Chapter 1

Preliminaries of nonlinear spin models
1.1 Introduction

The nonlinear dynamics is observed in many systems of different kinds, including living organisms that can be considered the most complicated dissipative structure. One of the relatively new applications of nonlinear dynamics is its increasing role in condensed matter system, where it appears as a tool for designing a new material, including those at the nanoscale. The very relevant case of nonlinearity is to consider the domain of phenomena relative to our quotidian lives. In such every day context, well described by classical physics, the nonlinear dynamics is common in place. Nowadays, the most successful technologies are led by the data storage industries. High speed data storage and good reliability are the main advantages of the nonlinear phenomena of the magnetic systems. In the case of magnetic data storage technology, the spin of electron is the key parameter. On the other hand, magnetic data storage devices are non-volatile because of the natural remanence characteristic of ferromagnetic materials.

The nonlinear systems provides the most practical techniques currently available for analyzing and identifying from random data measured at the input and output points of the nonlinear systems. For practicing scientists involved in aerospace, automotive, biomedical, electrical, mechanical, oceanographic, and other activities concerned with nonlinear system analysis. Nonlinear system is an essential reference in the field of physics. A physically important feature in nonlinear dynamical systems which has been of considerable interest in diverse area of physics is a solitary wave or a soliton. A soliton is a very special wave that travels without dispersion. A rare natural phenomenon, a soliton once formed flows essentially forever.

These solitons are solutions of nonlinear wave equations. A single soliton is just like a normal dispersionless wave that it does not change its shape in the course of time. It is known for its remarkable stability. When a soliton encounters another soliton of arbitrary size or velocity it can change beyond recognition for a short or long period, but ultimately it will revert to its original shape. Starting from the 19th century, researchers found that certain non-
linear partial differential equations (PDEs) admit exact solutions in the form of solitary waves, known today as solitons. There is a famous story of the Scottish engineer, John Scott Russell, who in 1834 observed a hump-shaped disturbance propagating undiminished down a canal. In 1844, he published this observation [1], writing in his own words as the following,

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation”.

Russell was so taken with this phenomenon that subsequent to his discovery, he built a thirty foot wave tank in his garden to reproduce the effect, which was precipitated by an initial sudden displacement of water. Russell found empirically that the velocity obeyed $v \approx \sqrt{g(h+u_m)}$, where $h$ is the average depth of the water and $u_m$ is the maximum vertical displacement of the wave. It is also found that a sufficiently large initial displacement would generate two solitons, and, remarkably, that solitons can pass through one another undisturbed. It was not until 1890 that Korteweg and de Vries published a theory of shallow water waves and obtained a mathematical description of Russell’s soliton.

Nonlinear PDEs which admit soliton solutions typically contain two important classes of terms which feed off each other to produce the effect:

\[ \text{DISPERSION} \neq \text{NONLINEARITY} \]
The effect of dispersion is to spread out pulses, while the effect of nonlinearities is often, to draw in the disturbances. In the case of wave propagation this dispersion tends to spread the waves where as the nonlinearity results in steepening of the waves thereby retaining the shape. In the 1970s it was realized that several of these nonlinear PDEs yield entire families of exact solutions, and not just isolated solitons. These families contain solutions with arbitrary numbers of solitons of varying speed and amplitude, and undergoing mutual collisions. This nonlinear phenomena play a pivotal role in magnetic systems.

1.2 Background of magnetism

Magnetism is one of the oldest phenomena in the history of natural science. Magnetism is a subject which has been studied for nearly three thousand years. Lodestones first attracted the attention of ancient scholars, and the navigational magnetic compass was the first technological product resulting from this study. Although an early form of compass was invented in China in the eleventh century, it was not until around 1600s that anything resembling a modern account of the working of the compass was proposed [2,3]. Progress in the last two centuries has been more rapid, and major results have emerged which connect magnetism with other physical phenomena. An understanding of the relationship between magnetism and electricity, an inextricable link, began in 1819 by Oersted’s accidental discovery. Maxwell’s careful derivation of a set of four equations in 1860s led to electromagnetism. This link was further developed by Einstein’s theory of relativity. However, it is the magnetism in condensed matter systems, accompanied by quantum mechanical understanding, including ferromagnets, spin glasses and low-dimensional systems, which is still of great interest today. Magnetic properties exhibited in macroscopic scale are fundamentally different from those of so called ordinary materials. Magnetism can be understood as a correlated phenomenon, involving the mutual cooperation of enormous numbers of particles on macroscopic length scales. Furthermore, the interest in answering fundamental questions in these systems runs in parallel with the technological drive to find new materials for
use as permanent magnets, electric and magnetic sensors, or in information storing applications. Magnetism is studied and developed with two principal aspects in mind. One is to enhance magnetic properties by finding new materials or adjusting composition of different elements within compounds that give magnetic properties. The other is to understand their electrical responses and electron transport through magnetic materials which have a long range order not existing in normal materials. Thus, understanding of magnetic properties drastically improved. However, fully understanding transport through magnetic materials, though potentially very interesting because of the presence of an extra degree of freedom due to long range magnetic ordering, remains very challenging.

Magnetic materials are in the heart of numerous appliances and devices used in every day life. From macroscopic magnetism to atomic scale magnetic phenomena (crystal field interaction, relativistic spin orbit coupling) their importance has led to intensive research for a long time [4]. Nowadays, it becomes clear that the future development in magnetism rests upon recent developments in the nanoworld [5]. Magnetic materials containing nanomagnets are expected to enhance the efficiency of widely established technologies such as data recording and to implement new functionalities of applicability in bioscience. The key for this expected improvement in technological capabilities is the unique combination of small size, exotic properties and processability of nanomagnets. Magneto-optical properties of nanomagnets have attracted also considerable attention in the last years due their potential use as magneto-optical storage materials, magneto-optical switches, modulators or sensors based on the Faraday effect [6]. Aside from these applications it seems clear that nowadays the application of nanomagnets, which are attracting more attention, is these in the data recording fields. Magnetic recording, invented over 100 years ago, has played a key role in the development of non-volatile information storage technologies, including audio, video and data [7]. It seems clear that one of the keys to improve magnetic storage is the increase in the areal density achieved by scaling down the particle size [8]. For many
Figure 1.1: Orientation of magnetic moments and its domain structure
years, information technology has been marked by an exponential growth in the ability to store and retrieve data so much so that the industry is reaching its limits on further improvement in data storage and retrieval with the materials at hand. Developing new materials requires research at the molecular and atomic level to understand their properties. Such studies require calculations involving even the complexities of quantum physics.

The development and the advancement of magnetic sources has changed considerably over the centuries due to the thrust of researchers in finding the mechanism involved in the process. The only form of magnetism known until the 19th century was ferromagnetism. Certain materials, when “magnetized”, would attract certain other materials. The only materials attracted by a magnet were those that could become magnetized themselves. Since only certain materials exhibited magnetic properties, scientists concluded that magnetism was an inherent property of materials. The more esoteric applications of magnetism are in the area of magnetic recording and storage devices in computers, and in audio and video systems. Magnetic storage devices work on the principle of two stable magnetic states represented by the '0' and '1' in the binary number system. Floppy disks have dozens of tracks on which data can be digitally written in or stored by means of a write-head and then accessed or read by means of a read-head. A write-head provides a strong local magnetic field to the region through which the storage track of the disk is passed. The read-head senses stray magnetic flux from the storage track of the disk as it passes over the head. Another example of digital magnetic storage and reading is the magnetic strip on the back of plastic debit and credit cards. The magnetic strip contains identification of data which can be accessed through, for example, an automatic teller machine.

Then, in the 19th century, scientists studying the relatively new field of electrical currents discovered that moving charges produce magnetic effects. A current traveling through a loop of wire creates a magnetic field along the axis of the loop. The direction of the field inside the loop can be found by curling the fingers of the right hand in the direction of the current through the loop;
the thumb then points in the direction of the magnetic field. With this discovery, magnetism appeared to occur in two different manners: ferromagnetism depending on the material, and electromagnetism caused by currents.

Even though there exist more specific categories depending on their arrangement of local magnetic moments within and their response to external magnetic field, magnetic materials are largely referred to ferromagnetic materials and magnetism to ferromagnetism. The models for magnetic moments in a ferromagnet can be subdivided into two categories: 1) relatively isolated magnetic moments, applicable to the f-electrons in rare earth ferromagnets, 2) delocalized moments or band ferromagnetism, which describe weakly ferromagnetic materials. Over the years, a substantial part of the research in magnetism has been devoted to the unification of these two limits into a general theory.

