1.1 SINGULARITIES IN A TWO-FLUID MEDIUM*

1. Introduction

Many authors have investigated different types of singularities that can be used in the one-fluid problem. Thorne (1953) and Rhodes-Robinson (1970) gave surveys of the fundamental line and point singularities submerged in a fluid of finite or infinite depth. The two-fluid problem was discussed by Gorgui and Kassem (1978), Mandal (1981) and Rhodes-Robinson (1980) the effect of surface tension being included by the last two authors.

In this section we shall discuss the basic line and point singularities when they are submerged in one of two fluids. The upper fluid is of finite constant depth 'h' with a free surface (FS); the lower fluid is of infinite depth. The time harmonic singularities are described by harmonic potential functions with period $2\pi f_0$ which satisfy the boundary conditions at the surface of separation (SS); in fact it is more convenient to use complex-valued potentials $\phi e^{-i\omega t}$, the actual potential being the real part. The potential must also satisfy limiting conditions in the neighbourhood of the singularity; it should behave

like a typical singular harmonic function, in the far field it should represent a spreading wave. Under these requirements, a unique solution will be found in all cases considered.

We note that our solution can be applied to cases when bodies are present in the fluids, whenever the two- or three-dimensional symmetry is such that the motion can be described by a series of singularities placed within the body in suitable positions. Whether the waves are generated by the body, or reflected by the body, does not matter.

2. Statement and formulation of the problem

Consider the irrotational motion of two non-viscous incompressible fluids under the action of gravity. Their SS is a horizontal plane, the lower fluid of infinite depth, the upper of finite height 'h'. Their motion is due to an oscillating singularity in one of the fluids; it is assumed to be simple harmonic with period $2\pi/\sigma$, so the velocity potentials $\varphi_1$ and $\varphi_2$ (of the lower and upper fluids respectively) will be too.

We take origin 0 in the mean SS, axis Oy pointing vertically downward into the lower fluid and chosen so it passes through the singularity which is then located at $(0,\eta)$ or $(0,-\eta)$ according as the singularity is in the lower or upper fluid respectively.
Then, for all $x$, except at the singular point. Also

$$\nabla^2 \varphi_1 = 0, \quad 0 \leq y \leq h \quad y > 0$$

$$\nabla^2 \varphi_2 = 0, \quad y < 0 \quad h < y < \infty$$

except at the singular point. Also

$$\frac{\partial \varphi_2}{\partial y} + K \varphi_2 = 0 \quad \text{on} \quad y = h$$

$$\nabla \varphi_1 \to 0 \quad \text{as} \quad y \to \infty$$

$$\frac{\partial \varphi_1}{\partial y} = \frac{\partial \varphi_2}{\partial y} \quad \text{on} \quad y = 0$$

and

$$K \varphi_1 + \frac{\partial \varphi_1}{\partial y} = g \left( K \varphi_2 + \frac{\partial \varphi_2}{\partial y} \right) \quad \text{on} \quad y = 0$$

when $K = \frac{g^2}{s^2}$, $s = \frac{\rho_2}{\rho_1}$, $g$ is the acceleration due to gravity, and $\rho_1$ is the lower and $\rho_2$ the upper fluid density.
Finally, $\varphi_1$ and $\varphi_2$ must satisfy the so-called radiation condition as $|x|\to \infty$. This condition is that the potential function represent diverging waves at a large distance from the singularity.

3. Submerged line singularity, upper fluid of finite depth

We first consider a line singularity placed at the point $(0,-\eta)$ in the upper fluid of depth $'h'$. Then

$$\varphi_2 \to \log R \text{ as } R = \left\{ \begin{array}{ll}
\frac{x^2 + (y + \eta)^2}{r} & j = 0, \\
\frac{x^2 + (y + \eta + h)^2}{r} & \end{array} \right. \to 0.$$

Now $\varphi_1$ and $\varphi_2$ can be represented as

$$\varphi_1 = \sum_{0}^{\infty} f_j \log R_j + \sum_{0}^{\infty} g_j \log R'_j + \int \sum_{-\infty}^{\infty} A(k)e^{-ky} \cos kx \, dk \quad (3.1)$$

$$\varphi_2 = \sum_{-\infty}^{\infty} c_j \log R_j + \sum_{-\infty}^{\infty} d_j \log R'_j + \int \left[ B(k) \cosh k(h+y) + C(k) \sinh k(h+y) \right] \cos kx \, dk \quad (3.2)$$

where $d_0 = 1, R_j^2 = x^2 + (y + 2j\eta - \eta)^2$ and $R'_j = x^2 + (y + 2j\eta + \eta)^2, j = 0, \pm 1, \pm 2, \ldots$ and $A, B, C, f_j, g_j, c_j, d_j$ are to be
found from the conditions (2.1), (2.2), and (2.3) and also the condition that the integrals are to be convergent. The radiation condition will be dealt with in the sequel.

The following integral representation will be needed in our calculations

\[
\frac{\partial}{\partial y} \log R_j = \begin{cases} 
\int_{-\infty}^{\infty} e^{-k(y+2jh-\eta)} \cos kx \, dk, & y > -2jh + \eta \\
\int_{-\infty}^{0} e^{k(y+2jh-\eta)} \cos kx \, dk, & y < -2jh + \eta 
\end{cases}
\]

\[
\frac{\partial}{\partial y} \log R'_j = \begin{cases} 
\int_{-\infty}^{\infty} e^{-k(y+2jh+\eta)} \cos kx \, dk, & y > -(2jh + \eta) \\
\int_{-\infty}^{0} e^{k(y+2jh+\eta)} \cos kx \, dk, & y < -(2jh + \eta) 
\end{cases}
\]

so that,

\[
\frac{\partial}{\partial y} \log R_j \bigg|_{y=-h} = \begin{cases} 
\int_{-\infty}^{\infty} e^{-k(2j-lh-\eta)} \cos kx \, dk, & j = 1,2,3,\ldots \\
\int_{-\infty}^{0} e^{k(2j-1h-\eta)} \cos kx \, dk, & j = 0,-1,-2,-3,\ldots
\end{cases}
\]
\[
\frac{\partial}{\partial y} \log R_j \bigg|_{y=-h} = \begin{cases} \\
\int_0^\infty e^{-k(2j-1h+n)} \cos kx \, dk, & j = 1,2,3,\ldots \\
-\int_0^\infty e^{k(2j-1h+n)} \cos kx \, dk, & j = 0,-1,-2,-3,\ldots
\end{cases}
\]

\[
\frac{\partial}{\partial y} \log R_j \bigg|_{y=0} = \begin{cases} \\
\int_0^\infty e^{-k(2jh-n)} \cos kx \, dk, & j = 1,2,3,\ldots \\
-\int_0^\infty e^{k(2jh-n)} \cos kx \, dk, & j = 0,-1,-2,-3,\ldots
\end{cases}
\]

\[
\frac{\partial}{\partial y} \log R_j \bigg|_{y=0} = \begin{cases} \\
\int_0^\infty e^{-k(2jh+n)} \cos kx \, dk, & j = 0,1,2,\ldots \\
-\int_0^\infty e^{k(2jh+n)} \cos kx \, dk, & j = -1,-2,-3,\ldots
\end{cases}
\]

Condition (2.1) gives

\[
K \left[ \sum c_j \log \left\{ x^2 + (2j-1h-n)^2 \right\}^{1/2} + \sum c_{-j} \log \left\{ x^2 + (2j+1h+n)^2 \right\}^{1/2} + \ldots \right]
\]
\[ + \sum_{j} d_{j} \log \left\{ x^{2} + (2j-1) h - \eta \right\}^{1/2} + \sum_{j} d_{-j} \log \left\{ x^{2} + (2j+1) h - \eta \right\}^{1/2} \]

\[ + \int_{0}^{\infty} B \cos kx \, dk \right] + \sum_{j} c_{j} \int_{0}^{\infty} e^{-k(2j-1) h - \eta} \cos kx \, dk \]

\[ - \sum_{j} c_{-j} \int_{0}^{\infty} e^{-k(2j+1) h + \eta} \cos kx \, dk \]

\[ + \int_{0}^{\infty} \cos kx \, dk = 0 \]  \hspace{1cm} (3.3)

from which we obtain

\[ c_{j+1} + d_{-j} = 0, \ j = 0, 1, 2, \ldots \]

\[ d_{j+1} + c_{-j} = 0, \ j = 0, 1, 2, \ldots \]

