On quasi Einstein manifolds

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Introduction

A non-flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is defined to be a quasi Einstein manifold if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the condition

\[
S(X, Y) = a g = (X, Y) + b A(X) A(Y)
\]

where \(a, b\) are scalars of which \(b \neq 0\) and \(A\) is a non-zero 1-form such that

\[
g(x, u) = A(x)
\]

for all vector fields \(X, U\) being a unit vector field.

In such a case \(a, b\) are called associated scalars. \(A\) is called the associated 1-form and \(U\) is called the generator of the manifold. An \(n\)-dimensional manifold of this kind shall be denoted by the symbol \((QE)\).

This paper deals with \((QE)\) \((n > 3)\) which are not conformally flat. The significance of the associated scalars \(a, b\) is pointed out by showing that in a \((QE)\) the Ricci tensor \(S\) has only two distinct eigenvalues \(a + b\) and \(a\) of which the former is simple and the latter is of multiplicity \(n - 1\), the generator \(U\) being an eigenvector corresponding to the eigenvalue \(a + b\). In a \((QE)\) the relation \(R(X, Y) \cdot S = 0\) does not in general hold. A necessary and sufficient condition is obtained in order that this relation may hold.

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A Riemannian manifold \((M^n, g)\) is said to be conformally conservative [1] if the divergence of its conformal curvature tensor is zero. Thus every conformally flat Riemannian manifold is conformally conservative, but the converse is not, in general, true. Since this paper deals with \((QE)_n\) \((n > 3)\) which are not conformally flat, it is meaningful to look for a sufficient condition in order that a \((QE)_n\) \((n > 3)\) may be conformally conservative. Two such sufficient conditions are considered, corresponding to the two cases when the associated scalars \(a\) and \(b\) are constants and when they are not so but their sum is zero.

1. Preliminaries

We consider a \((QE)_n\) with associated scalars \(a, b\), associated 1-form \(A\) and generator \(U\). Since \(U\) is a unit vector field,

\[(1.1)\quad g(U, U) = 1.\]

Again

\[(1.2)\quad S(X, Y) = ag(X, Y) + bA(X)A(Y).\]

Contracting (1.2) over \(X\) and \(Y\) we get

\[(1.3)\quad \gamma = na + b\]

where \(\gamma\) denotes the scalar curvature of the manifold. Putting \(Y = U\) in (1.2) we obtain

\[(1.4)\quad S(X, U) = (a + b)A(X).\]

Let \(L\) be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor \(S\). Then

\[(1.5)\quad g(LX, Y) = S(X, Y) \quad \forall X, Y.\]

Let \(R\) be the curvature tensor of \((QE)_n\). Then \(R(X, Y)\) may be regarded as a derivation of the tensor algebra at each point of the tangent space. Hence

\[(1.6)\quad [R(X, Y) \circ L](Z) = R(X, Y)LZ - L(R(X, Y)Z).\]
The transformations $R(X,Y)$ and $L$ are called the curvature transformation and the Ricci transformation respectively.

The commutativity of these two transformations is a consequence of the relation

\[(1.7) \quad R(X,Y) \circ L = 0.\]

This can be proved as follows: If (1.7) holds, then from (1.6) we get

\[(1.8) \quad R(X,Y) \circ LZ = LR(X,Y)Z \quad \forall Z.\]

Hence

\[(1.9) \quad R(X,Y) \circ L = L \circ R(X,Y).\]

This means that $R(X,Y)$ and $L$ commute. These formulas will be used in the sequel.

2. Significance of the associated scalars in a $(QE)_n \ (n > 3)$

We can express (1.4) as follows:

\[(2.1) \quad S(X,U) = (a + b)g(X,U).\]

From (2.1) we conclude that $a + b$ is an eigenvalue of the Ricci tensor $S$ and $U$ is an eigenvector corresponding to this eigenvalue.

