Z = U \) in (III.1.6) we get

\[ B(X) - B(U)A(X) = 0 \]

or,

(III.1.7) \[ B(X) = A(V)A(X) \quad [\therefore B(U) = g(U, V) = A(V)] \]

Next contracting (III.1.4) over \( Y \) and \( Z \) we get

\[ B(X)r - S(V, X) = 0 \]

or,

(III.1.8) \[ rB(X) = S(V, X) \]

Hence from (III.p.6) it follows that

\[ \text{dr}(X) = 0 \]

Hence \( r \) is constant.
Summing up we can state the following theorem.

**Theorem 11:** If in an \(A(QE)_n\) \(a + b \neq 0, A(V) \neq 0\), but \(\text{div} R = 0\), then the scalar curvature of \(A(QE)_n\) is constant and the 1-form \(B\) is proportional to the 1-form \(A\), the factor of proportionality being \(A(V)\) which is different from zero.

**Section-2**

**Conformally flat \(A(QE)_n\) (n>3)**

It is known [8,p236] that in a conformally flat \((M^n, g)\) (n>3)

\[(III.2.1) \quad (\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \frac{1}{2(n-1)} [\text{dr}(X)g(Y,Z) - \text{dr}(Z)g(Y,X)]\]

Using \((III.p.5)\) and \((III.p.6)\) this relation can be expressed as follows:

\[(III.2.2) \quad (n-1)[B(X)S(Y,Z) - B(Z)S(Y,X)] = [B(X)r - S(X,V)]g(Y,Z) - [B(Z)r - S(Z,V)]g(Y,X)\]

or,

\[(III.2.3) \quad (n-1)B(X)[S(Y,Z) - \frac{r}{n-1}g(Y,Z)] + S(X,V)g(Y,Z) = (n-1)B(Z)[S(Y,X) - \frac{r}{n-1}g(Y,X)] + S(Z,V)g(Y,X)\]

Putting \(Y = V\) in \((III.2.3)\) we get

\[(n-1)B(X)[S(V,Z) - \frac{r}{n-1}B(Z)] + S(X,V)B(Z) = (n-1)B(Z)[S(V,X) - \frac{r}{n-1}B(X)] + S(Z,V)B(X)\]

or,

\[(III.2.4) \quad (n-2)[B(X)S(V,Z) - B(Z)S(V,X)] = 0\]

Since \(n \neq 2\) we have

\[B(X)S(V,Z) - B(Z)S(V,X) = 0\]

or, \[B(X)[aB(Z) + bA(V)A(Z)] - B(Z)[aB(X) + bA(V)A(X)] = 0\]

[by \((III.p.2)\)]
Hence we have
\[ A(V)B(X)A(Z) - A(V)B(Z)A(X) = 0 \]
or,
\[ (III.2.5) \quad A(V)[B(X)A(Z) - B(Z)A(X)] = 0 \]
Putting \( Z = U \) in (III.2.5) we get
\[ (III.2.6) \quad A(V)[B(X) - A(V)A(X)] = 0 \]
If \( A(V) \neq 0 \), then it follows from (III.2.6) that
\[ (III.2.7) \quad B(X) = A(V)A(X) \]
We can therefore state as follows:

**Theorem 12:** If in a conformally flat \( A(QE)_n \) \((n>3)\), \( a + b \neq 0 \) and \( A(V) \neq 0 \), then the 1-form \( B \) is proportional to the 1-form \( A \), the factor of proportionality being \( A(V) \) which is different from zero.

**SECTION-3**

**Conformally conservative \( A(QE)_n \) \((n>3)\)**

In this section we enquire whether an \( A(QE)_n \) \((n>3)\) is conformally conservative [12]

Let
\[ (III.3.1) \quad H(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) - \frac{1}{2(n-1)} [dr(X)g(Y,Z) - dr(Z)g(Y,X)] \]
Then it is known [8,p236] that \( \text{div}C = 0 \) if and only if \( H(X,Y,Z) = 0 \)
In case of an \( A(QE)_n \) we can express (III.3.1) as follows:
From (III.3.2) it follows that in an $A(QE)_n$ ($n>3$) $H(X,Y,Z)$ is not, in general, zero.

Let us suppose that in an $A(QE)_n$,

\[ (III.3.3) \quad B(X)g(Y,Z) - B(Z)g(Y,X) = 0 \]

Putting $Y = U$ in (III.3.3) we get

\[ (III.3.4) \quad B(X)A(Z) - B(Z)A(X) = 0 \]

Next putting $Z = U$ in (III.3.4) we have

\[ (III.3.5) \quad B(X) - A(V)A(X) = 0 \]

Using (III.3.3) and (III.3.5), it follows from (III.3.2) that

\[ H(X,Y,Z) = b[A(V)A(X)A(Y)A(Z) - A(V)A(X)A(Y)A(Z) + B(X)g(Y,Z) - B(Z)g(Y,X)] \]
Thus if $A(V) \neq 0$ and $B(X)g(Y,Z) - B(Z)g(Y,X) = 0$

Then $H(X,Y,Z) = 0$

Again if $H(X,Y,Z) = 0$ and $A(V) \neq 0$ then from (III.2.5) it follows that

$B(X)g(Y,Z) - B(Z)g(Y,X) = 0$

Hence the conditions $A(V) \neq 0$ and $B(X)g(Y,Z) - B(Z)g(Y,X) = 0$ are both necessary and sufficient for $H(X,Y,Z)$ to be zero. This leads to the following result:

**Theorem 13:** An $A(QE)_n \ (n > 3)$ is conformally conservative if and only if $A(V) \neq 0$ and $B(X)g(Y,Z) - B(Z)g(Y,X) = 0$.

This chapter closes with this theorem.