PART D

WAVE SCATTERING BY SURFACE DISCONTINUITIES
Chapter 11

Carleman integral equations in the dock problem of surface water waves

11.1 Introduction

Two-dimensional water wave scattering problems involving discontinuities in the surface boundary conditions constitute an important special class in the linearised theory of water waves. As stated earlier in the general introduction several mathematical techniques have been employed in the literature for their solutions. For example, the Wiener-Hopf technique was used by Keller and Weitz (1953), Gol'dshtein and Marchenko (1989) while the modified residue calculus technique was used by Evans and Linton (1994), Linton (2001). Hermans (2003, 2004) used a method based on solving an integral equation along the region of the water surface which contains a floating dock or floating ice (thin elastic plate), by means of a superposition of exponential functions. Earlier Chakrabarti (2000a, 2000b) used Havelock type expansion of the velocity potential to reduce two such problems to some special forms of Carleman type
singular integral equations of the second kind over semi-infinite interval. These equations were then solved in closed forms by casting them into certain Riemann Hilbert problems on the positive real axis of the complex plane. Closed form expressions for the reflection and transmission coefficients for each problem were then derived.

In this chapter, the classical semi-infinite rigid dock problem, which is that of scattering of surface water waves by a semi-infinite rigid dock floating on the surface of deep water, first considered by Friedrichs and Lewy (1949) using a method related to the Laplace transformation (also see Stoker (1957), chapter 5, and Holford (1964), Appendix), is re-examined by reducing it to two Carleman type singular integral equations. Two procedures are employed for this purpose. The first is based on Fourier cosine inversion formula while the second is based on Havelock's inversion formula (see Ursell (1947)). The integral equations are solved by recasting them to Riemann Hilbert problems as in Chakrabarti (2000a, 2000b). Both the methods produce the same result for the reflection coefficient. The velocity potential can then be derived in closed form.

11.2 Mathematical formulation

Assuming linear theory and irrotational motion, the classical semi-infinite dock problem consists of solving the following boundary value problem described by the function \( \phi(x,y) \) (where the motion in water is described by the velocity potential \( \text{Re}\phi(x,y)e^{-i\omega t} \), the time-dependence factor \( e^{-i\omega t} \) being suppressed throughout):

\[
\nabla^2 \phi = 0, \quad -\infty < x < \infty, \quad y > 0
\]
\[
K\phi + \phi_y = 0 \text{ on } y = 0, \quad x < 0
\]  
(11.2.1)
(11.2.2)

where \( K = \omega^2/g, g \) being the gravity,

\[
\phi_y = 0 \text{ on } y = 0, \quad x > 0
\]  
(11.2.3)
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\[ r \frac{\partial \phi}{\partial r} = 0 \quad \text{as} \quad r = (x^2 + y^2)^{1/2} \to 0 \]  
(11.2.4)

\[ \phi \to 0 \quad \text{as} \quad y \to \infty, \]  
(11.2.5)

\[ \phi \to \begin{cases} 
  e^{-Ky+ikz} + Re^{-Ky-ikz} & \text{as} \quad x \to -\infty \\
  0 & \text{as} \quad x \to \infty
\end{cases} \]  
(11.2.6)

where \( R \) is the reflection coefficient (unknown) and is to be determined. Here
the dock is in the form of a semi-infinite rigid plate floating on infinitely deep
water and its lower part occupies the region \( y = 0, x > 0 (-\infty < z < \infty) \),
where \( y \)-axis is chosen vertically downwards and the plane \( y = 0, x < 0 (-\infty < 
\[ \text{and} \quad z < \infty \] \) coincides with the rest position of the free surface. The condition
(11.2.2) is the usual free surface condition and (11.2.3) is the condition on the
rigid dock, (11.2.4) is the edge condition (see Stoker (1957), chapter 5). The
term \( e^{-Ky+ikz} \) in (11.2.6) denotes incident wave field.