One fundamental object in magnetism is the magnetic moment \( \mu \). An orbital moment arises from a microscopic current loop formed by the motion of one or more electrical charges. This is also equivalent to a magnetic dipole, so-called because it behaves analogously to an electric dipole which simply consists of two opposite charges separated by a small distance. Therefore it is possible to imagine a magnetic dipole as an object which consists of two magnetic monopoles of opposite magnetic polarity separated by a small distance. Magnetic moments can also originate from spin, the intrinsic angular momentum of a quantum particle. The electron magnetic moment is proportional to its spin via the gyromagnetic ratio \( g \) and Bohr magneton \( \mu_B \):

\[
\mu = g \mu_B \Sigma
\]

Due to the nature of these objects, a full account require quantum mechanics including quantization of angular momentum, matrix related algebra which requires operator related calculations, and state mixing when any two objects are coupled together.

The macroscopic magnetic properties of materials are the consequence of the magnetic moments associated with individual electrons. Each electron in an atom has magnetic moments that originate from two sources. One is related
to its orbital motion around the nucleus; being a moving charge every electron may be considered to be a small current generating a very small magnetic field and having a magnetic moment along its axis of revolution. The second magnetic moment originates from the electron spin about its own axis. This spin magnetic moment is directed along the spin axis. Spin magnetic moments may only be in either an “upward” direction or in an antiparallel “downward” direction. By these means each electron in an atom may be thought of as being a small magnet having permanent orbital and spin magnetic moments. Electrons have a charge and a spin, but until recently both were considered separately. In classical electronics, charges are moved by electric fields to transmit information and are stored in a capacitor to save it. In magnetic recording, magnetic fields are used to read or write the information stored on the magnetization, which “measures” the local orientation of spins in ferromagnets. The recent expansion of hard disk recording owes much to this development. The new scientific breakthroughs open a new paradigm where magnetization dynamics and charge currents act on each other in nanostructured artificial materials.
Ultimately, “spin currents” could even replace charge currents for the transfer and treatment of information, allowing faster, low energy operations owing to spin electronics on its way.

The bulk permeability of a material is a quantitative description of how readily the material experiences magnetization, which is when an externally applied magnetic field causes the charges in a material to align their motion such that their magnetic moments align parallel or anti-parallel to the external magnetic field. A circulating charge produces a magnetic moment (m) in each atom, and the magnetic moments of atoms are the building blocks of natural magnetics.

1.3 History of magnetization

The spin of an electron, combined with its orbital angular momentum, results in a magnetic dipole moment and creates a magnetic field. In many materials (specifically, those with a filled electron shell), however, the total dipole moment of all the electrons is zero (i.e., the spins are in up/down pairs) only atoms with partially filled shells can experience a net magnetic moment in the absence of an external field. Ferromagnetic materials contain many atoms with unpaired spins. When these tiny magnetic dipoles are aligned in the same direction, they create a measurable macroscopic field. These permanent dipoles (often called simply “spins” eventhough they also generally include orbital angular momentum) tend to align in parallel to an external magnetic field, an effect called paramagnetism. A related but much weaker effect is diamagnetism, due to the orbital motion induced by an external field, resulting in a dipole moment opposite to the applied field. Ferromagnetism involves an additional phenomenon, however, the dipoles tend to align spontaneously, without any applied field. This is purely a quantum-mechanical effect. In a ferromagnet, they tend to align in the same direction because of the Pauli principle: two electrons with the same spin cannot occupy the same “position”, which effectively reduces the energy of their electrostatic interaction compared to electrons with opposite spin. Mathematically, this is expressed more precisely in terms of the spin-statistics theorem: because electrons are fermions with half-integer spin,
their wave functions are antisymmetric under interchange of particle positions. This can be seen in, for example, the Hartree-Fock approximation to lead to a reduction in the electrostatic potential energy. This difference in energy is called the “exchange energy”.

The exchange interaction is primarily responsible for the ordering of atomic moments occurring in magnetic solids. The aforementioned interaction described by classical electromagnetism usually plays only a marginal role. For instance, in iron (Fe) the exchange interaction between two atoms is about 1000 times stronger than that of classical interaction. There is a small number “exotic” ferromagnets in which the exchange interactions are exceptionally weak, and then the classical dipole-dipole interactions may become the dominant ones. However, such systems become ferromagnetic only at very low temperatures, usually below 1K. But if the Curie temperature in a given material is higher than a few Kelvins, then its ferromagnetism is surely produced by exchange interactions. In such systems the classical dipole-dipole interactions may only give rise to secondary effects, e.g., to weak magnetic anisotropy.

1.4 Revival of spin wave theory

Spin is one of the basic properties of elementary particles, as well as composite systems. Discoveries concerning spin have often led to the opening of a new area of physics. Spin is an intrinsic angular momentum associated with quantum mechanical particles. Unlike classical “spinning” objects, which derive their angular momentum from the rotation of their constituent parts, spin angular momentum is not associated with any rotating internal masses. For example, elementary particles, such as the electron, possess spin angular momentum, even though they are point particles. Also, unlike classical mechanical spinning, the spin is not described by a vector, but by a two-component object (for spin-1/2 particles): there is an observable difference in how it transforms under coordinate rotations. Mathematically, quantum mechanical spin is not described by a vector as in classical angular momentum. It is described using a family of objects known as spinors. There are subtle differences be-
between the behavior of spinors and vectors under coordinate rotations. For example, rotating a spin-1/2 particle by 360 degrees does not bring it back to the same quantum state, but to the state with the opposite quantum phase; this is detectable, in principle, with interference experiments. To return the particle to its exact original state, one needs a 720 degree rotation.

Spin is a key element in particle and nuclear physics, and has always played a paramount role in the study of fundamental symmetries, static-particle properties and the structure of fundamental interactions. Moreover, during the past 15 years, spin physics has enjoyed a true renaissance, with many enthusiastic young people both theoreticians and experimentalists entering the field, attracted by new ideas and experimental opportunities.

Recent developments in the field of magnetic data storage and magnetic sensors created much interest to the dynamic properties of small magnetic elements [9,10]. The study of quantized and localized spin-wave modes of small magnetic elements is very important for understanding the dynamic magnetic properties of such elements. In order to improve the properties of such laterally confined magnetic elements one needs to study the properties of thermally excited spin-wave modes in those systems. The idea of spin-wave modes as eigen-excitation of small magnetic elements sounds intuitive and might look very simple. In fact, lateral confinement should define boundary conditions for spin-wave amplitudes at the edges of the element. These conditions provide selection rules for the wave vector of the allowed modes. Thus, instead of continuous spectrum of spin waves (the spin-wave frequency is uniquely determined by its wave vector) one obtains a series of quantized spin-wave modes, each of them is observed within a given wave-vector interval.

The concept of dynamic eigenmodes of unconfined magnetic media called “spin waves” was introduced by Bloch in 1930. A spin wave represents a wave of spin precession propagating in a magnetically ordered medium. Early experimental evidence for the existence of spin waves came from measurements of thermodynamic properties of ferromagnets, in particular the temperature dependence of their saturation magnetization. The first direct observation of spin
precession was made using ferromagnetic resonance by Griffiths in 1946 for the case of uniform precession which can be treated as a spin wave with the zero wave vector. Later inelastic light scattering experiments performed by Fleury,
Porto, Cheesman, and Guggenheim in 1966 have confirmed the existence of spin waves. Being of great importance for basic research of magnetically ordered systems, spin waves play an important role for technological applications as well. In fact, since spin waves are eigenmodes of magnetic media, the dynamics of magnetization can be nicely analyzed based on the spin-wave approach. Apparently these properties are decisive for the functionality of magnetic sensors, memory elements, and radars.

The history of the spin-wave conception is inseparably linked with the reduction of the dimensions. Being first introduced for bulk media, spin waves in magnetic films have attracted enormous interest in 60s and 70s of the last century. It was found that the confinement caused by the finite thickness of a film results in a variety of new effects and even new types of spin waves. The so-called surface spin waves were introduced theoretically by Damon and Eshbach in 1961 and the observed experimentally by Grunberg and Metawe in 1977. Therefore, it is not surprising that the interest to spin waves in films was a forerunner of wide application of magnetic films in the information technology.

![Spin precessional motion](image)

**Figure 1.4: Spin precessional motion**

### 1.4.1 Ordering of magnetic moments

Different types of interactions described above produce different types of magnetic ground states such as ferromagnets, antiferromagnets, ferrimagnets, helical orders, spin glasses, and nuclear ordering. A ferromagnet possesses spontaneous magnetization in the absence of external applied field due to the effect of exchange interaction with positive exchange constants \(J_{ij} > 0\) to ensure ferromagnetic alignment. The origin of spontaneous magnetization was
explained with Weiss model. The temperature scale related to ferromagnetic ordering is called the Curie temperature, $T_C$, above which the spontaneous magnetization vanishes; it separates the disordered paramagnetic phase at $T > T_C$ from the ordered ferromagnetic phase at $T < T_C$. In case of iron, $T_C \approx 1000$ K. If the exchange constant $J_{ij}$ is negative, an alignment of nearest neighbor magnetic moments becomes such that zero net magnetization becomes preferrable. This is the case of antiferromagnetism. Very often this occurs in systems which have two interpenetrating sublattices with opposite direction of magnetic moments. So if the nearest neighbors of each magnetic moment on one sublattice come from the other sublattice, the system exhibits antiferromagnetic ordering. The transition temperature for antiferromagnetic ordering is known as Neel temperature, $T_N$. Typically, materials that show nonmagnetic atoms which mediate exchange interactions, i.e; superexchange interactions, exhibit antiferromagnetic ordering [11]. This is called G-type antiferromagnetism and also found in materials such as LaF eO$_3$ and LaCrO$_3$.

