Synopsis:

Summarising the main features of the quantum-decay problem and associated lifetime-width relation, often referred to as a manifestation of the time-energy uncertainty principle, we briefly survey the current situation in Sec. 5.1. Our work on the Mandelstam-Tamm inequality produces several useful new results. These include some properties of quantum decay and a transparent notion of physical time with its role in the time-energy uncertainty principle. This work is presented in Sec. 5.2. Discussions on the importance of the inequalities we find would concern us in Sec. 5.3, where we also remark on the differences between position-momentum-type uncertainties and the form of time-energy inequality we obtain. A non-trivial example is chosen to justify some of our assertions.
5.1. **Introductory Survey**

The problem of decay is actually related to time-asymmetry and arises in a wide variety of physical processes like radioactivity, spontaneous emission, autoionization, radiationless transitions, etc.

A strict quantum-mechanical explanation of the rather general exponential decay is, however, impossible\(^1,2\). So, this 'quantum decay' has attracted considerable recent attention. As we all know, the dynamics of a system (prepared in a definite state) governed by some Hamiltonian with purely discrete spectrum shows quantum recurrence\(^3\) on the one hand, and the system decays if the Hamiltonian concerned has a continuous spectrum\(^4\), on the other. In fact, both these features are of interest to us due to the possibility\(^5\) of another important phenomenon - quantum stochasticity or chaos\(^6\). Various models have been used to study these possibilities\(^7\). Some demonstrative analytical work on decay are also available in the literature\(^8\).

The quantum of decay\(^2\) is concerned with the quantity

\[
\Lambda_t = |\Phi e^{-i\mathbf{H}t/\hbar}\rangle \tag{5.1}
\]

where \(\Phi\) denotes the state of the system at \(t = 0\) just from when the dynamics of the system is governed by \(\mathbf{H}\). The non-decay probability \(p_t\) is then given by

\[
p_t = |\Lambda_t|^2. \tag{5.2}
\]

Denoting the common eigenstates of a complete set of commuting observables \((\mathbf{H}, \mathbf{A})\) by \(\Psi(E, \mathbf{A})\) and expanding \(\Phi\) in terms of \(\Psi(E, \mathbf{A})\), we get from (5.1)
\[ A_t = \int_{E_m}^{\infty} w(E) e^{-i\frac{E t}{\hbar}} dE, \quad w(E) = \int |\langle \phi | \Psi(E, \alpha) \rangle|^2 d\alpha \quad (5.3) \]

where \( E_m \) refers to the minimum of the energy spectrum. It is easy to see that \( P_t \) in (5.2) can turn out to be an exponential under very restrictive conditions. Thus, if we choose

\[ w(E) = \frac{\beta/2\hbar}{(E - \epsilon)^2 + \frac{1}{4} \beta^2}, \quad (5.4) \]

the well-known Breit-Wigner form (having width \( \Gamma = \beta \) at half-maximum of the (Lorentzian) line shape), and also assume that \( E_m = -\infty \) in (5.3), we obtain

\[ A_t = e^{-(i\epsilon + \beta/2)t/\hbar}, \quad (5.5) \]

which gives

\[ P_t = e^{-\beta t/\hbar} \quad (5.6) \]

and hence satisfies

\[ \Gamma \times \frac{1}{T_a} = \hbar \quad (5.7) \]

where \( T_a \) is the average life, equal to \( \hbar/\beta \) from (5.6). It is also widely accepted\(^9,^{10} \) that (5.7) is actually a manifestation of the time-energy uncertainty relation \(^* \) (TEUR). We may note at this point the following disturbing features: (i) the status of the width \( \Gamma \) at half-maximum of a lineshape function is very different from conventional root-mean-square deviations which

\[ ^* \text{A good reference list may be found in ref.} 10; \text{see especially the articles of Bohm and Allcock; see also M. Jammer, The Philosophy of Quantum Mechanics, Sec. 5.4. 1974 (Wiley : New York) and refs. therein.} \]
appear in usual uncertainty principles, (ii) actually,
\[ \Delta \mathcal{B} = \sqrt{\langle \phi | H^2 | \phi \rangle} - \langle \phi | H | \phi \rangle^2 \] turns out to be infinite in this case, implying \( \phi \notin D(H) \), (iii) here, we assume \( E_m = -\infty \) and such an assumption is usually said to be a severe one, and (iv) the form (5.7) is an equality, i.e., it is a certainty relation.