Using Havelock's expansion of water wave potential (cf. Ursell (1947)), one
can represent \( \phi(x,y) \) in the region \( x < 0 \) and \( x > 0 \) \( (y > 0) \), satisfying (11.2.1),
(11.2.2), (11.2.3), (11.2.5) and (11.2.6) as

\[ \phi(x, y) = e^{-Ky+ikz} + Re^{-Ky-ikz} + \frac{2}{\pi} \int_0^{\infty} \frac{A(\xi)}{\xi^2 + K^2} L(\xi, y)e^{\xi \xi} d\xi, \quad \text{for} \quad x < 0, \]  
(11.2.7)

\[ \phi(x, y) = \frac{2}{\pi} \int_0^{\infty} \frac{B(\xi)}{\xi} \cos \xi y e^{-\xi \xi} d\xi, \quad \text{for} \quad x > 0, \]  
(11.2.8)

where

\[ L(\xi, y) = \xi \cos \xi y - K \sin \xi y, \]

and \( A(\xi), B(\xi) \) are two unknown functions to be determined and are such
that the integrals involving these in the subsequent mathematical analysis are
convergent.

Continuity of \( \phi(x, y) \) and \( \phi_x(x, y) \) across \( x = 0 \) \( (y > 0) \) produces the relations

\[ (1+R)e^{-Ky} = \frac{2}{\pi} \int_0^{\infty} \frac{A(\xi)}{\xi^2 + K^2} L(\xi, y) d\xi = \frac{2}{\pi} \int_0^{\infty} \frac{B(\xi)}{\xi} \cos \xi y d\xi, \quad y > 0 \]  
(11.2.9)
and

\[ iK(1 - R)e^{-Ky} + \frac{2}{\pi} \int_0^\infty \frac{\xi A(\xi)}{\xi^2 + K^2} L(\xi, y)\,d\xi = -\frac{2}{\pi} \int_0^\infty \frac{B(\xi)}{\xi} \cos \xi y\,d\xi, \quad y > 0. \quad (11.2.10) \]

The relations (11.2.9) and (11.2.10) can be viewed in two different manners. First, the Fourier cosine inversion formula can be used in (11.2.9) and (11.2.10) to obtain two different forms of \(B(\xi)\) in terms of \(A(\xi)\) and \(R\). Elimination of \(B(\xi)\) will then produce a singular integral equation of Carleman type for \(A(\xi)\). Also the Havelock's inversion formula can be used in (11.2.9) and (11.2.10) to obtain two different forms of \(A(\xi)\) in terms of \(B(\xi)\) along with two other relations involving only \(B(\xi)\) and \(R\). Elimination of \(A(\xi)\) produces a singular integral equation of Carleman type for \(B(\xi)\). In the following analysis the relations (11.2.9) and (11.2.10) are attacked by both the above procedures.

In the successive mathematical analysis the following generalized identities will be required:

\[
\begin{align*}
\lim_{\xi \to 0} \int_0^\infty e^{-uy} \cos u\xi \cos \xi y\,dy &= \frac{\pi}{2} \{\delta(\xi - u) + \delta(\xi + u)\}, \\
\lim_{\xi \to 0} \int_0^\infty e^{-uy} \sin u\xi \sin \xi y\,dy &= \frac{\pi}{2} \{\delta(\xi - u) - \delta(\xi + u)\}, \\
\lim_{\xi \to 0} \int_0^\infty e^{-uy} \sin u\xi \cos \xi y\,dy &= \frac{u}{u^2 - \xi^2}.
\end{align*}
\quad (11.2.11)
\]

where \(u, \xi > 0\) and \(\delta(x)\) is the Dirac delta function.

### 11.3 Reduction to Carleman Equation for \(A(\xi)\):

Use of Fourier cosine inversion in (11.2.9) and (11.2.10) gives relations to solve for \(B(\xi)\):

\[
\frac{B(\xi)}{\xi} = \frac{(1 + R)K}{\xi^2 + K^2} + \frac{\xi A(\xi)}{\xi^2 + K^2} - \frac{2K}{\pi} \int_0^\infty \frac{uA(u)}{(u^2 - \xi^2)(u^2 + K^2)}\,du, \quad \xi > 0.
\quad (11.3.1)
\]
and