(3.4)

Since \( d_{0} = 1 \), we obtain

\[ c_{1} = -1 \]  \hspace{1cm} (3.5)
Condition (2.3) gives

\[
K \left[ \sum_{j=1}^{\infty} f_j \log \left\{ x^2 + (2jh-n)^2 \right\} + \frac{1}{2} + \sum_{j=1}^{\infty} g_j \log \left\{ x^2 + (2jh+n)^2 \right\} + \frac{1}{2} \right]
\]

\[+ \int_0^\infty A \cos kx \, dk + \int_0^\infty \sum_{j=1}^{\infty} e^{-k(2jh-n)} \cos kx \, dk\]

\[+ \int_0^\infty \sum_{j=1}^{\infty} e^{-k(2jh+n)} \cos kx \, dk - \int_0^\infty kA \cos kx \, dk\]

\[= sk \left[ \sum_{j=1}^{\infty} c_j \log \left\{ x^2 + (2jh-n)^2 \right\} + \frac{1}{2} + \sum_{j=1}^{\infty} c_{j+1} \log \left\{ x^2 + (2jh+n)^2 \right\} + \frac{1}{2} \right]
\]

\[+ \sum_{j=1}^{\infty} d_j \log \left\{ x^2 + (2jh-n)^2 \right\} - \sum_{j=1}^{\infty} d_{j+1} \log \left\{ x^2 + (2jh+n)^2 \right\} + \frac{1}{2}\]

\[+ \int_0^\infty (B \cosh kh + C \sinh kh) \cos kx \, dk \right] + s \int_0^\infty \left[ \sum_{j=1}^{\infty} c_j e^{-k(2jh-n)} \right]
\]

\[+ \sum_{j=1}^{\infty} c_j e^{-k(2jh-1)h-n)} + \sum_{j=1}^{\infty} d_j e^{-k(2jh+n)} + \sum_{j=1}^{\infty} d_j e^{-k(2j-1)h+n)}
\]

\[+ (1+d_1)e^{-k\eta} + k(B \sinh kh + C \cosh kh) \right] \cos kx \, dk.
\]

(3.6)
from which we obtain by considering the coefficients of the different logarithmic terms

\[
\begin{align*}
\forall s(c_j - c_{j+1}) &= f_j, \quad j = 1, 2, 3, \ldots \\
\forall s(d_j - d_{j+1}) &= g_j, \quad j = 1, 2, 3, \ldots \\
n(s(d_0 - d_1) &= -g_0
\end{align*}
\] (3.7)

and then (3.6) reduces to

\[
(k - K)A + s(K \cosh kh + k \sinh kh)B + s(K \sinh kh + k \cosh kh)C
\]
\[
= -2s \left[ d_1 e^{-kn} + \sum_{j=1}^{\infty} e^{-2kjh} (c_{j+1} e^{k\eta} + d_{j+1} e^{-k\eta}) \right].
\]

Condition (2.2) now gives

\[
\sum_{j=1}^{\infty} (1-s)(c_j e^{k\eta} + d_j e^{-k\eta}) e^{-2kjh} + \sum_{j=1}^{\infty} (1+s)(c_{j+1} e^{k\eta} + d_{j+1} e^{-k\eta}) e^{-2kjh}
\]
\[
+ d_1 (1+s)e^{-kn} - c_1 (1-s)e^{-kn} = -k(A+B \sinh kh + C \cosh kh).
\] (3.9)
Now for convergence of the integrals in the expressions for $\varphi_1$ and $\varphi_2$, $G(k)$ must be zero for $k = 0$, where $G(k)$ is the expression in the left side of (3.9), so that

$$\sum_{l=1}^{\infty} (1-s)(c_l + d_l) + \sum_{l=1}^{\infty} (1+s)(c_{l+1} + d_{l+1}) - c_l(1-s) + d_l(1+s) = 0.$$ 

This is satisfied by choosing

$$(1-s)c_j + (1+s)c_{j+1} = 0, \quad j = 1, 2, \ldots, \infty,$$

$$(1-s)d_j + (1+s)d_{j+1} = 0, \quad j = 1, 2, \ldots, \infty,$$

and

$$(1-s)c_1 = (1+s)d_1 \quad \Rightarrow \quad j = 0.$$

From (3.4), $c_0 = -d_1 = \mu$ where $\mu = \frac{1-s}{1+s}$ so that we obtain

$$c_j = (-1)^j \mu^{j-1}, \quad j = 1, 2, \ldots \quad \sqrt{\text{valid}}$$

$$d_j = (-1)^j \mu^j, \quad j = 1, 2, \ldots \quad \sqrt{\text{valid}}$$

$$c_0 = -d_1 = \mu$$

$$(3.10)$$
(3.8) can be written as

\[(k-K)A + s(k \cosh kh + k \sinh kh)B + s(k \sinh kh + k \cosh kh)C\]

\[
= 2s\mu \left[ e^{-k\eta} \frac{e^{-2kh} (e^{k\eta} + \mu e^{-k\eta})}{1 + \mu e^{-2kh}} \right]
\]

\[= \tilde{\alpha} \frac{e^{-k\eta} - e^{k\eta}}{1 + \mu e^{-2kh}} \]

From (3.3), we obtain

\[KB + kC = \frac{2e^{-kh} (e^{k\eta} + \mu e^{-k\eta})}{1 + \mu e^{-2kh}} = 0 \]  \hspace{1cm} (3.12)

From (3.9), we obtain

\[A + B \sinh kh + C \cosh kh = 0 \]  \hspace{1cm} (3.13)

Solving for \(A, B, C\) from (3.11), (3.12) and (3.13), we obtain

\[A = \frac{2e^{-kh}(e^{k\eta} + \mu e^{-k\eta}) \sinh kh}{K (1 + \mu e^{-2kh})} \times (\frac{k}{K} \sinh kh - \cosh kh) \left[ \frac{1}{k - K} \right. \]

\[+ \left. \frac{(s-1) \sinh kh}{\Delta(k)} \right] \tilde{F}_1 \]  \hspace{1cm} (3.14)
\[
B = \frac{1}{K} \left[ \frac{2e^{-kh}(e^{k\eta} + \mu e^{-k\eta})}{1 + \mu e^{-2kh}} - k\frac{1}{k - K} + \frac{(s - 1) \sinh kh}{\Delta(k)} \right] F_1
\]
\[
C = \left[ \frac{1}{k - K} + \frac{(s - 1) \sinh kh}{\Delta(k)} \right] F_1
\]

where
\[
\Delta(k) = \{k(1 - s) - sk\} \sinh kh - K \cosh kh
\]

and
\[
\Delta = \Delta(k) = \{k(1 - s) - sk\} \sinh kh - K \cosh kh
\]

Now, \(\Delta(k)\) has one simple pole at \(k = k_0 > 0\), say on the real axis of \(k\) (there are also poles \(k = k_1\)) and complex poles \(k_n = \alpha_n + i\beta_n\) where when \(s \to 0\), \(\beta_n \to k_n \) (where \(k_n \to (n-1)\pi \) as \(n \to \infty\) (cf. Rhodes-Robinson (1971)). The zeroes of the denominator of \(F_1\) are purely imaginary.

Hence \(A, B, C\) have simple poles \(k = k_0 \) and \(k = K\) on the positive real axis of \(k\). In the line integrals from
0 to $\infty$ we make indentation below these poles which account for the behaviors of the potential functions at infinity particularly as $|x| \to \infty$. This will be evident later.