It may be remarked here that the TEUR and quantum decay have one common characteristic – both of these require a continuous spectrum of \( H \); the former one, in addition, necessitates some \( H \) unbounded from below if the operator formulation of time is sought. So, naturally, the TEUR appears frequently in the context of decay. However, since \( \mathcal{F}_t \) can never be purely exponential in nature, unless there exists some kind of openness, it seems desirable to have an explicit form of the TEUR involving the lifetime (which, we feel, should appear understandably because we would be dealing with an unstable, i.e., decaying, state) for a closed but decaying quantum system. Recently, Bauer and Seller analyzed the problem in detail and concluded that neither the conventional Mandelstam-Tamm formulation nor the Wigner development (involving non-conventional averages) of the TEUR is suitable to arrive at the \( \Gamma - \tau \) relation for a decaying state; rather, they opined, to achieve this end one has to talk of 'spreads' in terms of 'equivalent-widths'. However, it is again difficult to understand why the width of the lineshape should be so fundamental a quantity to appear in the TEUR.

The properties of quantum decay have been studied by various authors, sometimes by using models, especially to investigate the non-exponential nature.
attitude seems to be the reverse - to derive quasi-exponential decay laws from first principles. The basic result indicates that the behaviour of \( P_t \) departs from an exponential form both for very small and large times, while for intermediate times \( P_t \) shows a quasi-exponential decay.

5.2. Decay And The MT Inequality:

The primary difficulty with the TEUR in the form of the well-known MT inequality (Messiah) given by

\[
T_{A_1} = \frac{\Delta A_1}{|\langle A_1 \rangle/\langle dt \rangle|} \geq \frac{\hbar}{2\Delta E} \tag{5.8}
\]

is that the time \( T_{A_1} \), though it is said to refer to some time characteristic of the observable corresponding to the operator \( A_1 \), does not correspond to the physical time \( t \) of evolution of the system with respect to some arbitrarily chosen time of its preparation in some particular 'packet' state; it possesses merely the dimension of time. Hence, although we can, in principle, find out the shortest time \( T_{A_m} \) from among the set of observables \( \{A_1\} \) for the system, \( T_{A_m} \) cannot probably 'be considered as a characteristic time of evolution of the system itself' (Messiah). It is also understandable, hence, that the MT time would not correspond to the lifetime for a decaying state; an explicit demonstrative calculation led Bauer and Hello to the same conclusion.

We know that a quantum system, prepared at \( t=0 \) in some non-stationary state \(|\phi\rangle\), evolves causally according to the equation

\[
\phi_t = e^{-ilHt/\hbar} |\phi\rangle, \phi \in \mathcal{D}(H) \tag{5.9}
\]
where \( H \neq H(t) \), is the Hamiltonian of the system. If \( |\phi\rangle \) is expressible as an integral over the continuous energy-eigenstates of \( H \) the system would decay and then \( P_t = |\langle \phi | \phi_t \rangle|^2 \) (see also (5.2)) shows immediately from (5.9) that it has to be an even function of time, a property which alone excludes not only the possibility of an exponential decay but also other monotone decreasing odd functions from being \( P_t \). Now, choosing \( A_1 \) of the form of a projector

\[
A_1 = |\phi\rangle \langle \phi| \quad (5.10)
\]
we find from (5.8) the inequality

\[
\left[ P_t (1 - P_t) \right]^2 / |dP_t / dt| \geq \hbar / 2 \Delta E \quad (5.11)
\]