Let $V$ be any other vector orthogonal to $U$. Then

\[(2.2) \quad g(U,V) = 0 \quad \text{i.e.} \quad A(V) = 0.\]

From (1.2) we obtain

\[S(X,V) = ag(X,V) + bA(X)A(V).\]

Hence in virtue of (2.2) we obtain

\[(2.3) \quad S(X,V) = ag(X,V).\]

From (2.3) we see that $a$ is an eigenvalue of the Ricci tensor and $V$ is an eigenvector corresponding to this eigenvalue. Since the manifold under
consideration is n-dimensional and \( V \) is any vector orthogonal to \( U \), it follows from a known result in linear algebra [2] that the eigenvalue \( a \) is of multiplicity \( n - 1 \). Hence the multiplicity of the eigenvalue \( a + b \) must be 1. So there are only two distinct eigenvalues of the Ricci tensor, namely \( a + b \) and \( a \), of which the former is simple and the latter is of multiplicity \( n - 1 \).

Hence we can state the following

**Theorem 1.** In a \((QE)_n\) \((n > 3)\), the Ricci tensor \( S \) has only two distinct eigenvalues \( a + b \) and \( a \) of which the former is simple and the latter is of multiplicity \( n - 1 \).

It is to be noted that if \( a + b = 0 \) then \( a \) cannot be zero, because \( b \neq 0 \).

3. \((QE)_n\) \((n > 3)\) satisfying the relation \( R(X,Y) \cdot S = 0 \)

We have

\[
(3.1) \quad [R(X,Y) \cdot S](Z,W) = -S[R(X,Y)Z,W] - S[Z,R(X,Y)W]
\]

\[
= -[ag(R(X,YZ),W) + bA(R(X,Y,Z)A(W)]
\]

\[
- [ag(R(X,YW),Z) + bA(R(XYW)A(Z)]
\]

[by (1.2)]

\[
= -b[A(R(X,Y,Z)A(W) + A(R(X,Y,W)A(Z)]
\]

Since \( b \neq 0 \), it follows from (3.1) that in a \((QE)_n\) \((n > 3)\) the relation \( R(X,Y) \cdot S = 0 \) does not, in general, hold. From (3.1) we see that if \( A(R(X,Y,Z)) = 0 \) then \( [R(X,Y) \cdot S](Z,W) = 0 \) \( \forall Z, W \).

Hence \( R(X,Y) \cdot S = 0 \). Let us now suppose that \( R(X,Y) \cdot S = 0 \). Then from (3.1) we get

\[
(3.2) \quad A(R(X,Y,Z))A(W) + A(R(X,Y,W))A(Z) = 0.
\]

Putting \( W = U \) in (3.2) we have

\[
A(R(X,Y,Z))A(U) + A(R(X,Y,U))A(Z) = 0
\]

or

\[
A(R(X,Y,Z)) + g(R(X,Y,U),U)A(Z) = 0.
\]

Hence \( A(R(X,Y,Z)) = 0 \) \( \therefore g(R(X,Y,U),U) = 0 \).
Thus we can state the following

**Theorem 2.** In a \((QE)_n\) \((n > 3)\), the relation \(R(X, Y) \cdot S = 0\) holds if and only if \(A(R(X, Y, Z)) = 0\). If \(R(X, Y) \cdot S = 0\), then

\[
S(R(X, Y, Z), W) + S(R(X, Y, W), Z) = 0
\]
or
\[
g[LR(X, Y, Z), W] + g[R(X, Y, W), LZ] = 0
\]
or
\[
g[LR(X, Y, Z), W] - g[R(X, Y, LZ), W] = 0
\]
or
\[(3.3)\]
\[
g[LR(X, Y, Z) - R(X, Y, LZ), W] = 0.
\]

From (3.3) we get \(LR(X, Y, Z) - R(X, Y, LZ) = 0\), i.e. \([R(X, Y) \circ L](Z) = 0\), \(\forall Z\). Hence

\[(3.4)\]
\[
R(X, Y) \circ L = 0.
\]

Again, if (3.4) holds, then \(R(X, Y) \cdot S = 0\). In virtue of (3.4) it follows that the curvature and the Ricci transformations commute [by (1.7)].