$$-B(\xi) = \frac{i(1 - R)K}{\xi^2 + K^2} + \frac{\xi^2 A(\xi)}{\xi^2 + K^2} - \frac{2K}{\pi} \int_{0}^{\infty} \frac{u^2 A(u)}{(u^2 - \xi^2)(u^2 + K^2)} du, \quad \xi > 0$$  \hspace{1cm} (11.3.2)

after using the results from the relations (11.2.11); the integrals being in the sense of Cauchy principal value. It may be noted that the relations (11.3.1) and (11.3.2) will be consistent if and only if

$$A(0) = 0 \quad \text{or} \quad 1.$$  \hspace{1cm} (11.3.3)

Elimination of $B(\xi)$ between (11.3.1) and (11.3.2) produces the Carleman type singular integral equation of the form

$$\xi C(\xi) - \frac{K}{\pi} \int_{0}^{\infty} \frac{C(u)}{u - \xi} du = -K \left( \frac{1}{\xi - iK} + \frac{R}{\xi + iK} \right), \quad \xi > 0,$$  \hspace{1cm} (11.3.4)

where

$$C(\xi) = \frac{2\xi A(\xi)}{\xi^2 + K^2}$$  \hspace{1cm} (11.3.5)

so that

$$C(\xi) = O(\xi^{-2}), \quad \text{as} \quad \xi \to \infty.$$  \hspace{1cm} (11.3.6)

The equation (11.3.4) when solved will determine the actual unknown function $A(\xi)$ provided the unknown constant $R$ can also be determined successfully. Then the function $B(\xi)$ can be determined by using any of the two relations (11.3.1) and (11.3.2), and then, the problem gets solved completely by using (11.2.7) and (11.2.8).

In order to solve the integral equation (11.3.4), let

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{0}^{\infty} \frac{C(u)}{u - \zeta} du, \quad \zeta = \xi + i\eta, \eta \neq 0$$  \hspace{1cm} (11.3.7)

where $\xi$ and $\eta$ are real. Then $\Phi(\zeta)$ is analytic in the complex $\zeta$-plane cut along the positive real axis.

If

$$\Phi^+(\xi) = \lim_{\eta \to 0^+} \Phi(\zeta), \quad \xi > 0,$$  \hspace{1cm} (11.3.8)
then by the Plemelj’s formulae, the equation (11.3.4) produces
\[ \Phi^+(\xi) - \frac{\xi + iK}{\xi - iK} \Phi^-(\xi) = -K \left\{ \frac{R}{\xi^2 + K^2} + \frac{1}{(\xi - iK)^2} \right\}, \xi > 0. \]  
\[ \text{(11.3.9)} \]

The relation (11.3.9) represents a Riemann Hilbert problem for the sectionally analytic function \( \Phi(\zeta) \) whose solution can be written as
\[ \Phi(\zeta) = -\frac{K}{2\pi i} \Phi_0(\zeta) \int_0^\infty \left\{ \frac{R}{u^2 + K^2} + \frac{1}{(u - iK)^2} \right\} \frac{1}{\Phi_0^+(u)(u - \zeta)} du \]  
\[ \text{(11.3.10)} \]

where
\[ \frac{\Phi_0^+(\xi)}{\Phi_0(\xi)} = \frac{\xi + iK}{\xi - iK}, \xi > 0, \]  
\[ \text{(11.3.11)} \]
giving
\[ \Phi_0(\zeta) = \exp \left[ \frac{1}{2\pi i} \int_0^\infty \ln \left( \frac{u + iK}{u - iK} \right) \frac{du}{u - \zeta} \right]. \]  
\[ \text{(11.3.12)} \]

Using Plemelj’s formulae once again, along with the relation (11.3.11), \( C(\xi) \) is obtained as
\[ C(\xi) = \Phi^+(\xi) - \Phi^-(\xi) = -\frac{K\xi}{\xi + iK} \left\{ \frac{R}{\xi^2 + K^2} + \frac{1}{(\xi - iK)^2} \right\} \]  
\[ -\frac{K^2 \Phi_0^+(\xi)}{\pi(\xi + iK)} \int_0^\infty \left\{ \frac{R}{u^2 + K^2} + \frac{1}{(u - iK)^2} \right\} \frac{1}{\Phi_0^+(u)(u - \zeta)} du, \xi > 0. \]  
\[ \text{(11.3.13)} \]