However, LaM nO$_3$, is known to exhibit A-type ordering. This is due to structural deformation causing alternating long and short Mn-O bonds within (100) planes. Thus, A-type ordering shows in-plane ferromagnetic and out-of-plane antiferromagnetic ordering at the same time [2].

### 1.5 Magnetic models

Different magnetic spin models can be explained by generalizing the following Heisenberg Hamiltonian,

$$H_e = -\sum_{<ij>} J_{ij} S_i \cdot S_j.$$ 

For this we rewrite the above equation by introducing directional coefficients $\alpha_1$, $\alpha_2$ and $\alpha_3$.

$$H_e = -J \sum_{<ij>} [\alpha_1 S_i^x S_j^x + \alpha_2 S_i^y S_j^y + \alpha_3 S_i^z S_j^z].$$

Depending on the values of $\alpha_1$, $\alpha_2$ and $\alpha_3$, various models can be explained as described in Table 1.1.

It is not to be forgotten that the above models have much wider applicability than just the description of magnetic ordering phenomena. For instance, the
Ising model also describes a binary alloy or a lattice gas [11]; the XY model also pertains to quantum fluids [11]. Therefore, the experimental study of these magnetic systems may continue at the same time to the understanding of the more universal aspects of critical phenomena [11]. The phenomenon of exchange anisotropy has become the basis for an important application in information storage technology with a high current level of world-wide research and development activities [11].

### 1.6 Magnetic Interactions

#### 1.6.1 Exchange Interaction

The tendency of spins to align themselves parallel or antiparallel to each other is attributed to the exchange interaction. This originates from the Columb interaction between electrons and the antisymmetry of the wave function given by the Pauli principle. The parallel alignment of the spins in the ferromagnet is the energetically favorable configuration, due to the negative value of the exchange integral. Exchange interactions are at their root nothing more than electrostatic interactions [12], arising because charges of the same sign lose energy when they are close together and save energy when they are apart. Exchange, however, is considered to be a main reason for a long range magnetic order. Origin of this interaction lies deeply in the idea of quantum mechanics, which can simply be described by Heisenberg model of Hamiltonian:

\[
H_{\text{ex}} = -\sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j. \tag{1.1}
\]
where $J_{ij}$ is the exchange constant between $i^{th}$ and $j^{th}$ spins and $S_i, S_j$ are the neighbouring spins. By increasing the distance between the atoms, the value of $J_{ij}$ decreases rapidly. Therefore, a good approximation of this interaction may consider only the nearest neighbors. A simple example of this would be a system with two electrons under Coloumb potential, which have spatial co-ordinates $r_1$ and $r_2$ respectively. The spatial wave function for this two-electron joint state can be written as a product of single electron states $\psi_a(r_1)$ and $\psi_b(r_2)$ with a consideration of the exchange symmetry. This leads to symmetric $\psi_S$ and antisymmetric $\psi_A$ spatial joint wavefunctions:

\[
\begin{align*}
\psi_S &\sim \psi_a(r_1)\psi_b(r_2) + \psi_a(r_2)\psi_b(r_1), \\
\psi_A &\sim \psi_a(r_1)\psi_b(r_2) - \psi_a(r_2)\psi_b(r_1).
\end{align*}
\]

(1.2)

For electrons the overall wavefunction must be antisymmetric so the spin part of the wavefunction must either be an antisymmetric singlet state $\chi_S$ ($S=0$) in the case of a symmetric spatial state or a symmetric triplet state $\chi_T$ ($S=1$) in the case of an antisymmetric spatial state. Therefore the total wavefunction can be written as either singlet or triplet case:

\[
\begin{align*}
\Psi_S &\sim \psi_S \chi_S, \\
\Psi_T &\sim \psi_A \chi_T.
\end{align*}
\]

(1.3)
and the energies of the two possible states are:

\[ E_S = \int \psi_S^* \hat{H} \psi_S d\tau_1 d\tau_2, \quad E_T = \int \psi_T^* \hat{H} \psi_T d\tau_1 d\tau_2. \] (1.4)

With the assumption that the spin parts of the wavefunctions are normalized, the difference between two energies is simply written as

\[ E_S - E_T = \psi_a^*(\mathbf{r}_1) \psi_b^*(\mathbf{r}_2) \hat{H} \psi_a(\mathbf{r}_2) \psi_b(\mathbf{r}_1) d\tau_1 d\tau_2, \] (1.5)

which is proportional to the exchange constant \( J \). Because this is a matrix element between two states that differ solely through the exchange of the coordinates of the two electrons, the singlet-triplet energy difference is referred to as an ‘exchange’ interaction. Exchange interactions are divided into two categories: direct and indirect exchange. If the electrons on neighboring magnetic atoms interact via an exchange interaction without the need for an intermediary, this is known as direct exchange. Though this seems the most obvious and simple for exchange interaction to take, the reality is really simple. Thus in many magnetic materials it is necessary to consider some kind of indirect exchange interaction. Another reason for the indirect exchange is because some atoms contain relatively localized \( d \)- and \( f \)-shell electrons. This allows to have large local moments since they are localized, interactions between them often need to be mediated. There are several different types of indirect exchange interactions known depending on the systems. For example, a number of ionic solids show magnetic ground states even when there seems to be no direct overlap between ions that exhibit magnetic moments since those ions are usually far away from each other. The exchange mechanism that governs these type of materials is known as superexchange. In this case, magnetic moments are exchanged via non-magnetic ions such as oxygens in MnO. In metals the exchange interaction between magnetic ions can be mediated by the conduction electrons. A localized magnetic moment spin-polarizes the conduction electrons and this polarization in turn couples to a neighboring localized magnetic moment that is away from the first magnetic moment. Again, there
is no direct coupling between magnetic moments and this is called Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction [2,11]. For some other metallic oxides, it is possible to have a ferromagnetic exchange interaction which occur because the magnetic ion can show mixed valency, that is, it can exist in more than one oxidation state. Some of the compounds containing Mn ions show this type of behavior when Mn ion exhibits mixed oxidation states of 3 and 4. In this type of materials double exchange mechanism is used to explain their ferromagnetic configuration. Underlying idea for this interaction is that the electron hopping from one type of oxidation state to another type of oxidation state in the neighbouring electron is energetically favorable but if this neighboring state has antialigned valence electrons spin to that of previous state valence electrons, then due to Hund’s rule conduction (hopping) would not occur. Thus the system ferromagnetically orders in order to save energy. It is also possible for the spin-orbit interactions to play a role in a similar manner to that of the oxygen atoms in superexchange. In this case, the excited state is not connected with oxygen but is produced by the spin-orbit interaction in one of the magnetic ions. Then an exchange interaction takes place between the excited state of one ion and the ground state of the other ion. This is known as anisotropic exchange interaction [2].

![Exchange Interaction Diagram](image)

**Figure 1.6:** A schematic presentation of exchange interaction
1.6.2 Anisotropic interaction

Anisotropy (usually characterized by the first uniaxial anisotropy constant $A$) is another physical quantity which is determined on the atomic scale. The anisotropy of most materials is of magneto crystalline origin, reflecting the competition between magnetostatic crystal-field interaction and spin-orbit coupling. The crystalline field reproduces the local symmetry of the crystal and acts on the orbits of the inner-shell $d$ and $f$ electrons. Due to the crystalline field the electron orbits depend on the anisotropic crystalline environment. The magnitude of the magneto crystalline anisotropy depends on the ratio of crystal field energy and spin-orbit coupling. In 3d atoms, the spin-orbit coupling $\lambda=50$ meV is much smaller than the crystal-field energy $E_0 \geq 1$eV, and the magnetic anisotropy plays a major role. Single-ion anisotropy is determined by the interaction between the orbital state of a magnetic ion and the surrounding crystalline field which is very strong. The anisotropy is the product of the quenching of the orbital moment by the crystalline field. This field has the symmetry of the crystal lattice. Hence the orbital moments can be strongly coupled to the lattice. This interaction is transferred to the spin moments via the spin-orbit coupling, giving a weaker $d$-electron coupling of the spins to the crystal lattice. When an external field is applied the orbital moments may remain coupled to the lattice whilst the spins are more free to turn. The magnetic energy depends upon the orientation of the magnetization relative to the crystal axes. In transition metals this contribution is generally much smaller than the shape anisotropy but can be comparable in magnitude in rare earth metals, hence the large interest in rare earth materials in thin film systems to tailor moment orientation such as perpendicular to magnetic anisotropy. Generally, however, a spin chain has to be described by two fields, corresponding to the two angles specifying the orientation of a classical vector of fixed length. In particular the dynamics of spins always involves all three components in a nontrivial way, and one therefore expects important differences between real spin chains and the simplifying sine-Gordon system. Fortunately, however, several one-dimensional models with a vector field $S(z,t)$ also allow for complete solution
of their mechanical behaviour. We have emphasized the structure and properties of some special solutions to the equations of motion for the Heisenberg chain with easy axis symmetry, because we wanted to demonstrate differences and similarities of soliton structures in magnetic chains with various symmetries. From a more formal point of view, the most remarkable property of the classical Heisenberg chains is their complete integrability, which first has been shown for the isotropic case by Lakshmanan and Takhtajan. We consider the magnetic chain with the Hamiltonian

$$H_a = A \sum_i (S_i^z)^2.$$  \hspace{1cm} (1.6)