The converse is also true. That is, if the curvature and the Ricci transformations commute, then (3.4) holds and therefore \(R(X, Y) \cdot S = 0\). This leads to the following result:

**Theorem 3.** In a \((QE)_n\) \((n > 3)\), the curvature and the Ricci transformations commute if and only if the relation \(A(R(X, Y, Z)) = 0\) holds.

4. \((QE)_n\) \((n > 3)\) with divergence-free conformal curvature tensor i.e. \(\text{div} C = 0\)

The conformal curvature tensor \(C\) of a Riemannian manifold \((M^n, g)\) is said to be conservative [1] if the divergence of \(C\) is zero. In such a case the manifold is said to be conformally conservative.

In this section we shall obtain two sufficient conditions for a \((QE)_n\) \((n > 3)\) to be conformally conservative. Let

\[(4.1)\]
\[
H(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) - \frac{1}{2(n-1)}[d\gamma(X)g(Y, Z) - d\gamma(Z)g(Y, X)].
\]
Then it is known [3] that $\text{div} C = 0$ if and only if $H(X,Y,Z) = 0$. We shall consider two types of $(QE)_n$ $(n > 3)$:

**Type I:** the associated scalars $a$ and $b$ are constants and therefore $\gamma$ is constant.

**Type II:** $a$ and $b$ are not constants but $a + b = 0$.

Type I: For this type $da(X) = 0$ and $db(X) = 0$ and therefore $d^\gamma(X) = 0$.

Now from (1.2) we get

$$(\nabla_X S)(Y,Z) = da(X)g(Y,Z) + db(X)A(Y)A(Z) + b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(\gamma)]$$

$$= b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(\gamma)].$$

Hence


Therefore (4.1) takes the following form:


From (4.3) we see that in this case $H(X,Y,Z)$ is not, in general, equal to zero.

We now impose the condition that the generator $U$ of the manifold is a recurrent vector field [4] with the associated 1-form $A$ not being the 1-form of recurrence. Then $\nabla_X U = B(X)U$, where $B$ is the 1-form of recurrence. Hence

$$g(\nabla_X U,Y) = g(B(X)U,Y)$$

or

$$(\nabla_X A)(Y) = B(X)A(Y).$$

In virtue of (4.4) we can express (4.3) as follows:

$$(4.5) \quad H(X,Y,Z) = b[B(X)A(Y)A(Z) + B(X)A(Z)A(Y) - B(Z)A(Y)A(X) - B(Z)A(X)A(Y)].$$
Since $(\nabla_X A)(U) = 0$, it follows from (4.4) that $B(X) = 0$. So $H(X, Y, Z) = 0$.
Hence we can state the following

**Theorem 4.** If in a $(QE)_n$ $(n > 3)$ the associated scalars are constants and the generator $U$ of the manifold is a recurrent vector field with the associated 1-form $A$ not being the 1-form of recurrence, then the manifold is conformally conservative.

Next we consider Type II. For this type, (4.6) $\gamma = (n - 1)a$ [by (1.3)]. Hence $\gamma$ is neither zero nor a non-zero constant. From (4.6) we get

$$d\gamma(X) = (n - 1)da(X).$$

Now

$$(\nabla_X S)(Y, Z) = da(X)g(Y, Z) - da(X)A(Y)A(Z)$$

$$- a[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)] \quad \because a + b = 0.$$  

Hence

$$(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X)$$

$$= da(X)g(Y, Z) - da(X)A(Y)A(Z)$$

$$- da(Z)g(Y, X) + da(Z)A(Y)A(X)$$

$$- a[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)]$$

$$+ a[(\nabla_Z A)(Y)A(X) + (\nabla_Z A)(X)A(Y)].$$

Again

$$\frac{1}{2(n - 1)}[d\gamma(X)g(Y, Z) - d\gamma(Z)g(Y, X)]$$

$$= \frac{1}{2}[da(X)g(Y, Z) - da(Z)g(Y, X)] \quad \text{by (4.7)].}$$

Hence

$$H(X, Y, Z) = \frac{1}{2} da(X)[g(Y, Z) - 2A(Y)A(Z)]$$

$$- \frac{1}{2} da(Z)[g(Y, X) - ZA(Y)A(X)]$$

$$+ a[(\nabla_Z A)(Y)A(X) - (\nabla_X A)(Y)A(Z)]$$

$$+ A(Y)\{(\nabla_Z A)(X) - (\nabla_X A)(Z)\}. $$
From (4.10) we see that $H(X, Y, Z)$ is not, in general, equal to zero. Hence we impose the conditions

(i) $U = \frac{1}{2a} \text{grad } a$

and

(ii) $\nabla_X U = -X + A(X)U$.