The integrals appearing in (11.3.13) can be evaluated by considering integrals of the form
\[ I(\zeta) = \int_\Gamma \frac{P(\tau)}{Q(\tau) \Phi_0(\tau)(\tau - \zeta)} d\tau \]  
\[ \text{(11.3.14)} \]
with \( \Gamma \) a positively oriented closed contour, consisting of a loop around the positive real axis and a circle of large radius with centre at the origin, in the complex \( \tau \)-plane, and \( P(\tau) \) and \( Q(\tau) \) are polynomials in \( \tau \). If these polynomials are such that the contribution to the integral in (11.3.14) over the circle of large radius vanishes, then
\[ I(\zeta) = \int_\Gamma \frac{P(u)}{Q(u) \left( \frac{1}{\Phi_0^+(u)} - \frac{1}{\Phi_0^-(u)} \right)} \frac{1}{(u - \zeta)} du = -2iK \int_0^\infty \frac{P(u)}{Q(u) \Phi_0^+(u)(u - iK)(u - \zeta)} du \]  
\[ \text{(11.3.15)} \]
after using the relation (11.3.11). Thus by using \( P(r) = 1 \) and \( Q(r) = -2iK(r + iK) \) it is observed that

\[
I_1(\zeta) = \int_0^\infty \frac{1}{(u^2 + K^2)\Phi_0^+(u)(u - iK)} du = -\frac{1}{2iK} \int \frac{d\tau}{(\tau + iK)\Phi_0(\tau)(\tau - \zeta)}
\]

\[
= -\frac{\pi}{K \zeta + iK} \left\{ \frac{1}{\Phi_0(\zeta)} - \frac{1}{\Phi_0(-iK)} \right\}.
\]

(11.3.16)

Similarly, by choosing \( P(\tau) = 1 \) and \( Q(\tau) = -2iK(\tau - iK) \) in (11.3.15),

\[
I_2(\zeta) = \int_0^\infty \frac{1}{(u-iK)^2\Phi_0^+(u)(u-iK)} du = -\frac{1}{2iK} \int \frac{d\tau}{(\tau - iK)\Phi_0(\tau)(\tau - \zeta)}
\]

\[
= -\frac{\pi}{K \zeta - iK} \left\{ \frac{1}{\Phi_0(\zeta)} - \frac{1}{\Phi_0(iK)} \right\}.
\]

(11.3.17)

Using Plemelj's formulæ

\[
\int_0^\infty \frac{1}{(u^2 + K^2)\Phi_0^+(u)(u - \xi)} du = \frac{1}{2} \{ I_1^+(\xi) + I_1^-(\xi) \}
\]

\[
= \frac{\pi}{K \xi + iK} \left\{ \frac{1}{\Phi_0(-iK)} - \frac{\xi}{(\xi - iK)\Phi_0^+(\xi)} \right\}
\]

(11.3.18)

and

\[
\int_0^\infty \frac{1}{(u-iK)^2\Phi_0^+(u)(u-\xi)} du = \frac{1}{2} \{ I_2^+(\xi) + I_2^-(\xi) \}
\]

\[
= \frac{\pi}{K \xi - iK} \left\{ \frac{1}{\Phi_0(iK)} - \frac{\xi}{(\xi - iK)\Phi_0^+(\xi)} \right\}.
\]

(11.3.19)

Using (11.3.18) and (11.3.19) in (11.3.13), \( C(\xi) \) is obtained as

\[
C(\xi) = -\frac{K \Phi_0^+(\xi)}{\xi^2 + K^2 \Phi_0(iK)} - \frac{KR}{(\xi + iK)^2 \Phi_0(-iK)} \Phi_0^+(\xi), \quad \xi > 0.
\]

(11.3.20)