Substituting the above results, we have

$$
\varphi_1 = \sum_1^\infty \left[ (-1)^j \mu_j^{j-1} + (-1)^j \mu_j^j \right] \log R_j + \sum_1^\infty (-1)^j \mu_j^{j+1} \log R_j' \\

- \int_0^\infty \frac{2}{K} e^{-kh} \frac{1}{1 + \mu e^{-2kh}} \sinh kh \ e^{-k\mu} \ cos \ kx \ dk \\

+ \int_0^\infty \frac{1}{k-K} \ (k-K \sinh kh - \cosh kh) e^{-k\mu} \ cos \ kx \ dk \\

+ \int_0^\infty \frac{1}{\Delta(k)} \ (s-1) \sinh kh \ k \sinh kh - \cosh kh) e^{-k\mu} \ cos \ kx \ dk

(3.19)

$$

$$
\varphi_2 = \sum_1^\infty \left[ (-1)^j \mu_j^{j-1} \ log \ R_j + \sum_1^\infty (-1)^j \mu_j^{j+1} \ log \ R_j' + \sum_1^\infty (-1)^j \mu_j^j \ log \ R_j' \\

+ \sum_1^\infty (-1)^j \mu_j^j \ log \ R_j' + \int_0^\infty \frac{2}{K} e^{-kh} \frac{1}{1 + \mu e^{-2kh}} \ cosh \ k(h+y) \ cos \ kx \ dk

.$$
\[ + \int_{0}^{\infty} \frac{1}{k - K} \left[ \sinh k(h+y) - \frac{k}{K} \cosh k(h+y) \right] F_1 \cos kx \, dk \]

\[ + \int_{0}^{\infty} \frac{1}{\Delta(k)} (s-1) \sinh kh \left[ \sinh k(h+y) - \frac{k}{K} \cosh k(h+y) \right] F_1 \cos kx \, dk \]

Now, as \( h \to \infty \) it is possible to obtain

\[ \phi_1 = \frac{2s}{1+s} \log R_0 + \frac{2s}{1+s} \int_{0}^{\infty} \frac{e^{-k(y+\eta)}}{k-M} \cos kx \, dk \]

\[ \phi_2 = \log R_0 + \frac{1-s}{1+s} \log R_0 - \frac{2s}{1+s} \int_{0}^{\infty} \frac{e^{k(y-\eta)}}{k-M} \cos kx \, dk \]

where \( \Delta(k) = \frac{1+s}{1-s} \) which are the results derived by Gorgui and Kassem (1978). Now to investigate the behavior of the integral for large \( |x| \), we put

\[ 2 \cos kx = e^{ik|x|} + e^{-ik|x|} \]

Then
where, $I_1 = \frac{1}{2} \left( \frac{k}{k-k} \sinh kh - \cosh kh \right) \frac{F_1}{k-k} e^{-ky}$.

For the first integral of (3.21), we consider in the complex $k$-plane a contour in the first quadrant bounded by the real axis of large length $X_1$ with an indentation below the pole $k = K$, an arc $\gamma$ of radius $X_1$ with center at the origin and the line joining the origin, with the point $X_1 e^{i\alpha}$ where $0 < \alpha < \frac{\pi}{2}$. Then for considering the behavior as $|x| \to \infty$, we only need to consider the behavior of the term arising from the residue at $k = K$, because the integral along the arc becomes exponentially small as $X_1 \to \infty$ and the integral along the line $0$ to $X_1 e^{i\alpha}$ ($0 < \alpha < \frac{\pi}{2}$) will have a factor $e^{-X_1 \sin \alpha |x|}$ which becomes exponentially small for large $|x|$. Hence making $X_1 \to \infty$ we find that as $|x| \to \infty$

$$\int_0^\infty I_\gamma e^{ik|x|} \, dk \to 2\pi i \text{ Residue of } I_\gamma e^{ik|x|} \text{ at } k = K.$$
For the second integral of (3.21), we consider in the complex $k$-plane a contour in the fourth quadrant bounded by the real axis from 0 to $X_1$ with an indentation below the pole $k = K$, an arc $\Gamma'$ of radius $X_1$ with center at the origin and the line joining the origin with the point $X_1 e^{-i\alpha}$ where $0 < \alpha < \frac{\pi}{2}$. Since now the singularities on the real axis are taken to be outside the contour and following a similar argument as above, we obtain that as $|x| \rightarrow \infty$

$$\lim_{|x| \rightarrow \infty} \int_0^\infty I_1 e^{-ik|x|} \, dk = 0.$$ Again,

$$\int_0^\infty \frac{1}{\Delta(k)} (s-1) \sinh kh \left( \frac{k}{k} \sinh kh - \cosh kh \right) e^{-ky} \cos kx \, dk$$

$$= \int_0^\infty I_2 e^{ik|x|} \, dk + \int_0^\infty I_2 e^{-ik|x|} \, dk$$

(3.23)

where $I_2 = \frac{1}{2} (s-1) \sinh kh \left( \frac{k}{k} \sinh kh - \cosh kh \right) \frac{F_1}{\Delta} e^{-ky}$. (3.24)

For the first integral of (3.24), we choose a similar contour
as was chosen for the integral with $I_1$ excepting that the indentation is now below $k = k_0$ instead of $K$. The contribution from the poles of $\Delta(k)$ which lie inside the contour has a factor $e^{-\beta_n|x|}$ where $\alpha_n + i\beta_n$ is a zero of $\Delta(k)$ in the first quadrant so that for large $|x|$ we may neglect it. The line may cross some singularities of $\Delta(k)$. To avoid this, if it crosses a zero of $\Delta(k)$, we indent the line about it so that it lies outside the region bounded by these contours and the contribution for this indentation will also contain a factor $e^{-\beta_m|x|}$ which becomes exponentially small for large $|x|$, $\alpha_m + i\beta_m$ being a singularity of this type. Hence, we find that as $|x| \to \infty$

$$\int_0^\infty I_2 e^{ik|x|} \, dk = 2\pi i \text{Residue of } I_2 e^{ik|x|} 	ext{ at } k = k_0.$$ 

By a somewhat similar argument as was used in the second integral in (3.21), we obtain

$$\int_0^\infty I_2 e^{-ik|x|} \, dk \to 0 \text{ as } |x| \to \infty.$$

Hence, we find that as $|x| \to \infty$, $\varphi_1$ tends to
\[-2\pi i \frac{e^{-K(h+y)}}{(1-2s)\sinh Kh - \cosh Kh} \left[ \mu e^{-K\eta} - \frac{e^{-K(\eta + \mu e^{-K\eta})}}{1 + \mu e^{-2Kh}} \right] \{\mu e^{-Kh} \infty - Kh/ KT)^, -KT^{-1/M_e - Kn} \infty e^{-K(h+y)} \sinh Kh - \cosh Kh \} + \sinh Kh + \cosh Kh} \right] e^{-K(h+y)} \infty 2\pi i(s-1)\sinh k_0 h(k_0/k) \sinh k_0 h\]

\[-\cosh k_0 h) \left[ \mu e^{-k_0 \eta} - \frac{e^{-k_0 h}(\eta + \mu e^{-k_0 \eta})}{1 + \mu e^{-2k_0 h}} \right] \{\mu e^{-k_0 h} \infty - k_0 \gamma \}

+ \frac{1}{k} \left\{ (K-k_0 + sk_0)\sinh k_0 h + sk \cosh k_0 h) \right\} \left\{ \begin{array}{l} \frac{i k_0}{k} \infty 1/2, \end{array} \right. \right.

\text{where}

\[D = \left\{ (1-2s)\sinh k_0 h - \cosh k_0 h \right\} \left\{ k_0 (1-s) - sk \right\} \cosh k_0 h + (1-s-hK)\sinh k_0 h \right\}. \] (3.26)

Similarly as \(|x| \to \infty\), \(\varphi_2\) tends to

\[-2\pi i \frac{e^{-K(h+y)}}{(1-2s)\sinh Kh - \cosh Kh} \left[ \mu e^{-K\eta} - \frac{e^{-K(\eta + \mu e^{-K\eta})}}{1 + \mu e^{-2Kh}} \right] \{\mu e^{-Kh} \infty - Kh/ KT)^, -KT^{-1/M_e - Kn} \infty e^{-K(h+y)} \sinh Kh - \cosh Kh \} + \sinh Kh + \cosh Kh} \right] e^{-K(h+y)} \infty 2\pi i(s-1)\sinh k_0 h(k_0/k) \sinh k_0 h\]
\begin{align*}
&+
\left(\sinh \theta - \cosh \theta\right) e^{iK|x|} + 2\pi i (s-1) \sinh k_0h \left[-\sinh k_0(h+y)\right] - \frac{k_0}{k} \cosh k_0(h+y) \left[\sum e^{-k_0\eta} \frac{e^{-k_0h} - \mu e^{-k_0\eta}}{1 + \mu e^{-2k_0h}} \right] e^{-k_0h} \\
&+ \frac{1}{K} \left( (k-k_0+sk_0) \sinh k_0h+sk_0 \cosh k_0h \right) \left[ek_0|x| - \frac{1}{D}\right], \quad (3.27)
\end{align*}

where $D$ is given by (3.26).