From (i) we get $g(X, U) = g \left( \frac{1}{2a} \text{grad } a, X \right)$ or

$A(X) = \frac{1}{2a} da(X)$

or

$$1 \cdot da(X) = a(A(X)).$$

Again from (ii) we get

$$\nabla_X A)(Y) = -g(X, Y) + A(X)A(Y).$$

In virtue of (4.11) and (4.12) we can express (4.10) as follows:

$$H(X, Y, Z) = aA(X)g(Y, Z) - 2aA(X)A(Y)A(Z)
- aA(Z)g(Y, X) - 2aA(X)A(Y)A(Z)
+ a[A(X)\{-g(Y, Z) + A(Y)A(Z)\}
- A(Z)\{-g(X, X) + A(X)A(Y)\}
+ A(Y)\{-g(X, Z) + A(X)A(Z)\}
+ g(X, Z) - A(X)A(Z)\}].$$

Hence $H(X, Y, Z) = 0$. 
Therefore we can state the following

**Theorem 5.** If in a \((QE)_n\) \((n > 3)\) the associated scalars are not constants but their sum is zero and the generator satisfies the conditions (i) and (ii), then the manifold is conformally conservative.

We shall next point out the geometric significance of the condition (ii), namely

\begin{equation}
\nabla_X U = -X + A(X)U.
\end{equation}

Let \(U^\perp\) denote the \((n - 1)\)-dimensional distribution in \((QE)_n\) orthogonal to \(U\). If \(X\) and \(Y\) belong to \(U^\perp\) where \(Y \neq \lambda X\), then

\begin{equation}
g(X, U) = 0
\end{equation}

and

\begin{equation}
g(Y, U) = 0.
\end{equation}

Since \((\nabla_X g)(Y, U) = 0\), it follows from (4.16) that by (4.14)

\begin{equation}
g(\nabla_X Y, U) = g(\nabla_X U, Y) = g(X, Y) - A(X)A(Y).
\end{equation}

Similarly from (4.15) we get

\begin{equation}
g(\nabla_Y X, U) = g(X, Y) - A(X)A(Y).
\end{equation}

Hence

\begin{equation}
g(\nabla_X Y, U) = g(\nabla_Y X, U).
\end{equation}

Now, \([X, Y] = \nabla_X Y - \nabla_Y X\). Therefore

\[g([X, Y], U) = g(\nabla_X Y - \nabla_Y X, U) = g(\nabla_X Y, U) - g(\nabla_Y X, U) = 0\]

by (4.19). Hence \([X, Y]\) is orthogonal to \(U\), i.e. \([X, Y] \in U^\perp\).

Thus the distribution \(U^\perp\) is involutive \([5]\). Hence from Frobenius' theorem \([5]\) it follows that \(U^\perp\) is integrable. This implies that the \((QE)_n\) is a product manifold.
We can therefore state the following

**Theorem 6.** If in a \((QE)_n\) \((n > 3)\) the associated scalars are not constants but their sum is zero and the generator of the manifold satisfies the conditions (i) and (ii) then this \((QE)_n\) is a product manifold.

Before concluding we would like to mention that the notion of a quasi-Einstein metric was introduced in 1996 by Chave and Valent [6]. They defined such a metric by the following constraint:

\[
S(X, Y) = ag(X, Y) + \frac{1}{2}[(\nabla_X A)(Y) + (\nabla_Y A)(X)],
\]

where \(S\) is the Ricci tensor of type \((0, 2)\), \(a\) is a scalar and \(A\) is a non-zero 1-form. Thus the notion of a quasi-Einstein manifold defined by us is different from that of a space of quasi-Einstein metric defined by Chave and Valent and our definition is new. This new notion was suggested by an unknown referee.