Thus from (11.3.5) it is found that

\[
A(\xi) = \frac{K}{2\xi} \left\{ \frac{\Phi_0^+(\xi)}{\Phi_0(iK)} + R \frac{\Phi_0^+(\xi)}{\Phi_0(-iK)} \right\}, \quad \xi > 0.
\]

(11.3.21)
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Again, using (11.3.21) in (11.3.1), $B(\xi)$ is seen to have the form

\[
B(\xi) = \frac{(1 + R)K\xi}{\xi^2 + K^2} - \frac{K\xi}{2(\xi^2 + K^2)} \left\{ \frac{\Phi_0(\xi)}{\Phi_0(iK)} + R\frac{\Phi_0(-iK)}{\Phi_0(-iK)} \right\} \\
+ \frac{K^2\xi}{\pi} \int_{0}^{\infty} \frac{1}{(u^2 - \xi^2)(\xi^2 + K^2)} \left\{ \frac{\Phi_0^+(u)}{\Phi_0(iK)} + R\frac{\Phi_0^-(u)}{\Phi_0(-iK)} \right\} du.
\]

(11.3.22)

The integrals in (11.3.22) can be evaluated by considering the integral

\[
J(\xi) = \int P(\tau) \frac{\Phi_0(\tau)}{Q(\tau)(\tau^2 - \xi^2)} d\tau
\]

(11.3.23)

where $\Gamma$ is the same as in (11.3.14) and $P(\tau), Q(\tau)$ are polynomials such that the contribution to the integral in (11.3.23) from the circle of large radius vanishes. Thus

\[
\int_{0}^{\infty} \frac{\Phi_0^+(u)}{(u^2 + K^2)(u^2 - \xi^2)} du = \frac{\pi}{K} \left\{ \frac{\Phi_0(\xi)}{2\xi(\xi - iK)} + \frac{\Phi_0(-\xi)}{2\xi(\xi + iK)} - \frac{\Phi_0(iK)}{(\xi^2 + K^2)} \right\},
\]

and

\[
\int_{0}^{\infty} \frac{\Phi_0^-(u)}{(u^2 + K^2)(u^2 - \xi^2)} du = \frac{\pi}{K} \left\{ \frac{\Phi_0^+(\xi)}{2(\xi^2 + K^2)} + \frac{\Phi_0^-(\xi)}{2(\xi + iK)} - \frac{\Phi_0(iK)}{(\xi^2 + K^2)} \right\}
\]

(11.3.24)

where $\zeta = \xi + i\eta (\xi > 0)$. Hence use of Plemelj's formulæ produces

\[
\int_{0}^{\infty} \frac{\Phi_0^+(u)}{(u^2 + K^2)(u^2 - \xi^2)} du = \frac{\pi}{K} \left\{ \frac{\Phi_0^+(\xi)}{2(\xi^2 + K^2)} + \frac{\Phi_0^-(\xi)}{2(\xi + iK)} - \frac{\Phi_0(iK)}{(\xi^2 + K^2)} \right\},
\]

and

\[
\int_{0}^{\infty} \frac{\Phi_0^-(u)}{(u^2 + K^2)(u^2 - \xi^2)} du = \frac{\pi}{K} \left\{ \frac{\Phi_0^+(\xi)}{2(\xi^2 + K^2)} + \frac{\Phi_0^-(\xi)}{2(\xi + iK)} - \frac{\Phi_0(iK)}{(\xi^2 + K^2)} \right\}
\]

(11.3.25)

Using the results of (11.3.25) in (11.3.22), ultimately $B(\xi)$ is obtained as

\[
B(\xi) = \frac{K}{2} \Phi_0(-\xi) \left\{ \frac{1}{(\xi + iK)\Phi_0(iK)} + \frac{R}{(\xi - iK)\Phi_0(-iK)} \right\}.
\]

(11.3.26)

Now from (11.3.1), it is evident that $B(0) = 0$. Using this in (11.3.26), it is found that

\[
R = \frac{\Phi_0(-iK)}{\Phi_0(iK)}.
\]