Thus $\varphi_1$ and $\varphi_2$ satisfy the radiation condition at infinity. Now as $h$ tends to infinity (3.25) and (3.26) take respectively the following forms

\begin{align*}
&\langle \psi_1 \rangle = 2\pi i \frac{s}{1+s} e^{e^{-k_0(y+\eta)}} e^{iK|x|}
\end{align*}

and

\begin{align*}
&\langle \psi_2 \rangle = -2\pi i \frac{s}{1+s} e^{e^{k_0(y-\eta)}} e^{iK|x|}
\end{align*}

where now $k_0 = \frac{1+s}{1-s} K = M$, and these agree with results of Gorgui and Kassem (1978).
4. Wave source submerged in lower fluid

In this case, there is a logarithmic type singularity at the point \((0, n)\).

Now \(\phi_1\) and \(\phi_2\) can be represented as

\[
\phi_1 = \sum_{j=0}^{\infty} c_j \log R_j + \sum_{j=0}^{\infty} d_j \log R_j + \int_0^\infty A(k) e^{-ky} \cos kx \, dk \tag{4.1}
\]

\[
\phi_2 = \sum_{j=-\infty}^{\infty} f_j \log R_j + \sum_{j=1}^{\infty} g_j \log R_j + \left[ B(k) \cosh k(h+y) + C(k) \sinh k(h+y) \right] \cos kx \, dk \tag{4.2}
\]

where \(\nu_0 = 1\). Condition (2.1) gives

\[
\begin{align*}
f_0 + g_1 &= 0 \\
f_1 &= 0 \\
f_{j+1} + g_{-j} &= 0, \quad j = 1, 2, \ldots \\
g_{j+1} + f_{-j} &= 0, \quad j = 1, 2, \ldots
\end{align*}
\tag{4.3}
\]
and

\[ KB + \sum_{1}^{\infty} f_j e^{-k(2j-1)h-n} - \sum_{0}^{\infty} f_{-j} e^{-k(2j+1)+\eta) + \sum_{1}^{\infty} g_j e^{-k(2j-1)h+n} - \sum_{1}^{\infty} g_{-j} e^{-k(2j+1)h-n) + kC = 0. \] (4.4)

Condition (2.3) gives

\[ d_0 = s f_0 - l \]

\[ c_j = s (f_j - f_{j+1}), \quad j = 1, 2, \ldots \] (4.5)

\[ d_j = s (g_j - g_{j+1}), \quad j = 1, 2, \ldots \]

and

\[ (k-K)A + s(K \cosh kh + k \sinh kh)B + s(K \sinh kh + k \cosh kh)C \]

\[ = 2s f_0 e^{-k\eta} - 2s e^{-k\eta} - 2s \sum_{1}^{\infty} f_{j+1} e^{-k(2jh-n)} - 2s \sum_{1}^{\infty} g_{j+1} e^{-k(2jh-n)} \] (4.6)
Condition (2.2) gives

\[ k(A + B \sinh kh + C \cosh kh) = -e^{-kh} + \sum_{l=1}^{\infty} (f_j - f_{j+1}) e^{-k(2jh - \eta)} \]

\[ + (sf_0 - 1)e^{-k\eta} + \sum_{l=1}^{\infty} (g_j - g_{j+1}) e^{-k(2jh + \eta)} \sum_{l=1}^{\infty} f_j e^{-k(2jh - \eta)} \]

\[ + f_0 e^{-k\eta} - \sum_{l=1}^{\infty} g_{j+1} e^{-k(2jh + \eta)} \sum_{l=1}^{\infty} g_j e^{-k(2jh - \eta)} \]

(4.7)

Now for convergence at \( k = 0 \), we obtain

\[ \sum_{l=1}^{\infty} [(s - 1)f_j - (s + 1)f_{j+1}] + \sum_{l=1}^{\infty} [(s - 1)g_j - (s + 1)g_{j+1}] \]

\[ + (s + 1)f_0 - 2 = 0 \].

(4.8)

This is satisfied by choosing
\begin{align*}
(s-1)f_j - (s+1)f_{j+1} &= 0, \quad j = 1, 2, \ldots \quad (4.9) \\
(s-1)g_j - (s+1)g_{j+1} &= 0, \quad j = 1, 2, \ldots \quad (4.9)
\end{align*}

\begin{align*}
f_0 &= \frac{2}{1+s},
\end{align*}

From (4.5), \( d_0 = -\mu = \sqrt{\frac{1-s}{1+s}} \).

From (4.3), (4.5) and (4.9), we obtain

\begin{align*}
&g_j = \frac{2}{1+s} (-1)^{j+1}, \quad j = 1, 2, \ldots \quad (4.10) \\
g_{-j} = 0, \quad j = 1, 2, \ldots \\
f_j = 0, \quad j = 1, 2, \ldots \\
f_{-j} = \frac{2}{1-s^2} (-1)^{j+1}, \quad j = 1, 2, \ldots \quad (4.10) \\
c_{-j} = 0, \quad j = 1, 2, \ldots \\
d_j = \frac{4s}{1-s^2} (-1)^{j+1}, \quad j = 1, 2, \ldots \quad (4.10)
\end{align*}
(4.4) can be written as

$$KB + kC \frac{2}{1+s} e^{-k(h+\eta)} + \frac{2\mu e^{-2kh}}{1 + \mu e^{-2kh}} \left[ e^{-k(h+\eta)} \frac{e - k(h-\eta)}{1+s} - \frac{e - k(h-\eta)}{1-s} \right] = 0. \quad (4.11)$$

(4.6) can be written as

$$(k-K)A + s(K \cosh kh + k \sinh kh)B + s(K \sinh kh + k \cosh kh)C$$

$$= -2\mu e^{-k\eta} - \frac{4s\mu e^{-k(2h+\eta)}}{(1-s)(1 + \mu e^{-2kh})} . \quad (4.12)$$

(4.7) gives

$$A + B \sinh kh + C \cosh kh = 0 . \quad (4.13)$$

Solving for $A, B, C$ from (4.11), (4.12) and (4.13), we obtain

$$A = \left( \frac{k}{K} \sinh kh - \cosh kh \right) \left[ \frac{1}{k-K} - \frac{(s-1)\sinh kh}{\Delta(k)} \right] \sigma_1,$$
\[
\frac{\sinh kh}{K} \left[ \frac{2e^{-k(h+\eta)}}{1+s} + \frac{2ue^{-2kh}}{1 + \mu e^{-2kh}} \left\{ \frac{e^{k(h-\eta)}}{1-s} - \frac{e^{-k(h+\eta)}}{1+s} \right\} \right] \quad (4.14)
\]

\[
B = -\frac{k}{K} \left[ \frac{1}{k-K} - \frac{(s-1)\sinh kh}{\Delta(k)} \right] G_1 + \frac{2e^{-k(h+\eta)}}{K(1+s)}
\]

\[
+ \frac{2ue^{-2kh}}{K(1 + \mu e^{-2kh})} \left[ \frac{e^{k(h-\eta)}}{1-s} - \frac{e^{-k(h+\eta)}}{1+s} \right] \quad (4.15)
\]

\[
C = \left[ \frac{1}{k-K} + \frac{(s-1)\sinh kh}{\Delta(k)} \right] G_1 \quad (4.16)
\]

where,

\[
G_1 = -2ue^{-k\eta} \left( 1 + \frac{2s}{1+s} \frac{e^{-2kh}}{1 + \mu e^{-2kh}} \right) - \frac{1}{K} \left[ (K-k+sk)\sinh kh + sk \cosh kh \right]
\]

\[
\left[ \frac{2e^{-k(h+\eta)}}{1+s} + \frac{2ue^{-2kh}}{1 + \mu e^{-2kh}} \left\{ \frac{e^{k(h-\eta)}}{1-s} - \frac{e^{-k(h+\eta)}}{1+s} \right\} \right] / (1-2s)\sinh kh - \cosh kh \quad (4.17)
\]
\[ \Delta(k) \] being given previously.