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**References**


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REFERENCES

ON TOTALLY UMBILICAL HYPERSURFACES OF A CONFORMALLY FLAT PSEUDO RICCI SYMMETRIC MANIFOLD.

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Introduction. The notion of a Pseudo Ricci Symmetric manifold was introduced by the first author [1] in 1988. According to him a non-flat Riemannian manifold \((M^n, g)\) \((n \geq 2)\) is said to be pseudo Ricci symmetric if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the condition

\[
(V_\alpha S)(y, z) = 2A(x)S(y, z) + A(y)S(x, z) + A(z)S(y, x),
\]

where \(A\) is a non-zero 1-form and \(V\) denotes the operator of covariant differentiation with respect to the metric tensor \(g\). In such a case \(A\) is called the associated 1-form and an \(n\)-dimensional manifold of this kind is denoted by the symbol \((PRS)_n\). Let \(g(x, U) = A(x)\) \(V\) vector fields \(x\). Then \(U\) is called the basic vector field of the manifold corresponding to the 1-form \(A\).

This paper deals with totally umbilical hypersurfaces of a conformally flat \((PRS)_n\) \((n \geq 3)\) with \(A\) as its associated 1-form and \(U\) as its unit basic vector field corresponding to the associated 1-form \(A\).

According to Chen and Yano [2] a Riemannian manifold \((M^n, g)\) \((n \geq 3)\) is said to be of quasi constant curvature if it is conformally flat and its curvature tensor \(R\) satisfies the condition

\[
g[R(x, y, z), w] = a[g(x, z)g(y, w) - g(y, z)g(x, w)]
+ b[g(x, z)B(y)B(w) - g(y, z)B(x)B(w)
+ g(y, w)B(x)B(z) - g(x, w)B(y)B(z)],
\]

where \(B\) is a non-zero 1-form such that

\[
g(x, V) = B(x) \quad \forall x,
\]

\(V\) being a unit vector field, and \(a, b\) are scalars of which \(b \neq 0\).

In such a case \(a\) and \(b\) are called associated scalars and \(V\) is called the generator of the manifold.

Using this notion of a manifold of quasi constant curvature it is shown in this paper that a totally umbilical hypersurface of a conformally flat \((PRS)_n\) \((n \geq 3)\) is a manifold of quasi constant curvature.

Further it is shown that such a totally umbilical hypersurface with commuting Ricci and curvature transformations is totally geodesic.

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1) Numbers in brackets refer to the references at the end of the paper.
§ 1. Preliminaries. Let $r$ denote the scalar curvature of a $(PRS)_n$ with $A$ as its associated 1-form and $U$ as its unit basic vector field. Further, let $L$ denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor $S$. Then

$$A(U) = 1$$

and $g(Lx, y) = S(x, y) \forall x, y$.

From (1) we have

$$\nabla S(y, z) - (\nabla S)(y, x) = A(x)S(y, z) - A(z)S(y, x).$$

Now, contracting (1.2) over $y$ and $z$ we get

$$dr(x) = 2A(x)r - 2A(Lx).$$

Next contracting (1) we get

$$dr(x) = 2A(x)r + 4A(Lx).$$

From (1.3) and (1.4) it follows that $A(Lx) = 0$. Hence from (1.3) we get

$$dr(x) = 2A(x)r.$$

These formulas will be used in the sequel.

§ 2. Totally umbilical hypersurfaces of a conformally flat $(PRS)$, $(n > 3)$. In this section we consider a hypersurface $(\bar{M}^{n-1}, \bar{g})$ of a conformally flat $(PRS)_n(n > 3)$ and denote the curvature tensor of the hypersurface by $\bar{R}$. Then for any vector fields $x, y, z, w$ tangent to $\bar{M}$ we have the following equation of Gauss ([3], p. 68)

$$g[R(x, y, z), w] = g[R(x, y, z), w] - g[B(x, y)B(y, z)] + g[B(y, w)B(x, z)],$$

where $R$ is the curvature tensor of $(PRS)_n$ and $B$ is the second fundamental form of $\bar{M}$. If

$$B(x, y) = \bar{g}(x, y)\mu$$

for any vector fields $x, y$ tangent to $\bar{M}$, where $\mu$ is the mean curvature vector of $\bar{M}$, then $\bar{M}$ is said to be totally umbilical ([3], p. 67).