(11.3.27)
Thus finally $A(\xi)$ and $B(\xi)$ are obtained as

$$A(\xi) = -\frac{K\Phi_0^+(\xi)}{\Phi_0(iK)(\xi + iK)} \quad (11.3.28)$$

and

$$B(\xi) = \frac{K\xi}{\xi^2 + K^2} \frac{\Phi_0(-\xi)}{\Phi_0(iK)} \quad (11.3.29)$$

Now, from (11.3.12), it is found that

$$\ln \Phi_0(z) = \frac{1}{\pi} \int_0^{\pi/2} \ln \left( \frac{z - K \tan \theta}{z} \right) d\theta$$

so that

$$\ln \left\{ \frac{\Phi_0(iK)}{\Phi_0(-iK)} \right\} = \frac{1}{\pi} \int_0^{\pi/2} \ln \left( \frac{i - \tan \theta}{i + \tan \theta} \right) d\theta = \frac{i\pi}{4}.$$ 

Hence $R$ is obtained as

$$R = e^{-i\pi} \quad (11.3.30)$$

This result can be identified with the result obtained by Holford (1964), eqn. (A12).

### 11.4 Reduction to Carleman Equation for $B(\xi)$:

Use of Havelock's expansion theorem (see Ursell(1947)) in the relations (11.2.9) and (11.2.10) produces

$$A(\xi) = B(\xi) - \frac{2K\xi}{\pi} \int_0^\infty \frac{B(u)}{u(\xi^2 - u^2)} du, \quad \xi > 0, \quad (11.4.1)$$

$$1 + R = \frac{4K^2}{\pi} \int_0^\infty \frac{B(u)}{u(u^2 + K^2)} du \quad (11.4.2)$$

and

$$\xi A(\xi) = -\xi B(\xi) + \frac{2K\xi}{\pi} \int_0^\infty \frac{B(u)}{\xi^2 - u^2} du, \quad \xi > 0, \quad (11.4.3)$$

$$1 - R = \frac{4iK}{\pi} \int_0^\infty \frac{B(u)}{u^2 + K^2} du \quad (11.4.4)$$
where the integrals in (11.4.1) and (11.4.3) are in the sense of CPV.

Elimination of $A(\xi)$ between (11.4.1) and (11.4.3) leads to

$$B(\xi) - \frac{K}{\pi} \int_0^\infty \frac{B(u)}{u(\xi - u)} du = 0, \quad \xi > 0$$

which is equivalent to

$$\xi B(\xi) + \frac{K}{\pi} \int_0^\infty \frac{B(u)}{u - \xi} du = c, \quad \xi > 0$$

(11.4.5)

where

$$c = \frac{K}{\pi} \int_0^\infty \frac{B(u)}{u} du$$

so that $c$ is an unknown constant. The equation (11.4.5) can be regarded as a Carleman type singular integral equation for $B(\xi)$.

To solve the equation (11.4.5), let

$$\Lambda(\zeta) = \frac{1}{2\pi i} \int_0^\infty \frac{B(u)}{u - \zeta} du, \quad \zeta = \xi + i\eta, \eta \neq 0,$$

(11.4.6)

then $\Lambda(\zeta)$ is analytic in the complex $\zeta$-plane with a cut along the positive real axis from 0 to $\infty$. Using Plemelj's formulæ, the equation (11.4.6) reduces to

$$(\xi + iK)\Lambda^+(\xi) - (\xi - iK)\Lambda^-(\xi) = c, \quad \xi > 0.$$ (11.4.7)

This is a Riemann-Hilbert problem for the sectionally analytic function $\Lambda(\zeta)$, and its solution is given by

$$\Lambda(\zeta) = \frac{\Lambda_0(\zeta)}{2\pi i} \int_0^\infty \frac{c}{\Lambda_0^+(u + iK)(u - \zeta)} du$$ (11.4.8)

where $\Lambda_0(\zeta)$ satisfies

$$\frac{\Lambda_0^+(\xi)}{\Lambda_0^-(\xi)} = \frac{\xi - iK}{\xi + iK}, \quad \xi > 0$$ (11.4.9)