We now assume that $A > 0$, which guarantees that localized solutions with finite energy are asymptotically confined to the $x - y$ plane, that is the easy plane. The single-ion anisotropy provides the easy-plane character of the system at sufficiently low temperatures, whereas at higher temperatures the system becomes more and more equivalent to the isotropic Heisenberg chain. Akhiezer and Borovik have given a solutions of this type for easy-plane and easy axis ferromagnets and antiferromagnets [13]. Nakamura and Sasada [14] and Tjon and Wright [15] gave an analysis of soliton solutions in the isotropic Heisenberg chain. A more detailed study of solitons in magnets with easy plane anisotropy is worth doing for one or more reason. At present, the nonlinear properties of quasi one dimensional ferromagnets are best investigated experimentally on samples of CsNiF$_3$ and TMMC, which have easy plane anisotropy in the three dimensional ordering temperature [16]. The magnetic dipolar anisotropy, or shape anisotropy, is mediated by the dipolar interaction. This interaction is long range and so its contribution dependent upon the shape of the sample. Hence shape anisotropy becomes important in thin films and often produces inplane alignment of moments. Three interactions are responsible for the magnetocrystalline anisotropy energy: the single ion anisotropy, the dipole-dipole interaction, and the anisotropic part of the exchange interaction. The single ion anisotropy arises from the effect of the crystal field on the energy level of the single ion and thus contributes only to the magnon-phonon processes. The
dipole-dipole interaction and the anisotropic part of the exchange interaction, however, contribute to both magnon-phonon processes.

The energy of a ferromagnet depends on the direction of the magnetization relative to the structural axes of the material. There are different types of anisotropy which contribute to the free energy of the ferromagnet, such as magneto-crystalline anisotropy, shape anisotropy and magneto-elastic anisotropy. The anisotropy energy is usually small compared to the exchange energy. Nevertheless, the direction of the magnetization is determined only by the anisotropy in absence of an external magnetic field. The exchange interaction just tries to align the magnetic moments, regardless of direction. The most common type of anisotropy is magnetocrystalline anisotropy, which is caused by the spin-orbit interaction of the electrons due to the reorientation of orbitals and the creation of crystallographic axes.

1.6.3 Zeeman Energy

The interaction of the magnetization and the external field \( H \) gives the Zeeman energy. In general, it is important for the magnetic excitations caused by an external field. The magnetization will align with the external field, when \( H \) overcomes the contributions from the crystalline-anisotropy and demagnetization fields.

\[
H_z = -g\mu_B H - S_i, \tag{1.7}
\]

where the quantities \( g \) and \( \mu_B \) are, respectively, the gyromagnetic ratio and the Bohr magneton.

1.6.4 Dzyaloshinskii-Moriya interaction

These are compounds whose magnetic behaviour at high temperature can be ascribed to an antiferromagnetic material but which show a weak spontaneous magnetization below a transition temperature. This phenomenon, known as weak ferromagnetism, was explained by Dzyaloshinskii [17] and Moriya [18]
as an antiferromagnetic structure in which there exists a canting angle between the magnetic moments of different sub-lattices with opposite signs of the magnetic moments. This non-collinearity of the magnetic moments results in a net magnetization. There are two main mechanisms that can produce the non-collinearity of the magnetic moments. In the first one, the non-collinearity of the magnetic moments reflects the non-collinearity of the direction of the local magnetic anisotropy. The second mechanism is due to the combination of the spin-orbit coupling and the magnetic interaction between two magnetic centres. This mechanism can be accounted by a new term in the Hamiltonian:

\[ H_{\text{DM}} = - \langle i,j \rangle D \cdot S_i \times S_j \]  

(1.8)

The canted antiferromagnet with a Dzyaloshinskii-Moria antisymmetric exchange interaction [19-22] has been considered in a small-D vector. The conditions allowing canting between two magnetic moments are very strict. In relation to the crystal field, the nuclear and the magnetic unit cells must be the same and the sites whose magnetic moments are non-collinear must not be related by an inversion centre. In other words, weak ferromagnetism is favoured in molecular magnetism due to the higher propensity for low-symmetry molecules to adopt structures without inversion centre. The organic magnets with the highest transition temperatures are weak ferromagnets, in particular the 1,3,5-Triphenyl-6-Oxoverdazyl (TOV) compound [23]. The magnitude of the spin canting is related to the magnetic anisotropy. As a consequence, organic free radicals tend to exhibit very low canting angles and, consequently, very low values of the magnetization. Such an interaction was first suggested from purely symmetry grounds by Dzyaloshinskii. The theory is an extension of the superexchange theory to include the effect of spin-orbit coupling.

The essence of the Dzyaloshinski-Moriya interaction manifests itself in the comparative discussion on the magnetism of cuprates and nickelates. The spatial pattern of the DM interaction is determined by the crystal structure. The spin arrangement is fixed by the single-ion anisotropy because the strength of the single ion anisotropy overwhelms the strength of the anisotropic exchange interactions of both superexchange and direct-exchange mechanisms.
The weak ferromagnetism is realized when the spin system can obtain the energy gain by laying the fixed spin arrangement in accordance with the determined spatial pattern of the DM interaction. The weak ferromagnetism in ordinary Dzyaloshinskii-Moriya antiferromagnets, e.g., $\alpha - \text{Fe}_2\text{O}_3$, MnCO$_3$, CoCO$_3$, CrF$_3$, etc., emerges in the spin systems which involves single ion anisotropy. In the ordinary DM antiferromagnets, the DM interaction cannot be detected in the magnetic properties such as weak ferromagnetism and spin-wave gaps if there is no cooperation of the single-ion anisotropy. The problem is to provide a mechanism of constructing the spin arrangement in which the DM interaction can contribute to the gain in energy of the spin system.

It is well known that antisymmetric DM interaction are the cause of so-called weak ferromagnetism in some otherwise compensated antiferromagnets [24]. In these materials, DM interactions produce static canting between spins and, as a consequence of this, the spins are no longer ordered antiferromagnetically in their equilibrium state. Recently, DM interactions become an object of some interest in ferromagnets [25-26]. Liu [25] has examined all possible phases of the ordering of spins in ferromagnets with DM interactions, and it has been shown that besides helical spin configurations also a collinear configuration is possible if a uniaxial anisotropy energy is present which is greater than the DM energy. On the other hand, Melcher [26] has studied the effect of DM interactions on the dispersion law of spin waves in collinear DM ferromagnets. This case was restricted to the situation where both the magnetization and the DM vector are parallel to each other. The effect of DM interaction on the process of spin-wave propagation in the collinear DM ferromagnet, taking into account the arbitrary orientation of magnetization with respect to the DM vector. It is shown that DM interactions cause a dynamical phase shift between neighbouring spins, and that this shift add to the normal phase difference of the precession related to spin-wave propagation. This additional phase shift will be called a dynamical-canting phase. It will also be shown that the dynamical-canting phase depends on the angle between the magnetization and the DM vector.
1.6.5 Biquadratic exchange interaction

For high-spin systems (S > 1) the biquadratic exchange interaction [27] should be considered. In Heisenberg and Ising spin systems, the existence of higher order spin coupling is one of the major important phenomena in ferromagnetic spin system. It has been made clear that these interactions have significant effects on the magnetic properties in compounds containing ion-group ions or rare earth metal ions. The appearance of biquadratic exchange interaction was given by Anderson and Kittel [27] in a spin system with biquadratic exchange interaction with positive sign as well as with Heisenberg type exchange interaction, the ferromagnetic or the antiferromagnetic spin structure changes abruptly. This interaction gives new low-lying nonlinear excitations supporting the importance of solitary-waves as elementary excitations in the one dimensional systems [28]. The Hamiltonian of this interaction can be given as

\[ H_B = - \sum_{i,j} K (\mathbf{S}_i \cdot \mathbf{S}_j)^2. \]  

where K is the biquadratic exchange parameter. It has significant effects on the temperature dependence of the specific heat of magnetic systems. This interaction has proved to give essential quantitative modifications for the thermodynamics of the Heisenberg ferromagnet [29, 30]. It was pointed out that the higher-order exchange interactions are smaller than the bilinear ones for the 3d group ions [31], and comparable with the bilinear ones in the rare-earth compounds [32, 33]. On the other hand, in solid helium and in such phenomena as the quadrupolar ordering of molecules in solid hydrogen and liquid crystals, the higher-order exchange interactions turned out to be the main ones [34].