Thus using the above results, we obtain

\[ \varphi_1 = \log R_0 + \frac{s-1}{s+1} \log R_0' + \frac{4s}{1-s^2} \sum_{l=1}^{\infty} (-1)^l \mu_l \log R_j' + \]

\[ + \int_0^\infty \frac{1}{k-k'} \left( \frac{k}{k'} \sinh kh - \cosh kh \right) G_1 e^{-ky} \cos kx \, dk \]

\[ + \int_0^\infty \frac{1}{\Delta(k)} (s-1) \sinh kh \left( \frac{k}{k'} \sinh kh - \cosh kh \right) G_1 e^{-ky} \cos kx \, dk \]

\[ - \int_0^\infty \frac{\sinh kh}{k} \left[ \frac{2e^{-k(h+\eta)}}{1+s} + \frac{2\mu e^{-2kh}}{1+\mu e^{-2kh}} \left( \frac{e^{k(h-\eta)}}{1-s} - \frac{e^{-k(h+\eta)}}{1+s} \right) \right] \cos kx \, dk \]

\[ (4.18) \]

\[ \varphi_2 = \frac{2}{1+s} \log R_0 + \frac{2}{1-s} \sum_{l=1}^{\infty} (-1)^l \mu_j \log R_j' + \]

\[ + \int_0^\infty \left[ \sinh k(h+y) - \frac{k}{k'} \cosh k(h+y) \right] \frac{G_1}{k-k'} \cos kx \, dk + \]
\[ + \sum_{\nu, \sigma} \frac{1}{\Delta(k)} (s-1) \sinh kh \left[ \sinh k(h+y) \right. \]

\[- \frac{k}{k} \cosh k(h+y) \right] G_1 \cos kx \, dk + \int_0^\infty \left( \frac{2e^{-k(h+y)}}{K(1+s)} \right) \]

\[+ \frac{2\mu e^{-2kh}}{K(1+\mu e^{-2kh})} \left\{ \frac{e^{-k(h+\eta)}}{1-s} - \frac{e^{-k(h+\eta)}}{1+s} \right\} \cosh k(h+y) \right] \cos kx \, dk, \ (4.19) \]

Now as \( h \to \infty \), we obtain

\[ \varphi_1 = \log R_0 - \frac{1-s}{1+s} \log R_0 - \frac{2}{1+s} \int_0^\infty \frac{e^{-k(y+\eta)}}{k-M} \cos kx \, dk \]

\[ \varphi_2 = \frac{2}{1+s} \log R_0 + \frac{2}{1+s} \int_0^\infty \frac{e^{k(y-\eta)}}{k-M} \cos kx \, dk \]

which are the results derived by Gorgui and Kassem (1978).

Proceeding similarly as in the previous case we see that the above potentials have the following forms as \( |x| \to \infty \).
\[-2\pi i (e^{-\kappa h}) \left[ \mu e^{-K \eta} (1 + \frac{2s}{1+s} \frac{e^{-2\kappa h}}{1+\mu e^{-2\kappa h}}) + \ldots (se^{\kappa h}) \right] \]

\[
\left\{ \frac{e^{-K(h+\eta)}}{1+s} + \frac{\mu e^{-2\kappa h}}{1+\mu e^{-2\kappa h}} \left( \frac{e^K(h-\eta)}{1-s} - \frac{e^{-K(h+\eta)}}{1+s} \right) \right\} e^{-K \gamma e^{iK|x|}} / (2s-1) \sinh \kappa h
\]

\[+ \cosh \kappa h \right] + 2\pi i (1-s) \sinh \kappa_0 h \left( \frac{k_0}{K} \sinh \kappa_0 h - \cosh \kappa_0 h \right) \]

\[
\left[ \mu e^{-\kappa_0 \eta} (1 + \frac{2s}{1+s} \frac{e^{-2\kappa_0 h}}{1+\mu e^{-2\kappa_0 h}}) + \frac{1}{K} (K-k_0 + sk_0) \sinh \kappa_0 h + sK \cosh \kappa_0 h \right] \]

\[
\left\{ \frac{e^{-\kappa_0 (h+\eta)}}{1+s} + \frac{\mu e^{-2\kappa_0 h}}{1+\mu e^{-2\kappa_0 h}} \left( \frac{e^{\kappa_0 (h-\eta)}}{1-s} - \frac{e^{-\kappa_0 (h+\eta)}}{1+s} \right) \right\} e^{-\kappa_0 \gamma e^{i\kappa_0 |x|}} / D
\]

where

\[D = \left[ (1-2s) \sinh \kappa_0 h - \cosh \kappa_0 h \right] \left[ h \ k_0 (1-s) - sK \cosh \kappa_0 h \right] + (1-s-hK) \sinh \kappa_0 h \]
and

\[
\frac{-\pi i e^{-K(h+y)}}{(1-2s)\sinh Kh - \cosh Kh} \left[ -2\mu e^{-K\eta} \left( 1 + \frac{2s}{l+s} \frac{e^{-2Kh}}{1+\mu e^{-2Kh}} \right) \right.
\]

\[
- \frac{e^{-K(h+\eta)}}{1+s} \left( \frac{e^{K(h+\eta)}}{1+s} \right)^{l+\mu e^{-2Kh}} \right]
\]

\[
e^{iK|x|} + \pi i (s-1) \sinh k_0 h x \left. \right\} \sinh k_0 (h+y) \]

\[
- \frac{k_0}{K} \cosh k_0 (h+y) \left[ -2\mu e^{-k_0 h} \left( 1 + \frac{2s}{l+s} \frac{e^{-2k_0 h}}{1+\mu e^{-2k_0 h}} \right) \right.
\]

\[
- \frac{1}{K} \left\{ (K-k_0 + s k_0) \sinh k_0 h + s K \cosh k_0 h \right\} \left( \frac{2}{l+s} \frac{e^{-k_0 (h+\eta)}}{1+s} \right) \frac{2\mu e^{-2k_0 h}}{1+\mu e^{-2k_0 h}}
\]

\[
\left\{ \left( \frac{e^{k_0 (h-\eta)}}{1-s} - \frac{e^{k_0 (h+\eta)}}{1+s} \right) \right\} e^{i k_0 |x|} \right]_D
\]

(4.21)

where D is given by (3.26).
Now, as \( h \) tends to infinity, (4.20) and (4.21) take respectively the following forms:

\[
\frac{2\pi i}{1+s} e^{-M(y+\eta)} iM|xl
\]

and

\[
\frac{2\pi i}{1+s} e^{M(y-\eta)} iM|xl
\]

5. Submerged point singularities, upper fluid of finite depth

Here, \( \varphi_2 \sim \frac{P_n(\cos \theta)}{R_{0}^{n+1}} \) as \( R_{0}' = \left\{ r^2 + (y+\eta)^2 \right\}^{1/2} \to 0, \)

\( n = 0, 1, 2, \ldots \)

where \( r \) is the distance from the \( y \) axis, and \( \theta = \tan^{-1}\left( \frac{r}{y-\eta} \right) \). We assume

\[
\varphi_1 = \int_0^\infty A(k) e^{-ky} J_0(kr) \, dk \quad (5.1)
\]
\[
\varphi_2 = \frac{P_n(\cos \theta)}{R_0^{n+1}} + \int_0^\infty \left\{ B(k) \cosh k(h+y) + C(k) \sinh k(h+y) \right\} J_0(kr) \, dk.
\]