so that $\Lambda_0(\zeta)$ is given by

$$\Lambda_0(\zeta) = \exp \left[ \frac{1}{2\pi i} \int_0^\infty \ln \frac{u - iK}{u + iK} - 2\pi i \frac{u - iK}{u - \zeta} du \right].$$ (11.4.10)
Using Plemelj's formulae, $B(\xi)$ is obtained as

$$B(\xi) = \Lambda^+(\xi) - \Lambda^-(\xi) = \frac{c\xi}{\xi^2 + K^2} + \frac{cK}{\pi} \frac{\Lambda_0^+(\xi)}{\xi + iK} \int_0^\infty \frac{1}{\Lambda_0^+(u)(u + iK)(u - \xi)} \, du, \xi > 0.$$  

(11.4.11)

The integral in (11.4.11) can be evaluated by considering the integral

$$M(\zeta) = \int_\Gamma \frac{1}{\Lambda_0(\tau)(\tau - \zeta)} \, d\tau$$  

(11.4.12)

where $\Gamma$ is the same as in (11.3.14). Now it is found that

$$M(\zeta) = 2iK \int_0^\infty \frac{1}{\Lambda_0^+(u)(u + iK)(u - \zeta)} \, du + D_1$$  

(11.4.13a)

where $D_1$ is the contribution to the integral in (11.4.12) over the circle of large radius and centre at the origin.

But, by the residue theorem, $M(\zeta)$ is obtained as

$$M(\zeta) = \frac{2\pi i}{\Lambda_0(\zeta)}.$$  

(11.4.13b)

Using Plemelj’s formulae, it is found from (11.4.13a) and (11.4.13b), that

$$\frac{1}{2\pi i} \int_0^\infty \frac{1}{\Lambda_0^+(u)(u + iK)(u - \xi)} \, du = \frac{1}{4iK} \left( \frac{1}{\Lambda_0^+(\xi)} + \frac{1}{\Lambda_0^-(\xi)} \right) + \frac{D_1}{4\pi K}, \xi > 0.$$  

(11.4.14)

Substitution of (11.4.14) in (11.4.11) yields

$$B(\xi) = \frac{c\xi}{\xi^2 + K^2} - \frac{1}{2\xi - iK} \left( 1 + \frac{\Lambda_0^+(\xi)}{\Lambda_0^-(\xi)} \right) + \frac{cD_1}{\pi} \frac{\Lambda_0^+(\xi)}{\xi - iK}, \xi > 0,$$  

(11.4.15)

where $D = \frac{cD_1}{\pi}$ is an unknown constant.

To determine the unknown constants $D$ and $R$, the relation (11.4.15) is used in the relations (11.4.2) and (11.4.4). This gives rise to the relations

$$1 + R = \frac{4DK^2}{\pi} \int_0^\infty \frac{\Lambda_0^+(\xi)}{\xi(\xi - iK)(\xi^2 + K^2)} \, d\xi$$  

(11.4.16)
The integrals in (11.4.16) and (11.4.17) can be evaluated by considering integrals of the form (11.3.14). Thus

\[ 1 - R = \frac{2D}{K} \left\{ \Lambda_0(iK) + \Lambda_0(-iK) \right\} \]

and

\[ 1 - R = \frac{2D}{K} \left\{ \Lambda_0(-iK) - \Lambda_0(iK) \right\} \]

so that

\[ R = \frac{\Lambda_0(iK)}{\Lambda_0(-iK)}, \quad D = \frac{K}{2} \frac{1}{\Lambda_0(-iK)}. \] (11.4.18)

Hence finally,

\[ B(\xi) = \frac{K}{2} \frac{\Lambda_0^+(\xi)}{\Lambda_0(-iK)(\xi - iK)}, \quad \xi > 0. \] (11.4.19)

Now comparing (11.4.10) and (11.3.12), it may be seen that

\[ \Lambda_0(\xi) = \frac{1}{\Phi_0(\xi)} \] (11.4.20)

after noting the relation

\[ \ln \left( \frac{u - iK}{u + iK} \right) + \ln \left( \frac{u + iK}{u - iK} \right) = 2\pi i, \] (11.4.21)

for \( u > 0, \) with \( K > 0. \)

Hence from (11.4.18) it is found that

\[ R = \frac{\Phi_0(-iK)}{\Phi_0(iK)} \] (11.4.22)

which coincides exactly with (11.3.27).