1.6.6 Dipole- dipole interaction

It is necessary to think about interactions between electrons because these allow the magnetic moments or spins in a solid to communicate with each other and produce the long range order that makes magnetic materials unique.
Figure 1.7: A Schematic presentation of biquadratic interaction

Simplest form of interaction between electrons that can be considered is the dipole-dipole interaction. It can be easily estimated that the magnitude of this effect for the case of two Bohr magnetons separated by $1 \text{Å}$ is approximately $10^{-23} \text{ J}$ which is equivalent to roughly 1 K in temperature. Since many materials order magnetically at much higher temperatures (around 1000 K), the
magnetic dipole interaction is too weak to be playing a dominant role in those systems.

The interaction between unpaired electrons from different radicals is a necessary requirement for obtaining long-range magnetic order in the solid. Therefore, these strategies for stabilizing free radicals based on the isolation of the unpaired electrons must be employed in moderation. In other words, the kinetic stability and the strength of the magnetic interaction of the free radicals are two opposing requirements that must be balanced in the design of purely organic systems that show bulk magnetic behaviour. A consequence of this balance is the low ordering of temperatures and in these kinds of materials, almost all of them quite below liquid helium temperature (4.2K), which greatly limits the applications of such systems. In organic molecular systems the two main sources of magnetic anisotropy are magnetic dipolar interactions and spin-orbit coupling. Both sources usually have a small magnitude and purely organic magnets are considered as isotropic Heisenberg systems. The weakness of the dipolar interactions is due to the delocalization of the unpaired electrons. On the other hand, the unpaired electrons are located in molecular orbitals composed of p atomic orbitals with no orbital moment. Therefore, the spin-orbit coupling anisotropy arises from the coupling of the ground state with excited states through orbital angular momentum, and is a second-order term. This low magnetic anisotropy introduces another difficulty in the design of purely organic magnets, since in Heisenberg systems long-range magnetic ordering is only possible if the magnetic interactions are propagated in a three-dimensional network. There are only three requirements for obtaining long-range magnetic ordering in purely organic solids: the existence of magnetic centres, the magnetic interaction among these magnetic centres and the communication of these magnetic interactions throughout the solid. In practice it proves extremely difficult to achieve all these goals in any one single system. In particular, the last two requirements imply a crucial role of the unpaired electron distribution in the free radicals and the packing of these free radicals in the crystal. Among the systems showing long-range magnetic order,
those which exhibit a spontaneous magnetization which are of particular interest, since they can potentially lead to applications, e.g., magnetic sensors and magneto-optic applications. Many different phenomena are inherently caused by non-equilibrium dynamics and long-range interactions, which in turn have a nonlinear origin. Dipole-dipole interactions arise from localized permanent dipoles. The two opposite charges are on the same molecules and, thus, neutralize each other so there is no net charge on the molecule as there is an ion. However, the individual charges can interact with the individual charges of opposite sign on another molecule, and the interaction can be extremely strong. The energy of dipole-dipole interactions can approach the energy of a weak chemical bond. Due to the high energy of the dipole-dipole interactions, molecules can associate with each other (e.g., methanol-water association); this type of association is often called ‘hydrogen bonding’. The energy of interaction is directly proportional to the product of the polarizability of the molecule and the square of the dipole moment. The interaction energy is also inversely proportional to the sixth power of the distance between the interacting charges.

The dipole-dipole interaction between the atoms has given rise to a myriad of fascinating phenomena [35-39]. Especially, in the field of quantum information, the dipole-dipole interaction plays an important role in generating the entanglement. In numerical calculations based on a finite system, the dipole-dipole interaction can be fully taken into account [40-44]. Testing the magnetic orderings in experimental systems [45-47] is of great interest not only from the viewpoint of fundamental physics but also for applications, including magneto-logic devices [48]. However, there exists an obstacle in realizing the ordering using microstructured dipolar magnets. That is, the magnetocrystalline and shape anisotropies dominantly determine the orientation of the magnetic moments, and so the influences of the dipole-dipole interaction are hard to recognize [45, 46]. The Heisenberg Hamiltonian of the dipole-dipole interaction is given by

\[
H_D = \frac{(g \mu_B)^2}{2} \langle i, j \rangle - \frac{S_i \cdot S_j}{r_{ij}^3} - \frac{3(S_i \cdot r_{ij})(S_j \cdot r_{ij})}{r_{ij}^6}, \quad (1.10)
\]
where \( r_{ij} = r_i - r_j \) is the vector connecting two magnetic sites \( i \) and \( j \). This interaction is responsible for the shape-dependent demagnetization of ordinary ferromagnets.

### 1.6.7 Octupole-dipole interaction

The Heisenberg model of the Hamiltonian with higher order spin-spin interactions can be considered as the starting point for understanding the physical systems. The octupole-dipole interaction is the third higher order interaction and it was introduced by Toru Moriya [18] which can also play an important role in the description of hyperfine structures. It can be expressed in terms of the Heisenberg Hamiltonian as

\[
H_B = -J_0 \sum_{<i,j>} (S_i \cdot S_j)(S_j \cdot \hat{k})^2.
\]  

(1.11)
The degree of overlapping of wavefunctions will be effective when the site $j$ is its nearest neighbour. $J_0$ represents the octupole-dipole parameter and $\hat{k}=(0,0,1)$. So far we have discussed about the origin and properties of the spin system and these spin systems have their intrinsic localized modes due to the effect of dispersion or diffraction and higher order nonlinearity which underlies the basic dynamics of the spin states that are of quite interest in recent years. Hence, we give a detailed insights based on the physical aspects of localization.

1.7 Localization phenomena

Both nonlinearity and discreteness have played important roles in spin dynamics [49]. The nonlinearity cannot be treated as a perturbation in all cases, as evidenced by the appearance of domain walls, kinks and solitons [50]. An important advance in dealing with nonlinearity in condensed matter physics has been the introduction of the soliton as a new type of elementary excitations. It has been suggested [51] that solitons, which had been extensively studied [52] in fluids, plasmas and optics, may be present as thermal excitations in quasi-one-dimensional materials as well, and should be treated as a new type of elementary excitation in addition to spatially extended plane wave-like modes. Since then nonlinear excitations have attracted wide interest in many branches of condensed matter physics, for example, in lattice dynamics [53], electronic polymers [54], molecular crystals [55] and magnetic systems [56, 57]. Classically, these nonlinear excitations are solutions of integrable nonlinear partial differential equations [58, 59] which can be used to describe some realistic physical systems within the continuum approximation. The paradigm of such nonlinear excitations has provided a rather useful framework for investigating a large number of phenomena in condensed matter physics, especially the thermodynamic and transport properties of low dimensional materials [60, 61]. In particular, solitary excitations in one-dimensional magnetic systems have been extensively investigated. In general, there are no exact solutions to the equations of motion derived from Heisenberg ferro and
antiferromagnetic Hamiltonian’s. The classical continuum limit approximation of 1-D easy-plane magnets demonstrate the existence of sine-Gordon kink excitations in addition to spin waves [62, 63]. Another type of continuous nonlinear excitations that can be supported by a magnetic chain is a breather, which can be visualized as a magnon bound state [64, 65]. Perhaps because these nonlinear excitations have infinite lifetime in integrable systems but are found to be unstable in non-integrable continuous systems, historically, most attention in nonlinear dynamics was devoted to integrable continuous models. Among these integrable models are the (1+1)-dimensional sine-Gordon equation, the KdV equation, and the nonlinear Schrödinger equation, to name a few of the best-known examples [58-59, 66]. In strongly nonlinear discrete systems, the spatial size of nonlinear excitations can become comparable to the lattice spacing; hence, the discreteness of the underlying physical systems is expected to have a significant effect on the properties of nonlinear excitations in condensed matter physics.