Remembering that \( \Delta E \) has to be finite (for \( \| H \phi \| \) is finite, \( \phi \in \mathcal{D}(H) \) ), the above inequality shows several interesting features:

(a) Rearranging (5.11), we obtain

\[
|dP_t / dt| \leq (2 \Delta E / \hbar) \left[ P_t (1 - P_t) \right]^2 \quad (5.12)
\]
which marks the time \( t_h \) when \( P_t = \frac{1}{2} \), i.e., the half-life, as a significant time, and not the average life, for then only the right hand side attains its maximum value so that

\[
|dP_t / dt| \leq \Delta E / \hbar \quad (5.13)
\]
always holds; the equality in (5.13), however, may hold only at \( t_h \). The message of (5.13) is that no unstable quantum system can decay completely within a time \( \hbar / \Delta E \). This time, though is a rather crude estimate (see below), clearly establishes a limit to the
instability of a decaying quantum system in time-sense.

(b) From (5.12), we obtain some characteristics of quantum decay for different regions of time (note that as \( t \to 0 \), \( P_t \to 1 \) and as \( t \to \infty \), \( P_t \to 0 \)):

\[
\frac{dP_t}{dt} = 0, \quad t = 0 \quad (5.14a)
\]

\[
d \cos^{-1} P_t/\Delta E/\hbar, \quad t \to 0 \quad (5.14b)
\]

\[
d P_t^2/dt \leq \Delta E/\hbar, \quad t \to \infty \quad (5.14c)
\]

and, in general,

\[
d \cos^{-1} P_t/\Delta E/\hbar, \quad 0 < t < \infty \quad (5.15)
\]

These results are likely to be of interest in small- and large-time studies of the behaviour of \( P_t \).

(c) Integrating (5.15) directly, one obtains

\[
t \geq \left( \frac{\hbar}{\Delta E} \right) \cos^{-1} P_t^{\frac{\gamma}{2}} \quad (5.16)
\]

so that the minimum-time limit, stated earlier, turns out to be precisely \( \pi \hbar /2 \Delta E \). Also, (5.16) leads straightforwardly to the inequality

\[
P_t \geq \cos^2 (\Delta E t/\hbar), \quad 0 \leq t \leq \pi \hbar /2 \Delta E \quad (5.17)
\]

obtained also by Fleming through a different and lengthy route. From (5.17) we find, using \( P_{T_h} = \frac{1}{2} \), that

\[
\Delta E T_h \geq \pi \hbar /4, \quad 0 < T_h \leq \pi \hbar /2 \Delta E. \quad (5.18)
\]

On the other hand, if \( T_h > \pi \hbar /2 \Delta E \), we have

\[
P_t \geq 0 > 1 - 2 \Delta E t/\pi \hbar, \quad t > \pi \hbar /2 \Delta E \quad (5.19)
\]
from which it again follows that
\[ \Delta E T_h > \frac{\pi \hbar}{4}, \frac{p_n}{T_h} > \frac{\pi \hbar}{2 \Delta E}. \] (5.20)

Joining (5.18) and (5.20), we obtain the desired TBUR for a decaying quantum system:

\[ \Delta E T_h > \frac{\pi \hbar}{4}. \] (5.21)

(d) We can use (5.12) to obtain also a lower bound to \( P_t \) over a time-interval not contained in (5.17) as follows. For

\[ t < T_h, \frac{1}{2} < P_t < 1, \] so that the inequality

\[ |dP_t/dt| \leq 2 \Delta E P_t/\hbar \] (5.22)

holds and this, on integration, gives

\[ P_t \geq e^{-2 \Delta E t/\hbar}, \quad 0 \leq t \leq T_h. \] (5.23)

Thus, for unstable packets with not-too-small lifetimes (i.e. \( T_h > \pi /2 \Delta E \)), the inequality (5.23) would be useful in studying the small-time behaviour of \( P_t \) (for a discussion, see \( \text{O} \)hirardi et al.22) in the range \( \pi /2 \Delta E \leq t < T_h \).