It is known [1] that in a conformally flat $(PRS)_n(n > 3)$ the curvature tensor $R$ satisfies the following condition:

$$g[R(x, y, z), w] = \frac{r}{(n-1)(n-2)} [g(y, z)g(x, w) - g(x, z)g(y, w)]$$

$$+ \frac{r}{(n-1)(n-2)} [T(x)T(z)g(y, w) - T(y)T(x)g(x, w)]$$

$$+ T(y)T(w)g(x, z) - T(x)T(w)g(y, z)],$$

where

$$T(x) = \frac{A(x)}{\sqrt{A(U)}},$$

$A$ being the associated 1-form and $U$ being the basic vector field corresponding to $A$. If $x, y, z, w$ are vector fields tangent to $\bar{M}$, then using (2.3) we can express (2.1) as follows:
(2.5) \[ \bar{g}[\bar{R}(x, y, z), w] = \frac{r}{(n-1)(n-2)} [g(y, z)g(x, w) - g(x, z)g(y, w)] \\
+ \frac{r}{(n-1)(n-2)} [T(x)T(z)g(y, w) - T(y)T(z)g(x, w)] \\
+ T(y)T(w)g(x, z) - T(x)T(w)g(y, z)] \\
+ \bar{g}[B(x, w)B(y, z)] - \bar{g}[B(y, w)B(x, z)]. \]

Since by hypothesis \( \bar{M} \) is totally umbilical, (2.2) holds. Hence (2.5) takes the following form:

(2.6) \[ \bar{g}[\bar{R}(x, y, z), w] = \frac{r}{(n-1)(n-2)} [g(y, z)g(x, w) - g(x, z)g(y, w)] \\
+ \frac{r}{(n-1)(n-2)} [T(x)T(z)\bar{g}(y, w) - T(y)T(z)\bar{g}(x, w)] \\
+ T(y)T(w)\bar{g}(x, z) - T(x)T(w)\bar{g}(y, z)] \\
= a[\bar{g}(x, z)\bar{g}(y, w) - \bar{g}(x, w)\bar{g}(y, z)] + b[T(x)T(z)\bar{g}(y, w) - T(y)T(z)\bar{g}(x, w)] \\
+ T(y)T(w)\bar{g}(x, z) - T(x)T(w)\bar{g}(y, z)], \]

where

(2.7) \[ a = -\frac{r}{(n-1)(n-2)}[\mu^2] \quad \text{and} \quad b = \frac{r}{(n-1)(n-2)}. \]

and \( T(x) = A(x) \) in invirtue of (2.4) and (1.1).

Since in a conformally flat \((PRS)\), \( r \) cannot be zero, it follows from (2.7) that \( b \) cannot be zero.

Again \( T(x) = g(x, U) \), where \( U \) is a unit vector field. Comparing (2.6) with (2) and (3) we conclude that \( \bar{M} \) is a manifold of quasi constant curvature with associated scalars \( a, b \), given by (2.7) and generator \( U \). This leads to the following theorem.

**Theorem 1.** A totally umbilical hypersurface of a conformally flat \((PRS)\,(n>3)\) is a manifold of quasi constant curvature.

§ 3. Totally umbilical hypersurface of a conformally flat \((PRS)\) with commuting Ricci and curvature transformations of the hypersurface. In this section we investigate the effect of commuting Ricci and curvature transformations of a totally umbilical hypersurface of a conformally flat \((PRS)\,(n>3)\) on the second fundamental form of the hypersurface.