A major advance of the theory of nonlinear excitations in discrete lattices in late 80s and early 90s was the discovery that some localized vibrations in perfectly periodic but non-integrable lattices can be stabilized by lattice discreteness [67-70]. This realization has led to extensive studies of the features associated with intrinsic localization in various nonlinear nonintegrable lattices, and it has proven to be a conceptual and practical breakthrough [71-75]. These localized excitations are called either Intrinsic localized modes (ILMs) with the emphasis on the fact that their formation involves no disorder and that they extend over a nano-length scale, or discrete breathers with the emphasis on their similarity to exact breather soliton solutions in nonlinear continuum theories. Although it is well known that no bound state or localized mode exists in a continuum 3D space for a scalar field [76], ILMs in discrete lattices are not confined to certain lattice dimensions [70, 77-80]. These unusual modes can occur at any site and may be stationary or move slowly through the lattice [71]. One key element for realistic lattices is the existence of gaped linear dispersion relations. Depending on the nature of the interparticle forces, a va-
riety of interesting ILMs can exist, with spatial mode patterns ranging in type from alternating (zone center) to staggered (zone boundary). A number of useful existence and stability criteria [73] has recently been formalized in terms of currently emerging in nonlinear crystal dynamics [71, 75], magnetic systems [81, 82], electron-phonon systems [83], friction [84], etc., The potential for these self-localized oscillatory excitations in equilibrium and nonequilibrium classical and quantum discrete lattices is now extensive and this thrust is becoming a major activity in nonlinear condensed matter research. The challenge at this writing is that these excitations are yet to be equally identified in experiment.

In magnetic systems, both exchange interactions between spins and spin anisotropy (either single-ion or dipole-dipole) are intrinsically nonlinear. Since the strength of the internal effective field acting on a spin always decreases with increasing the spin deviation from its equilibrium direction, the nonlinearity in magnetic systems is generally soft. Intrinsic localized spin wave modes (ILSMs) are expected to exist in perfect but nonintegrable discrete magnetic chains because of the intrinsic nonlinearity in the exchange and anisotropy interactions. Here we consider the simplest magnetic system that can possibly support ILSMs, namely, ferromagnetic chains. Although no ILSM can occur in isotropic ferromagnetic chains with only nearest-neighbor exchange interaction because the nonlinearity in the Heisenberg exchange interaction is intrinsically soft, both even-parity and odd-parity of ILSMs appear in Heisenberg ferromagnetic chains with easy plane anisotropy [82, 85] when a strong magnetic field is applied perpendicular to this plane. Responding to their vibrational counterpart [71-75], these highly localized ILSMs involve only a few lattice sites and have amplitude-dependent frequencies which lie outside the harmonic plane-wave bands. The existence of such ILSMs for a ferromagnetic chain with nearest-neighbor interactions requires that the strength of the single-ion anisotropy and the external magnetic field exceed certain critical values so that the resulting ILSM frequencies can appear above the linear spin wave band. The production of ILSMs by the application of an external
magnetic field by making use of an experimental parameter not available with crystal lattice systems.

1.8 Techniques adopted for the study

1.8.1 Classical spin models

Many crystals have an ordered magnetic structure. This means that in the absence of an external magnetic field, the mean magnetic moment of at least one of the atoms in each unit cell of the crystal is non-zero. In the simplest type of magnetically ordered crystals, i.e., ferromagnets, the mean magnetic moments of all the atoms have the same orientation provided that the temperature of the ferromagnet does not exceed a critical value, i.e., the Curie temperature. For this reason ferromagnets have spontaneous magnetic moments, i.e., non-zero macroscopic magnetic moments, even in the absence of an external magnetic field. The dynamical behavior of a spin is determined by the equation of motion, which can be derived from the quantum theory [86]. The time evolution of a spin observable \( S \) is determined by its commutator with the Hamiltonian operator \( H \):

\[
i\hbar \frac{d\mathbf{S}}{dt} = [\mathbf{S}, \mathbf{H}].
\]  

The Hamiltonian, which describes the interaction of the spin with the external magnetic field, given by its flux \( B \), can be expressed as:

\[
H = -\frac{g\mu_B}{\hbar} \mathbf{S} \cdot \mathbf{B},
\]  

where \( \mu_B \) is the Bohr magneton \((\mu_B < 0)\) and \( g \) is the gyromagnetic factor for a free electron. The \( z \)-component of the commutator in Eq. (1.12) can be derived as:

\[
[S_z, H] = -\frac{g\mu_B}{\hbar} [S_z, S_x B_x + S_y B_y + S_z B_z]
= -\frac{g\mu_B}{\hbar} [S_z, S_x] B_x + [S_z, S_y] B_y
= ig\mu_B \cdot B_y S_x + B_x S_y
= ig\mu_B (\mathbf{S} \times \mathbf{B})_z,
\]
with the help of commutation rules for spin operators:

\[ [S_i, S_j] = i\hbar \delta_{ij} S_k, \]

and the corresponding expressions can be derived for the other two components of the spin, which lead to the spin equation of motion:

\[ \frac{d}{dt} <S> = \frac{g\mu_B}{\hbar} (S \times B). \]  

(1.14)

The derived equation of motion for one spin can be further generalized for the case of homogeneous magnetization within the macrospin model, considering the relation between the magnetization \( M \) and \( <S> \):

\[ M = \frac{g\mu_B}{\hbar} <S>. \]  

(1.15)

Therefore, the analogous equation of motion of the magnetization in an external field \( H \) is observed as in case of one spin:

\[ \frac{d}{dt} M = -\gamma \mu_B M \times H = \gamma_0 M \times H, \]  

(1.16)

where the gyromagnetic ratio \( \gamma = g\mu_B / \hbar \) is introduced and \( \gamma_0 > 0 \). Eq. (1.16) is known as the Landau-Lifshitz (L-L) equation. The specific properties of the L-L equation are observed for a constant magnetic field \( H \), due to the vector product specifications:

\[ \frac{d}{dt} M^2 = 0, \quad \frac{d}{dt}(M \cdot H) = 0. \]  

(1.17)

The first equation states that the absolute value of the magnetization does not change during the precession around a constant external field. The second equation implies that the angle between the magnetization and the external field is constant during the rotation. The schematically presented magnetization rotation as shown in Fig. (1.9), in which the top of the magnetization vector marks out a circle around \( H \). The magnetization vector \( M \) precesses around the magnetic field \( H \) with the frequency \( \omega = \gamma_0 H \). This frequency is also known as the Larmor frequency. The derived equation of motion for the magnetization in an external field \( H \) can be generalized for the case in which the anisotropy fields and the exchange interaction contribute to the magnetic energy by simply replacing \( H \) with the effective magnetic field \( H_{\text{eff}} \). The theoretical and experimental study of spin dynamics governed by the Landau-Lifshitz equation has been the focus of considerable research for many years.
1.8 Techniques adopted for the study

1.8.2 Space curve formalism

The essential features of quantum mechanics can be discussed using two-level systems. The visualization, however, of the complex vector space is rather difficult, which makes the study of the geometry of quantum evolution non-
trivial. On the other hand, a variety of nonlinear phenomena [87] and certain topological properties of unit vector fields [88] have been studied using a space curve formalism. This has led to new insights for some geometric properties of the phase of the wave function. In the case of a two-level system, the Hilbert space is a real four-dimensional space. We will restrict ourselves here to the normalized wavefunctions. In this case, the Hilbert space becomes $S^3$. Feynman et al., [89] have noticed that in this case the Schrödinger equation may be written in the form of a three-dimensional vector equation. Integrable equations have been shown to associate with motions of curves or surfaces. The pioneering work is due to Hasimoto, [90] who sets up one-to-one correspondence between the integrable Schrödinger equation and the binormal motion of a space curve driven by its curvature and torsion. Utilizing the Hasimoto transformation, Lamb [87] obtained the mKdV and sine-Gordon equations from the motion of curves in $R^3$. Langer and Perline [91, 92] further proved that the dynamics of non-stretching vortex filament in $R^3$ gives the NLS hierarchy. In an intriguing paper, Goldstein and Petrich [93] related the mKdV equation and its hierarchies to motions of non-stretching curves on the plane in $R^2$. Doliwa and Santini [94] discovered that the NLS hierarchy and complex mKdV equation arise from motions on $S^3(R)$, where the radius $R$ plays the role of the spectral parameter. Nakayama, Segur and Wadati [95] obtained the sine-Gordon equation by considering a nonlocal motion of curves in $R^2$.