(e) Focussing our attention on the 'intermediate times' over which \( P_t \) is usually said to follow an exponential character, we denote by \( T \) the time which corresponds to the extremum of the function \( -\ln P_t/t \) and find that

\[ T = \frac{P_T \ln P_T}{dP_t/dt} \bigg|_T \] (5.24)

which, by virtue of (5.12), leads to the following inequality.
Understandably, it is only over a region around \( T \) where \( P_\perp \) would behave in an almost exponential fashion. Noting that the bracketed part of (5.25) is bounded from above and that for small enough times \((- \ln P/t) \Delta E^2/\hbar^2 \), coupled with (5.13), one can now make the following remark. If \( \Delta E \) is large, \( T \) may be sufficiently small and the region over which \( P_\perp \) behaves near-exponentially may be small, but, for small \( \Delta E \), \( T \) would be considerably large, so also would be the desired near-exponential region. From a detailed analysis, emphasizing particularly the preparation of the unstable state, Fonda et al. also arrived at the same conclusions.

5.3. Results And Discussion:

We thus see that the MT TSUR can be suitably employed to obtain several useful results in the context of decay.

As regards the TSUR, we may here emphasize the following important points. Consider (a system in) some state \( \phi \) at some particular instant of time (so that \( \phi \) depends on co-ordinates only); it satisfies the \( \Delta x - \Delta p \) uncertainty; in fact, we need not specify the system (i.e., the Hamiltonian) for this purpose. For example, the function \( \phi = e^{-kx^d} \) would have some value of the product \( \Delta x \Delta p_x \geq \hbar/2 \), irrespective of whether the function represents some state of a harmonic oscillator, anharmonic oscillator, free particle (packet), or else. However, to understand the significance of the TSUR for \( \phi \), we have to specify the system.
choose some system such that \( \Phi \) concerned is expressible in terms of only the continuum eigenstates of the \( H \) corresponding to this system and that \( \| H \Phi \| \) is finite, i.e. \( \Phi \in D(H) \). The \( H \) to be chosen can be otherwise completely arbitrary. Then, considering the instant of time concerned as the 'time-zero' of preparation of the system defined by \( H \) in state \( \Phi \), we note that the system would decay and the TLUK states that

(1) the time required for the system to decay completely \( (T_0) \) obeys

\[
T_0 \Delta E \geq \pi \frac{\hbar}{2}
\]

(see eq.(5.16)) whatever may be the chosen \( H \) (obeying, of course, the two conditions mentioned), or,

(2) more precisely, the half-life for the decaying system obeys

\[
T_h \Delta E \geq \kappa \frac{\hbar}{4}
\]

(see eq.(5.21)). Let us note that we have always inequalities greater than \( \hbar/2 \).

Thus, comparing with the \( \Delta x - \Delta p \) uncertainty, here, three features are important - the 'simultaneity' loses significance, a restriction on \( H \) is imposed and the TLUK involves inequality greater than \( \hbar/2 \) in magnitude.

On the other hand, some inequalities we obtain are also useful in the context of decay, e.g., those given by eqs.(5.14), (5.23), etc. (see also the discussion below (5.25). An example to be considered below would reveal some additional features of the problem.