(5.2)

We use the following integral representations

\[
\frac{P_n(\cos \theta)}{R_0^{n+1}} = \frac{1}{n!} \int_0^\infty k^n e^{-k(y+\eta)} J_0(kr) \, dk, \quad y > -\eta
\]

\[
= (-1)^n \int_0^\infty k^n e^{k(y+\eta)} J_0(kr) \, dk, \quad y < -\eta.
\]

(5.3)

From conditions (2.1), (2.2) and (2.3), we obtain

\[
KB + kc = \frac{(-1)^{n+1}}{n!} k^n(k+\eta) e^{-k(h-\eta)}
\]

(5.4)

\[
A + B \sinh kh + C \cosh kh = \frac{k^n}{n!} e^{-k\eta}
\]

(5.5)

and
\[ (k-K)A + s(K \cosh kh + k \sinh kh)B + s(K \sinh kh + k \cosh kh)C \]

\[ = \frac{s}{n!} k^n e^{-k\eta} (k-K). \]  \hspace{1cm} (5.6)

Solving for \( A, B, C \) we obtain

\[ A = \frac{k^n}{n!} \left[ e^{-k\eta} + \frac{(-1)^n}{k} (k+K) e^{-k(h-\eta)} \sinh kh \right] + \frac{W_1}{k-K} \]

\[ + \left( \frac{k}{K} \sinh kh \cosh kh \right) + \frac{W_1}{\Delta} (s-1) \sinh kh \left( \frac{k}{K} \sinh kh \cosh kh \right) \]

\[ B = \frac{(-1)^n k^n (k+K)}{n! K} e^{-k(h-\eta)} - \frac{k}{k-K} \left[ \frac{1}{k-K} + \frac{(s-1) \sinh kh}{\Delta} \right] W_1 \]

\[ C = \left[ \frac{1}{k-K} + \frac{(s-1) \sinh kh}{\Delta} \right] W_1 \]  \hspace{1cm} (5.7)

where
\[
\frac{k^n}{n!} \left[ (s-1)(k-K)e^{-k\eta} + \frac{(-1)^n}{K} (k-K) \sinh kh \\
+ sK \cosh kh \right] (k+K)e^{-k(h-\eta)} \right].
\]

\[
W_1 = \frac{1}{(1 - 2s) \sinh kh - \cosh kh}
\]

and \( \Delta \) is the same expression used in sub-section 3, so that we obtain

\[
\varphi_1 = \int_0^\infty \frac{k^n}{n!} \left[ e^{-k\eta} + \frac{(-1)^n}{K} (k-K) \sinh kh e^{-k(h-\eta)} \right] e^{-ky} J_0(\mu r) \, dk
\]

\[
+ \int_0^\infty \frac{W_1}{k-K} \left( \frac{k}{K} \sinh kh - \cosh kh \right)e^{-ky} J_0(\mu r) \, dk
\]

\[
+ \int_0^\infty \frac{W_1}{\Delta} (s-1) \sinh kh \left( \frac{k}{K} \sinh kh - \cosh kh \right)e^{-ky} J_0(\mu r) \, dk
\]

\[
(5.9)
\]

and,

\[
\varphi_2 = \frac{P_n(\cos \theta)}{R_0^{n+1}} + \int_0^\infty \frac{(-1)^{n+1}}{n!} \frac{k^n(k+K)}{K} e^{-k(h-\eta)} \cosh (h+y)j_0(\mu r) \, dk
\]

\[
+ \int_0^\infty \frac{W_1}{k-K} \left[ \sinh (h+y) - \frac{k}{K} \cosh (h+y) \right] j_0(\mu r) \, dk
\]
Now, as \( h \) tends to infinity, \( \varphi_1 \) and \( \varphi_2 \) take respectively the following forms:

\[
\frac{-2sM}{n!(l+s)} \int_0^\infty \frac{k^n}{k-M} e^{-k(y+\eta)} J_0(kr) \, dk \tag{5.11}
\]

and

\[
\frac{P_n(\cos \theta)}{R_0^{n+1}} + \int_0^\infty \frac{k^n}{n!} \left[ 1 + \frac{2sM}{(l+s)(k-M)} \right] k(y-\eta) J_0(kr) \, dk \tag{5.12}
\]

where \( M = K \frac{l+s}{1-s} \), which are the results derived by Gorgui and Kassem (1978). Now putting \( 2J_0(kr) = H_0^{(1)}(kr) + H_0^{(2)}(kr) \) and rotating the contour in the integral involving \( H_0^{(1)}(kr) \) in the first quadrant and in the integrals involving \( H_0^{(2)}(kr) \) in the fourth quadrant, we can reduce the integrals into suitable forms from which it is seen that as \( r \to \infty \), \( \varphi_1 \) and \( \varphi_2 \) respectively take the following forms:
\[
2\pi i \left(\frac{-1}{n!}\right)^n K^{n+1} \frac{e^{K(n-h-y)} H_0^{(1)}(kr)}{(2s-1)\sinh Kh + \cosh Kh} \\
+ \frac{k_0}{n!} (s-1)\sinh k_0 h \left(\frac{k_0}{K} \sinh k_0 h - \cosh k_0 h\right) \left(\frac{2}{e^{K(n-h-y)}} - K\right) e^{-k_0 n} \\
+ \frac{(-1)^n}{K} \left\{ (K-k_0+sk_0) \sinh k_0 h \right\} \\
+ sk_0 \cosh k_0 h \left\{ (k_0+K)e^{-k_0(h-\eta)} \right\} e^{-k_0 y} H_0^{(1)}(k_0 r)/D \quad (5.13)
\]

where D is given by (3.26) and

\[
2\pi i \left(\frac{-1}{n!}\right)^n K^{n+1} \frac{-\cosh K(h+y)}{(1 - 2s)\sinh Kh - \cosh Kh} \\
+ \frac{k_0}{n!} (s-1)\sinh k_0 h \left[ \sinh k_0 (h+y) - \frac{k_0}{K} \cosh k_0 (h+y) \right]
\]
\[(s-1)(k_0-K)e^{-k_0\eta} + \frac{(-1)^n}{K} \{ (K-k_0+sk_0) \sinh k_0h +
+ sk \cosh k_0h \} (k_0+K)e^{-k_0(h-\eta)} H_0^1(k_0r) / D \]  

(5.14)

where \( D \) is given by (3.26).

Now as \( h \) tends to infinity, (5.13) and (5.14) take respectively the following forms:

\[
\frac{\pi i}{2sk} (s-1) \frac{M^n}{n!} (M-K)^2 e^{-M(y-\eta)} H_0^1(Mr)
\]

and

\[
\frac{\pi i}{2sk} \frac{M^n}{n!} (1-s)(M-K)^2 e^{M(y-\eta)} H_0^1(Mr).
\]

6. Multipole submerged in lower fluid

Here, \( \varphi_1 \sim \frac{P_n(\cos \theta)}{R_{n+1}} \) as \( R_o = \left[ r^2 + (y-\eta)^2 \right]^{1/2} \rightarrow 0, n=1,2,\ldots \),

where \( \theta = \tan^{-1} \left( \frac{r}{y-\eta} \right) \).
We assume

\[
\varphi_1 = \frac{P_n(\cos \theta)}{R_o^{n+1}} + \int_0^\infty A(k)e^{-ky} J_0(kr) \, dk 
\]

(6.1)

\[
\varphi_2 = \int_0^\infty \left[ B(k) \cosh k(h+y) + C(k) \sinh k(h+y) \right] J_0(kr) \, dk, \quad (6.2)
\]

We use

\[
\frac{P_n(\cos \theta)}{R_o^{n+1}} = \frac{1}{n!} \int_0^\infty k^n e^{-k(y-\eta)} J_0(kr) \, dk, \quad y > n,
\]

\[
= \frac{(-1)^n}{n!} \int_0^\infty k^n e^{k(y-\eta)} J_0(kr) \, dk, \quad y < n.
\]

Proceeding much as in the previous case, we obtain

\[
\varphi_1 = \frac{P_n(\cos \theta)}{R_o^{n+1}} + \int_0^\infty \frac{(-1)^n}{n!} k^n e^{-k\eta} J_0(kr) \, dk
\]

\[
+ \int_0^\infty \frac{V}{k-k^2} \left( \frac{\sinh kh - \cosh kh}{k} \right) e^{-ky} J_0(kr) \, dk +
\]
+ \int_0^\infty \frac{V}{\Delta} (s-l) \sinh kh \left( \frac{k}{K} \sinh kh - \cosh kh \right) e^{-ky} J_0(\kappa r) \, dk \quad (6.4)

and

\varphi_2 = \int_0^\infty \frac{V}{k-K} \left[ \sinh k(h+y) - \frac{k}{K} \cosh k(h+y) \right] J_0(\kappa r) \, dk

+ \int_0^\infty \frac{V}{\Delta} (s-l) \sinh kh \left[ \sinh k(h-y) - \frac{k}{K} \cosh k(h+y) \right] J_0(\kappa r) \, dk \quad (6.5)

where, \( V = \frac{2(-1)^n K^n e^{-k\eta}}{n! \left[ 1 - 2s \sinh kh - \cosh kh \right]} \) \quad (6.6)

and \( \Delta(k) \) is the same expression used previously.