Now the following results are proved:

\[ \frac{\Lambda_0(-iK)}{\Phi_0(-iK)} = \frac{1}{2} \] (11.4.23)
and

\[ \frac{\Lambda_0^+(\xi)}{\Phi_0(-\xi)} = \frac{\xi}{\xi + iK}. \] (11.4.24)

To show (11.4.23), it may be noted from (11.3.12) and (11.4.10) that

\[
\Phi_0(-iK) = \exp \left[ \frac{1}{2\pi i} \int_0^\infty \ln \left( \frac{u+iK}{u-iK} \right) \frac{du}{u} \right],
\]

\[
\Lambda_0(-iK) = \exp \left[ \frac{1}{2\pi i} \int_0^\infty \ln \left( \frac{u-iK}{u+iK} \right) - 2\pi i \frac{d}{du} \right].
\]

Using the result (11.4.21), it is found that

\[
\frac{\Lambda_0(-iK)}{\Phi_0(-iK)} = \exp \left[ -\frac{1}{\pi i} \int_0^\infty \ln \left( \frac{u+iK}{u-iK} \right) \frac{u}{u^2 + K^2} du \right]
\]

\[
= \exp \left[ -\frac{2}{\pi} \int_0^{\pi/2} \left( \frac{\pi}{2} - \theta \right) \tan \theta d\theta \right]
\]

\[
= \frac{1}{2}.
\]

To show (11.4.24), the results in (11.4.10) and (11.4.21) are used to obtain

\[
\Lambda_0^+(\xi) = \exp \left[ -\frac{1}{2} \ln \left( \frac{\xi - iK}{\xi + iK} \right) - \frac{1}{2\pi i} \int_0^\infty \ln \left( \frac{u+iK}{u-iK} \right) \frac{du}{u+\xi} \right], \xi > 0.
\]

Also, from (11.3.12), it is seen that

\[
\Phi_0(-\xi) = \exp \left[ \frac{1}{2\pi i} \int_0^\infty \ln \left( \frac{u+iK}{u-iK} \right) \frac{du}{u+\xi} \right], \xi > 0.
\]

Hence

\[
\frac{\Lambda_0^+(\xi)}{\Phi_0(-\xi)} = \left( \frac{\xi - iK}{\xi + iK} \right)^{1/2} \exp \left[ -\frac{1}{\pi i} \int_0^\infty \ln \left( \frac{u+iK}{u-iK} \right) \frac{du}{u^2 - \xi^2} \right]. \] (11.4.25)

The integral in (11.4.25) can be evaluated and its value is \(2\pi i \ln \left\{ \frac{(\xi^2 + K^2)^{1/2}}{\xi} \right\}\). Substituting this value in (11.4.25), the result in (11.4.24) is obtained.
Using (11.4.23) and (11.4.24) it is seen that the two expressions as given by the relations (11.3.28) and (11.4.19) actually represent the same function $B(\xi)$.

11.5 Conclusion

A mixed boundary value problem associated with two-dimensional Laplace's equation, arising in the classical problem of surface water wave scattering by a semi-infinite rigid dock floating on the surface of infinitely deep water, is re-examined here by reducing it to some special forms of Carleman type singular integral equation of the second kind over semi-infinite intervals. The principal idea behind the reduction of the original boundary value problem lies in utilizing standard Fourier analysis involving Fourier cosine expansion and Havelock's expansion of integrable functions and related results. The resulting singular integral equations of the Carleman type are solved in closed forms by casting them into certain Riemann Hilbert problems over the positive real axis of the complex plane involved in the procedure adopted here. Closed form expression for the reflection coefficient for the semi-infinite dock problem is derived.

The present method is useful in the study of water wave scattering problems involving surface discontinuities arising due to the presence of a floating strip in the form of rigid dock or floating ice particles (inertial surface) or a thin ice sheet modelled as a thin elastic plate. This has been demonstrated in the chapters 12 and 13.