We shall study now the behaviour of a normalised Gaussian
packet,

\[ \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\delta^\gamma}{\delta^\gamma} e^{-\frac{x^2}{2\delta^2}} e^{ikx} \, dk \]  \hspace{1cm} (5.27)

in field-free space. Noting that \( \phi(x) \) can be written in the form

\[ \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\delta^\gamma}{\delta^\gamma} e^{-\frac{x^2}{2\delta^2}} e^{ikx} \, dk \]  \hspace{1cm} (5.27)

we find, with \( H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \),

\[ e^{-iHt/\hbar} \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\delta^\gamma}{\delta^\gamma} e^{-\frac{k^2}{2} \left( \delta^2 + \frac{i\hbar t}{m} \right)} e^{ikx} \, dk \]

\[ = \frac{\delta^\gamma}{\delta^\gamma} \frac{1}{\delta'} e^{-\frac{x^2}{2\delta'^2}}, \quad \delta' = \left( \delta^2 + \frac{i\hbar t}{m} \right), \]

so that

\[ A_t = \langle \phi | e^{-iHt/\hbar} | \phi \rangle = \frac{1}{\delta' \sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\delta'^2}} e^{-\frac{x^2}{2\delta^2}} \, dx \]

\[ = \frac{1}{\sqrt{1 + \frac{i\hbar t}{2m\delta^2}}} \]  \hspace{1cm} (5.29)

which gives

\[ P_t = \left( 1 + \frac{\hbar^2 t^2}{4m^2 \delta^4} \right)^{-\gamma/2} \]

\[ = \left( 1 + 2 \Delta \hbar^2 t^2 / \hbar^2 \right)^{-\gamma/2} \]  \hspace{1cm} (5.30)

where

\[ \Delta \hbar = \frac{\hbar^2}{\sqrt{8} m \delta^2}. \]

From (5.32), we immediately obtain
\[ \Delta \Theta \frac{T_n}{n} = \sqrt{\frac{3}{2}} \frac{n}{\Delta S} \] (5.33)

which is in conformity with the TEUR given by (5.21). Also, we see from (5.31) that for large \( t \), \( P_t \propto t^{-1} \) and thus in this particular case the more-or-less expected general behaviour, \( P_t \propto t^{-3} \), is not obeyed. For this packet, the near-exponential behaviour (see the discussion around eq.(5.25)) prevails around \( T \sim \sqrt{2} \frac{n}{\Delta S} \). A look at Table 1 would show this near stationary character around \( T \).

Finally, we wish to mention that Fleming\textsuperscript{21} also obtained an uncertainty-type relation using (5.17) and involving an 'average life' of a decaying state which may not always be finite (his definition of average life has been used by some other authors also (see ref.21); our example gives a meaningless result for his average life) which thus limits the validity of his result. In this respect, our derived relationship\textsuperscript{24} is definitely more general and hence advantageous.
Table 1. Hear-stationary behaviour around the potential decay region

- **Note**: $t = \frac{1}{2} \Delta t$.

<table>
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</tr>
</tbody>
</table>
REFERENCES

5. For a time-periodic $\hat{H}$ with discrete quasienergy spectra, this has been questioned by T. Hogg and B.A. Huberman, Phys. Rev. Letts. 48, 711, 1982.
12. Decay requires, $\phi_0$ be expressed in terms of continuum eigenstates of $\hat{H}$, as the Fock-Krylov theorem demands; whereas, the existence of a time-operator satisfying $[T, \hat{H}] = i\hat{H}$, like $[x, p_x] = i\hbar$, would require a continuous spectrum of energy from $-\infty$ to $+\infty$.
13. The relevant refs. may be found from Sec.1.3, (CH. 1, refs. 102-105); see also N.J. Newton, Ann. Phys. (NY) 124, 527, 1980 and M. Bauer, ibid, 150, 1, 1983.
18. See, e.g. ref.11; ref.16; A.M. Lane, Phys. Letts. 94A, 359, 1983; for lifetime calculations, see, e.g., W. Tobocman, Phys. Rev. C17, 2205, 1978 and refs. therein.

20. For a summary of results, see refs. 2 and 11; later works include: Hack\textsuperscript{17} (large-t study without Payley-Weiner theorem), Lane\textsuperscript{18} (small-t study in relation to Golden-rule decay), etc.