Now, as \( h \) tends to infinity, \( \varphi_1 \) and \( \varphi_2 \) take respectively the following forms:

\[
\frac{P_n(\cos \theta)}{R_0^{n+1}} + \frac{(-1)^n}{n!} \int_0^\infty \left\{ 1 + \frac{2K}{(1+s)(k-M)} \right\} k^n e^{-k(y-\eta)} J_0(\kappa r) \, dk \quad (6.7)
\]

and

\[
\frac{2(-1)^n M}{n! (1+s)} \int_0^\infty \frac{k^n}{k-M} e^{k(y-\eta)} J_0(\kappa r) \, dk \quad (6.8)
\]
where \( M = K \frac{1+s}{1-s} \) which are the results derived by Gorgui and Kassem (1978).

Proceeding much as in the previous case, we find that as \( r \to \infty \), \( \varphi_1 \) and \( \varphi_2 \) respectively take the following forms:

\[
\frac{K}{K} \quad \text{and} \quad \frac{K}{K}
\]

\[
2\pi \frac{(-1)^n}{n!} \frac{k^{n+1}}{K} \left( e^{-kh} - K(y+\eta) \right) \frac{H_0^{(1)}(kr)}{(1-2s)\sinh Kh - \cosh Kh}
\]

\[
+ \frac{2\pi \frac{(-1)^n}{n!} (s-1)k^0 \sinh k_0 \sinh k_0 h - \cosh k_0 h e^{k_0 (y+\eta)} H_0^{(1)}(k_0 r)}{(1-2s)\sinh Kh - \cosh Kh}
\]

(6.9)

and

\[
\frac{K}{K}
\]

\[
2\pi \frac{(-1)^n}{n!} \frac{k^{n+1}}{K} \left( e^{-k(h+y)} \right) \frac{H_0^{(1)}(Kr)}{(1-2s)\sinh Kh - \cosh Kh}
\]

\[
+ \frac{2\pi \frac{(-1)^n}{n!} (s-1)k^0 \sinh k_0 \sinh k_0 (h+y)}{(1-2s)\sinh Kh - \cosh Kh}
\]

\[
- \frac{k_0}{k} \cosh k_0 (h+y) \left[ e^{-k_0 \eta} H_0^{(1)}(k_0 r) \right]
\]

(6.10)
where $D$ is given by (3.26).

Now, as $h \to \infty$, (6.9) and (6.10) take respectively the following forms:

$$\pi i K M^n \frac{(-1)^n}{s^n} \frac{M}{K} \frac{M(y+\eta)}{(y+\eta)} \left( \frac{M}{K} - 1 \right) e^{M(y+\eta)} \left( \frac{M}{K} \right)_{(\eta)} H_0'(\eta)$$

and

$$\pi i K M^n \frac{(-1)^n}{s^n} \frac{M}{K} \frac{M(y-\eta)}{(y-\eta)} \left( \frac{M}{K} - 1 \right) e^{M(y-\eta)} \left( \frac{M}{K} \right)_{(\eta)} H_0'(\eta).$$

7. Conclusion

Integral representation of the potential function in a two layered fluid medium where the upper layer is of finite depth with a free surface and lower layer is of infinite depth have been obtained. The different results reduce to the known results of Gorgui and Kassem (1978) when the FS in the upper layer is taken to infinity (i.e., $h \to \infty$). We note that these authors thought there was no difficulty except longer equations and a bulkier result in adding the surface tension term in their problem, closely related to ours.
1.2 SINGULARITIES IN A THREE-LAYERED FLUID MEDIUM

1. Introduction

Different types of singularities that can be used in solving one-fluid problems concerning scattering or generation of surface waves of small amplitudes by obstacles present in the fluid have been surveyed in some detail initially by Thorne (1953) who neglected the effect of surface tension and later by Rhodes-Robinson (1970) who included it. The singularities are mainly submerged in a one-fluid medium of finite or infinite depth. The study of internal waves at the surfaces of separation of a multi-layered fluid medium necessitates the consideration of different types of singularities in the fluid. For the two-fluid case velocity potentials describing different types of singularities were obtained by Gorgui and Kassem (1978) when the upper fluid is unbounded and the lower fluid is of either finite or infinite depth, and by Kassem (1982) when both the fluids are of finite depths, the surface tension effect being neglected in all the cases. The effect of surface tension is included in problems considered independently by Rhodes-Robinson (1980) and Mandal (1981) when both the fluids are

*The content of this section has been published in J. Ind. Inst. Sci 65 No.11 Nov, 1984, 223-243*
unbounded and in section 2.1 when the upper fluid is unbounded and the lower fluid is of finite depth.

These two-fluid problems naturally motivates us to extend the results for a multi-layered medium. For this reason a three-layered fluid medium is considered where the upper fluid is unbounded, the middle fluid is of finite depth and the lower fluid is of infinite depth, the two surfaces of separation being horizontal planes of infinite extent. In the present section, we give a discussion of the basic line and point singularities oscillating with small amplitudes present in each of the three fluids. The time harmonic singularities are described by harmonic potential functions which are typical singular solutions of Laplace's equation in the neighbourhood of the singularities. Under the given boundary conditions at the two mean surfaces of separation and the radiation condition that there are only outgoing waves in the far field, unique solution will be found for each type of singularity concerned, the proofs depending upon the use of appropriate integral representations for singular harmonic functions. Detailed method of calculations for finding the different potential functions in different media is given in the case of a line singularity present in the middle fluid only. For other cases the final results are mostly stated.
2. Statement and formulation of the problem

We consider the irrotational motion of three non-viscous fluids under the action of gravity. The middle fluid is of finite depth \( 'h' \) while the upper and lower fluids are unbounded. The two mean surfaces of separation are horizontal planes of infinite extent. The motion is due to a singularity oscillating harmonically with small amplitudes in one of the three fluids. The motion in each case can be described by velocity potentials which are simple harmonic in time with period \( 2\pi/\sigma \) and thus it is more convenient to use complex valued potentials \( \varphi_j \exp(-i\omega t) \) (\( j = 1,2,3 \)) of which the actual velocity potentials are real parts, where the subscripts 1,2,3 are used for lower, middle and upper media respectively.