A space curve in three-dimensional space is described in parametric form by a position vector $\mathbf{r} = \mathbf{r}(s)$, where $s$ is the usual arclength variable. The unit tangent $\mathbf{t} = \mathbf{r}_s$, the principal normal $\mathbf{n}$ and the binormal $\mathbf{b}$ form an orthonormal triad of unit vectors that satisfy the Frenet-Serret equations [96]

$$
\mathbf{t}_s = \kappa \mathbf{n}, \quad \mathbf{n}_s = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \mathbf{b}_s = -\tau \mathbf{n}.
$$

(1.18)

where $\kappa$ and $\tau$ are geometric parameters that represent, respectively, the curvature and torsion of the space curve. A curve can be uniquely represented by specifying $\kappa$ and $\tau$, for a stated orientation. If this space curve evolves in time $u$, then $\mathbf{r} = \mathbf{r}(s, u)$. We write the evolution equations of the triad quite generally,
in a form similar to Eq. (1.18): [97]

\[ t_u = g \tau + h \mathbf{b}, \quad \mathbf{a}_u = -g \tau + \tau_0 h, \quad b_u = -ht - \tau_0 a. \] (1.19)

Here \( g, h \) and \( \tau_0 \) are geometric parameters which are (in general) functions of \( s \) and \( u \). The above equation describe the evolution in \( u \) of the Frenet frame \((\mathbf{t}, \mathbf{a}, \mathbf{b})\) on the curve. For non-stretching curves, the triad must satisfy the compatibility conditions

\[ \tau_{su} = \tau_{us}, \quad \mathbf{a}_{su} = \mathbf{a}_{us}, \quad b_{su} = b_{us}. \] (1.20)

Using Eqs. (1.18) and (1.19), the compatibility conditions become

\[ k_u = g_s - \tau h, \quad \tau_u = (\tau_0)_s + k h, \quad h_u = \kappa \tau_0 - \tau g. \] (1.21)

Further, Lamb formulation is the introduction of the following complex transformation \( \psi \), called the Hasimoto transformation, defined [90] as \( \psi(s, u) = \kappa e^{i h ds}. \) Hence obtain the associated moving curve parameters that correspond to an integrable, shape-preserving curve motion. This in turn unravels a certain special geometric structure of the given soliton-bearing NLPDE. The above mapping is to provide a motivation for a general formulation to investigate the interesting curve evolutions that get associated with soliton-supporting equations.

### 1.8.3 Semi-classical approach

The soliton solution for spin chains have been investigated by several different approaches. In the classical approach, general single soliton solutions are obtained for a continuum version of classical Heisenberg chain. In quantum spin system, a bosonic representation of spin operators turns out to be a very suitable method for studying the solitary waves, because they allow quantum corrections in a systematic way. In spin coherent representation, one can work directly with the operators, make no approximation to the Hamiltonian, and can develop an exact nonlinear equation for the quantum system, but this
method seems to be limited because it works with a limited number of interaction terms in the Hamiltonian. To avoid this limitation, it is necessary to use a truncated Holstein-Primakoff expansion for the spin operators [98, 99]. Then the Hamiltonian can be bosonized. It is well known that in quantum spin system the classical quantity $S_c = \hbar S$ and the condition $S_c = \lim_{\hbar \to 0} \frac{\hbar}{S} (\hbar S)$ states the semiclassical limit. Semi classical treatments use a truncated Holstein-Primakoff expansion with a small parameter $\varphi = 1/\sqrt{S}$ for obtaining the properly truncated Hamiltonian. The method of quantization based upon the creation and annihilation operators, obtained from the spin operators with the Holstein-Primakoff transformation.

### 1.8.4 Holstein-Primakoff method

In order to account the interaction among spin waves automatically, it is convenient to use the Holstein-Primakoff method. We now introduce operators $S_{l\pm}$ and $n_I$ by

$$S_{l\pm} = S_{lx} \pm iS_{ly}, \quad n_I = S - S_{lz}. \quad (1.22)$$

The eigenvalues of $n_I$ are integers from 0 to $2S$ according to the eigenvalue of $S_{lz}, M = -S, -S + 1, \ldots, S$. Let $\psi(M)$ be the eigenfunction for the eigenvalue $M$ of $S_{lz}$. Then we obtain

$$S_l - \psi(M) = \frac{(S + M)(S - M + 1)\psi(M - 1)}{2S} \cdot \frac{1}{(n_I + 1)(1 - \frac{n_I}{2S})^{1/2}} \psi(M - 1). \quad (1.23)$$

Therefore, by using $n_I$ for $M$, we obtain

$$S_l - \psi(n_I) = \sqrt{2S} (n_I + 1)^{1/2} (1 - \frac{n_I}{2S})^{1/2} \psi(n_I + 1). \quad (1.24)$$

Similarly, we have

$$S_l + \psi(n_I) = \sqrt{2S} (1 - \frac{n_I - 1}{2S})^{1/2} n_I^{1/2} \psi(n_I - 1) \quad (1.25)$$
Introducing new Bose operators \( a_i^\dagger \) and \( a_i \) by \( a_i^\dagger a_i = n_i \), we find the following relations:

\[
\begin{align*}
  a_i^\dagger a_i \psi(n_i) &= n_i \psi(n_i), \\
  a_i^\dagger \psi(n_i) &= \sqrt{n_i} \psi(n_i + 1), \\
  a_i \psi(n_i) &= \sqrt{n_i} \psi(n_i - 1). 
\end{align*}
\]

Therefore, on comparing (1.24-1.25,1.26), we obtain

\[
\begin{align*}
  S_{l-} &= \sqrt{\frac{25}{S} a_i^\dagger 1 - \frac{a_i^\dagger a_i}{2S} 1/2,} \\
  S_{l+} &= \sqrt{\frac{25}{S} 1 - \frac{a_i^\dagger a_i}{2S} 1/2} a_i, \\
  S_{lz} &= S - a_i^\dagger a_i. 
\end{align*}
\]

The eigen value of \( a_i^\dagger a_i \) is any integer \((0,1,2,...)\), while that of \( n_i \) is restricted to the values \(0,1,2,...2S\). However, as evident from (1.27), the matrix element of \( S_{l-} \) and \( S_{l+} \) between \( 2S \) and \( 2S+1 \) vanishes automatically, assuring that there is no mixing of states with \( n_i > 2S \) into those with \( n_i \leq 2S \).

Writing the exchange interaction as

\[
H = -2J \sum_{i<j} (S - a_i^\dagger a_j)(S - a_j^\dagger a_i) + S a_i^\dagger a_j^\dagger 1 - \frac{a_i^\dagger a_i}{2S} 1/2 - \frac{a_j^\dagger a_j}{2S} 1/2
\]

and expanding formally the square roots, we have

\[
H = -2J S^2 + 2JS \sum_{i<j} (a_i^\dagger a_i + a_j^\dagger a_j - a_i^\dagger a_j - a_j^\dagger a_i) - 2J \sum_{i<j} a_i^\dagger a_j^\dagger a_j^\dagger a_i + \frac{1}{4} (a_i^\dagger a_i^\dagger a_j^\dagger a_j + a_i^\dagger a_j^\dagger a_j^\dagger a_i + a_j^\dagger a_i^\dagger a_i^\dagger a_j + a_j^\dagger a_i^\dagger a_j^\dagger a_j) + \cdots
\]

This is clearly an expansion in terms of \( 1/S \). The first, second and third terms are proportional to the second, first and zeroth powers of \( S \), respectively. Since the second term is quadratic in \( a_j \) and \( a_j^\dagger \), it can be easily digonalized by the
transformation
\[ a_{\mu}^\dagger = \frac{1}{\sqrt{N}} - e^{+i\mu R_0 a_0^\dagger}, \]
\[ a_{\mu}^\dagger = \frac{1}{\sqrt{N}} - e^{-i\mu R_0 a_0^\dagger}. \] (1.30)

The transverse component is expressed in terms of \( n_i \) by taking the z-component of \( S \) as a basis. Conversely, one may take the transverse component as a basis and express the z-component in terms of the transverse one, as was shown by Anderson [100]. In this case an iterative approximation leads to the following expression for \( a_i^\dagger \) and \( a_i \) in terms of \( S_{l\pm} \)

\[ a_i^\dagger = \frac{1}{2S} S_{l-} + \frac{1}{4S} \frac{1}{(2S)^3/2} S_{l-} S_{l-} S_{l+} + ..., \]

\[ a_i = \frac{1}{2S} S_{l+} + \frac{1}{4S} \frac{1}{(2S)^3/2} S_{l-} S_{l+} S_{l+} + ..., \]

Substituting this relation into the third equation of (1.27), we can expand \( S_{lz} \) in terms of the transverse component as

\[ S_{lz} = S - \frac{1}{2S} S_{l-} S_{l+} + .... \] (1.31)

The Heisenberg equation of motion for the Boson operator is then written as

\[ i\hbar \frac{dS_{l\pm}}{dt} = [S_j, H]. \] (1.32)

The boson operator acts on the lattice sites and satisfies the Heisenberg equation of motion.