According to Theorem 1 \( \bar{M} \) is a manifold of quasi constant curvature for which the associated scalars \( a \) and \( b \) are given by

(3.1) \[ a = -\frac{r}{(n-1)(n-2)}[\mu^2] \quad \text{and} \quad b = \frac{r}{(n-1)(n-2)}. \]

Hence

It is known ([4], p. 222) that a sufficient condition for a manifold of quasi constant curvature with associated scalars \( a, b \), to have commuting Ricci and curvature transfor-
On totally umbilical hypersurfaces.

If $M$ has commuting Ricci and curvature transformations, then according to the above-mentioned sufficient condition $a+b$ must be zero because $M$ is of quasi constant curvature. Hence from (3.1) we get $|\mu|^2=0$ from which it follows that $\mu=0$. Hence from (2.2) we get

$$B(x, y) = 0$$

i.e. $B$ is identically zero.

It is known ([3], p. 67) that if the second fundamental form of a hypersurface of a Riemannian manifold $(M^n, g)$ is identically zero, then the hypersurface is totally geodesic. Hence from (3.2) it follows that the totally umbilical hypersurface $M$ under consideration is totally geodesic. Thus we can state the following:

**Theorem 2.** If a totally umbilical hypersurface of a conformally flat $(PRS)_n(n>3)$ has commuting Ricci and curvature transformations, then the hypersurface is totally geodesic.

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REFERENCES

CURVATURE COLLINEATION
SYMMETRY OF THE GÖDEL-TYPE
SPACE-TIMES.

By A. B. Shamardan, M. R. A. Mostarak and Y. A. Abd-Eltwab.

§ 1. Introduction. The Gödel-type space-time has a metric of the form: \[ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 + 2Mdx^0dx^3 - L(dx^3)^2\]

where \(M\) and \(L\) are functions of \(x^3\) only. The metric (1.1) admits some symmetric properties if the functions \(M\) and \(L\) satisfy certain conditions. For example, A. K. Raychaudhuri and S. N. Guha Thakurta [6] showed that the metric (1.1) is homogeneous (in the sense that it admits four parameter group of isometries) provided the functions \(M\) and \(L\) satisfy the two conditions: \(D^3D - (D^3)^2 = \text{constant}\), and \(\frac{M}{D} = \text{constant}\), where \(D^3 = L - M^2\) and the comma indicates ordinary partial derivative.

Several symmetry properties have been obtained and discussed later including homothetic and conformal motions, affine collineations, curvature collineations and finally curvature inheritance [3], [5]. A. M. Abd-Alla [1] succeeded in obtaining a vector field \(\xi^a\) in the Mong's form relative to which the homogeneous Gödel-type space-times possesses curvature collineation symmetry property [1]. Y. A. Abd Eltwab and H. F. At-del-Hameed [4] succeeded in obtaining vector fields relative to which the homogeneous Gödel-type space-times have curvature inheritance property [4]. In this paper we considered the curvature collineation symmetry property in the general Gödel-type space-times (not necessary homogeneous). We obtained sufficient conditions for the metric (1.1) to admit a curvature collineation symmetry property. These conditions lead to new Gödel-type space-times which are indeed universes. Besides, we obtained new vector fields \(\xi^a\) relative to which the new Gödel-type Universes possesses curvature collineation symmetry property.

§ 2. Preliminaries. A Riemannian space is said to admit a curvature collineation (CC) if there exists a vector \(\xi^a\) for which \(\mathcal{L}_\xi R_{bkm} = 0\) or

\[\mathcal{L}_\xi R_{bkm} = R_{bkm}^m - R_{mbk}^m \xi^m + R_{mbk}^n \xi^n + R_{mbk}^n \xi^n + R_{mbk}^m \xi^m = 0\]

where \(R_{bkm}\) is the Riemann curvature tensor and \(\mathcal{L}_\xi\) denotes the Lie derivative with respect to the vector field \(\xi^a\) ([3]). A CC is related to a special conformal motion which implies the existence of a covariant constant vector field. Precisely, in a Riemann space with vanishing scalar curvature \(R\), a covariant conservation law generator is obtained as a consequence of the existance of a CC. The nonvanishing components of the Riemannian curvature tensors

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1) Numbers in brackets refer to the references at the end of the paper.