1.8.5 Glauber's coherent state representation

The Glauber's coherent state representation for Bose operators [101]

\[ |\alpha\rangle = \prod_j |\alpha_j\rangle, \] (1.33)

where

\[ |\alpha_j\rangle = \exp \left[ -\frac{1}{2} |\alpha_j|^2 \sum_{m=0}^{\infty} \frac{\alpha_j^m}{\sqrt{m!}} m \right], \] (1.34)
with $\langle \alpha | \alpha \rangle = 1$. The semiclassical approach allows us to consider the projection of spins which can be continuously distributed along the $z$-axis. The states represented in Eq. (1.33) are the eigen states of the operators $a_i$ with eigen value $\alpha_i$.

$$a_i^\dagger |\alpha\rangle = \alpha_i^* |\alpha\rangle,$$

$$a_i |\alpha\rangle = \alpha_i |\alpha\rangle. \quad (1.35)$$

For the system in the state $|\alpha\rangle$, one finds the equation for the average $\langle \alpha | a_j | \alpha \rangle$. Where $|\alpha_i\rangle$ is the coherent state eigen vector for operator $a_i$ and $\alpha_i$ is the coherent amplitude in this representation. The states $|\alpha_i\rangle$ are nonorthogonal and overcomplete. The advantages of using a spin-coherent representation to describe spin dynamics arising from nonlinear evolution equations for spin operators [102].

1.8.6 Algebraic methods

The theory of solitary waves has attracted much interest in recent years for treatment of PDEs describing nonlinear and evolution concepts. The nonlinear problems are characterized by dispersive effects, dissipative effects, and diffusion process. The nonlinear evolution equations attracted a huge size of research works to establish a variety of solutions of distinct physical structures. Several methods have been developed aiming to achieve useful progress by developing more solutions, and to facilitate the calculations. The methods are tangent hyperbolic function method, sine-cosine function method, Jacobi elliptic function method, and methods involving symbolic computation. However, some of these analytical solutions methods are not easy to use because of the tedious work that it requires. In what follows, the sine-cosine, the tanh method and Jacobi elliptic function ansatz will be reviewed briefly.

1.8.6.A Tangent hyperbolic function method

The tanh method is developed by Malfliet [103-105] where the tanh is used as a new variable, since all derivatives of a tanh are represented by a tanh itself.
Consider the general nonlinear partial differential equation depending on two physical fields say \( x \) and \( t \)

\[
H(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, ...) = 0. \tag{1.37}
\]

Restricting our attention to travelling waves, we consider its travelling solutions

\[ u(x, t) = u(\xi), \quad \xi = x - ct \text{ or } \xi = x + ct, \]

where \( c \) is the velocity of wave and the equation becomes an ordinary differential equation. In order to seek the travelling wave solutions, we introduce a finite series of \( \text{tanh} \)-function in the form of

\[
u(\xi) = a_0 + \sum_{i=1}^{N} a_i \phi^i + b_i \phi^{-i}, \quad \phi' = b + \phi^2.
\]

where \( b \) is a parameter to be determined, \( \phi = \phi(\xi) \) and \( \phi' = \frac{\phi}{\xi} \). The parameter \( N \) can be found by balancing the highest order derivative term with the nonlinear term. Substituting these equations in the ordinary differential equation will yield a system of algebraic equations with respect to \( a_i, b_i, b \) and \( c \) since all the coefficients of \( \phi^i \) have to vanish. Then we can determine \( a_0, a_i, b_i, b \) and \( c \). The Riccati equation has the following general solutions,

(a) If \( b < 0 \)

\[
\phi = -\sqrt{-b} \tan(\sqrt{-b} \xi), \\
\phi = -\sqrt{-b} \coth(\sqrt{-b} \xi),
\]

it depends on the initial conditions.

(b) If \( b = 0 \)

\[
\phi = -1/\xi,
\]

(c) If \( b > 0 \)

\[
\phi = \sqrt{b} \tan(\sqrt{b} \xi), \\
\phi = -\sqrt{b} \cot(\sqrt{b} \xi).
\]
1.8.6.B Sine-cosine function method

We now describe the sine-cosine function method for the above nonlinear partial differential equation (1.37). The main steps of this method are as follows [106]:

(i) To find the travelling wave solutions to equation (1.37), we introduce the wave variable \( \xi = (x - ct) \), so that \( u(x, t) = u(\xi) \).

(ii) This enables us to use the following transformations

\[
\frac{\partial}{\partial t} = -c \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial t^2} = c^2 \frac{d^2}{d\xi^2}, \quad \frac{\partial}{\partial x} = \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2}.
\]  

(1.38)

One can immediately reduce the nonlinear PDE into a nonlinear ODE. Now the ordinary differential equation associated with equation (1.37) becomes

\[
Q(u, u_\xi, u_{\xi\xi}, \ldots) = 0,
\]  

(1.39)

where \( u_\xi \) denotes \( \frac{du}{d\xi} \). Equation (1.39) is then integrated by setting the constant of integration to be zero.

(iii) The solution of equation (1.39) is expressed in the following form

\[
u(x, t) = \lambda \sin^\beta(\mu \xi),
\]  

(1.40)

or in terms of a cosine function as

\[
u(x, t) = \lambda \cos^\beta(\mu \xi).
\]  

(1.41)

where \( \lambda, \mu \) and \( \beta \) are the parameters that have to be determined, \( \mu \) and \( c \) are the wave number and the wave speed respectively [107]. We write the derivatives of equation (1.40) as follows:

\[
u(\xi) = \lambda \sin^\beta(\mu \xi),
\]

\[
u^n(\xi) = \lambda^n \sin^{n\beta}(\mu \xi),
\]

\[
u^n_\xi = n\mu \lambda^n \cos(\mu \xi) \sin^{n-1}(\mu \xi),
\]

\[
u^n_{\xi\xi} = -n^2 \mu^2 \beta^2 \lambda^n \sin^{n\beta}(\mu \xi) + n\mu \lambda^n \beta(\eta \beta - 1)\sin^{n\beta-2}(\mu \xi).
\]  

(1.42)
and the derivatives of (1.41) becomes

\[ u(\xi) = \lambda \cos^\beta (\mu \xi), \]
\[ u^n(\xi) = \lambda^n \cos^{n\beta} (\mu \xi), \]
\[ u^n_\xi = -n\mu\beta \lambda^n \sin(\mu \xi) \cos^{n\beta - 1} (\mu \xi), \]
\[ u^n_{\xi\xi} = -n^2 \mu^2 \beta^2 \lambda^n \cos^{n\beta} (\mu \xi) + n\mu^2 \lambda^n \beta(n\beta - 1) \cos^{n\beta - 2} (\mu \xi). \]

(1.43)

and so on for other derivatives.

(iv) We substitute equation (1.42) or (1.43) into equation (1.39), and the resultant equations contain the trigonometric functions such as \( \sin^R(\mu \xi) \) or \( \cos^R(\mu \xi) \). The parameters \( \lambda, \mu \) and \( \beta \) are then determined by balancing the exponents of each pair of sine or cosine terms. We then collect all the coefficients of the terms with the same power in \( \sin^k(\mu \xi) \) or \( \cos^k(\mu \xi) \), where these coefficients have to vanish. It yields a system of algebraic equations among the unknowns \( \lambda, \mu \) and \( \beta \) which will be determined.

The algorithms described above certainly works well for a large class of very interesting nonlinear equations. The main advantage of the method is that the great capability of reducing the size of computational work compared to existing techniques such as the inverse scattering method, Hirota’s bilinear method and the truncated Painlevé analysis.

1.8.6.C Jacobi elliptic function method

For the above-mentioned partial differential equation to construct Jacobi elliptic function solutions, we expand its solution in the form

\[ u(x, t)u(\xi) = \sum_{i=0}^{n} a_i sn^i \xi \]

(1.44)

where \( \xi = kx + ct \), \( sn \xi = sn(\xi \mid m) \) and \( m \) is called a modulus. The parameter \( n \) can be fixed by balancing the highest order derivative terms with nonlinear terms. The jacobi elliptic functions \( sn(\xi \mid m), cn(\xi \mid m), dn(\xi \mid m) \) are doubly periodic and
possess properties of triangular functions, namely [108, 109],

\[
\begin{align*}
\text{sn}^2 \xi + \text{cn}^2 \xi &= 1, & \text{dn}^2 \xi &= 1 - m^2 \text{sn}^2 \xi, \\
(\text{sn} \xi)' &= \text{cn} \xi \text{dn} \xi, & (\text{cn} \xi)' &= -\text{sn} \xi \text{dn} \xi, \\
(\text{dn} \xi)' &= -m^2 \text{sn} \xi \text{cn} \xi,
\end{align*}
\]  

(1.45)

when \( m \to 1 \), the Jacobi functions degenerate to the hyperbolic functions. i.e.,

\[
\begin{align*}
\text{sn} \xi &\to \tanh \xi, & \text{cn} \xi &= \sec \xi, & \text{dn} \xi &= \sec \xi.
\end{align*}
\]

when \( m \to 0 \), the Jacobi functions degenerate to the hyperbolic functions. i.e.,

\[
\begin{align*}
\text{sn} \xi &\to \sin \xi, & \text{cn} \xi &= \cos \xi, & \text{dn} \xi &= 1.
\end{align*}
\]

We only consider the expansion in terms of the Jacobi functions \( \text{sn} \xi \) and \( \text{cn} \xi \). We can get solitary wave solution based on different Jacobi elliptic functions for some nonlinear wave equations.