The origin 0 is taken on the mean surface of separation of the middle and lower fluids and the axis \( \Theta y \) pointing vertically downwards into the lower fluid is chosen in such a way that it passes through the singularity, so that the point at which the velocity potential has a singularity is taken conveniently as any one of the points \((0,\eta), (0,-\eta), (0,-2\eta+\eta) (\eta > 0)\) according as the singularity is in the lower, middle or upper fluid respectively. The velocity potentials then satisfy
\[ \nabla^2 \varphi_1 = 0 \ , \ y > 0 \]

\[ \nabla^2 \varphi_2 = 0 \ , \ -h < y < 0 \]

\[ \nabla^2 \varphi_3 = 0 \ , \ y < -h \]

except at the point of singularity. The linearised surface of separation conditions are

\[ K \varphi_1 + \frac{\partial \varphi_1}{\partial y} = s_1 (K \varphi_2 + \frac{\partial \varphi_2}{\partial y}) \text{ on } y = 0, \quad (2.1) \]

\[ K \varphi_2 + \frac{\partial \varphi_2}{\partial y} = s_2 (K \varphi_2 + \frac{\partial \varphi_3}{\partial y}) \text{ on } y = -h, \quad (2.2) \]

\[ \frac{\partial \varphi_1}{\partial y} = \frac{\partial \varphi_2}{\partial y} \text{ on } y = 0, \quad (2.3) \]

\[ \frac{\partial \varphi_2}{\partial y} = \frac{\partial \varphi_3}{\partial y} \text{ on } y = -h, \quad (2.4) \]

where \( K = \sigma^2/g, \ s_1 = \rho_2/\rho_1, \ s_2 = \rho_3/\rho_2, \ g \) being the
gravity, \( \rho_1, \rho_2 \) and \( \rho_3 \) being the densities of the lower, middle and upper fluids respectively (\( \rho_1 > \rho_2 > \rho_3 \)). Also

\[ \text{grad } \varphi_1 \to 0 \quad \text{as } y \to +\infty \]  

\[ \text{grad } \varphi_3 \to 0 \quad \text{as } y \to -\infty \]

There is another condition to be satisfied by \( \varphi_j \) (\( j = 1,2,3 \)) as \( |x| \to \infty \) which is the so-called radiation condition. This states that the potential function should represent diverging waves at a large distance from the singularity.

3. Line singularity submerged in the middle fluid of finite depth

Let a line singularity be placed at the point \( (0,-\eta) \) in the middle fluid. Then

\[ \varphi_2 \sim \log R_0' \quad \text{as } R_0' = \left\{ x + (y+\eta)^2 \right\}^{1/2} \to 0 \quad (3.1) \]

Now \( \varphi_1, \varphi_2, \varphi_3 \) can be represented as
\[\varphi_1 = \sum_{j=1}^{\infty} f_j \log R_j + \sum_{j=0}^{\infty} g_j \log R_j + \int_{-\infty}^{\infty} A(k) \exp(-ky)\cos kx \, dk \] (3.2)

\[\varphi_2 = \sum_{j=-\infty}^{\infty} c_j \log R_j + \sum_{j=0}^{\infty} d_j \log R_j \]

\[+ \int_{-\infty}^{\infty} \left[ B(k) \cosh k(h+y) + C(k) \sinh k(h+y) \right] \cos kx \, dk \] (3.3)

\[\varphi_3 = \sum_{j=0}^{\infty} P_j \log R_j + \sum_{j=-1}^{\infty} q_j \log R_j + \int_{-\infty}^{\infty} D(k) \exp(ky) \cos kx \, dk \] (3.4)

where

\[R_j^2 = x^2 + (y+2jh-n)^2, \quad R_j^2 = x^2 + (y+2jh+n)^2, \quad j = 0, \pm 1, \pm 2, \ldots\]

Because of (3.1) we choose \(d_0 = 1\). Conditions (2.5) and (2.6) are automatically satisfied. \(A(k), B(k), C(k), D(k)\) and \(f_j, g_j, c_j, d_j, P_j, q_j\) are to be so chosen that the conditions (2.1), (2.2), (2.3) and (2.4) are satisfied and the different integrals converge. The radiation condition will be dealt with in the sequel.
The following integral representations will be needed in our calculations:

\[ \frac{\partial}{\partial y} \left( \log R_j \right) = \pm \int_0^{\infty} \exp \left\{ \pm k(y + 2j h - \eta) \right\} \cos kx \, dk, \quad y \geq -2jh + \eta \]

\[ \frac{\partial}{\partial y} \left( \log R'_j \right) = \pm \int_0^{\infty} \exp \left\{ \pm k(y + 2jh + \eta) \right\} \cos kx \, dk, \quad y \geq -(2jh + \eta) \]

where the upper signs are for \( > \) cases and the lower signs are for the \( < \) cases. Hence

on \( y = -h \), \[ \frac{\partial}{\partial y} \left( \log R_j \right) = \pm \int_0^{\infty} \exp \left\{ \pm k \left\{ (2j - 1)h - \eta \right\} \right\} \cos kx \, dk, \]

on \( y = -h \), \[ \frac{\partial}{\partial y} \left( \log R'_j \right) = \pm \int_0^{\infty} \exp \left\{ \pm k \left\{ (2j - 1)h + \eta \right\} \right\} \cos kx \, dk, \]

on \( y = 0 \), \[ \frac{\partial}{\partial y} \left( \log R_j \right) = \pm \int_0^{\infty} \exp \left\{ \pm k \left\{ (2jh - \eta) \right\} \right\} \cos kx \, dk, \]

where the upper signs are for \( j = 1, 2 \ldots \) and the lower
signs for $j = 0, -1, -2 \ldots$, and

\[
\frac{3}{2y} \left( \log R'_j \right) = \pm \int_0^\infty \exp \left[ \pm k \left( 2jh+\eta \right) \right] \cos kx \, dk
\]

where the upper sign is for $j = 0, 1, 2, \ldots$ and the lower sign is for $j = -1, -2 \ldots$.

After using these integral representations in appropriate places the condition (2.1) gives

\[
K \left[ \sum_{j} f_j \log \left\{ x^2 + (2jh+\eta)^2 \right\}^{1/2} + \sum_{\eta} g_{\eta} \log \left\{ x^2 + (2jh+\eta)^2 \right\}^{1/2} \right. \\
+ \int_0^\infty \cos kx \, dk \left. + \int_0^\infty \left[ \sum_{j} f_j \exp \left\{ -k(2jh+\eta) \right\} \right. \\
+ \sum_{\eta} \exp \left\{ -k(2jh+\eta) \right\} - kA \right] \cos kx \, dk \\
= s_1 K \left[ \sum_{j} c_j \log \left\{ x^2 + (2jh+\eta)^2 \right\}^{1/2} + \sum_{\eta} c_{-\eta} \log \left\{ x^2 + (2jh+\eta)^2 \right\}^{1/2} + \right. \\
+ \left. \int_0^\infty \cos kx \, dk \right]
\]
\[ \sum_{j=0}^{\infty} d_j \log \left\{ \frac{x^2 + (2jh+\eta)^2}{x^2 + (2jh-\eta)^2} \right\} \left[ 1 + \sum_{j=0}^{\infty} d_j \right] \]

\[ + \int_0^\infty (C \cosh kh + B \sinh kh) \cos kx \, dk \]

\[ + s_1 \int_0^\infty \left[ \sum_{j=0}^{\infty} c_j \exp\left\{ -k(2jh-\eta) \right\} - \sum_{j=0}^{\infty} c_j \exp\left\{ -k(2jh+\eta) \right\} \right] \cos kx \, dk \]

\[ + \sum_{j=1}^{\infty} d_j \exp\left\{ -k(2jh+\eta) \right\} - \sum_{j=1}^{\infty} d_j \exp\left\{ -k(2jh-\eta) \right\} \]

\[ + k(B \sinh kh + C \cosh kh) \right\} \cos kx \, dk \]

(3.5)

By equating the coefficients of similar logarithmic terms (3.5) gives

\[
\begin{align*}
    f_j &= s_1(c_j + d_j), \quad j = 1, 2, \ldots \\
    g_j &= s_1(d_j + c_j), \quad j = 0, 1, 2, \ldots
\end{align*}
\]

(3.6)

Since \( d_0 = 1 \) we obtain \( g_0 = s_1(c_0 + 1) \).
Again, the condition (2.2) similarly gives

\[
\begin{align*}
K \left[ \sum_1^\infty c_j \log \left\{ x^2 + (2j-1 \ h-\eta)^2 \right\} \right]^{1/2} &+ \sum_0^\infty c_{-j} \log \left\{ x^2 + (2j+1 \ h+\eta)^2 \right\}^{1/2} \\
+ \sum_1^\infty d_j \log \left\{ x^2 + (2j-1 \ h+\eta)^2 \right\}^{1/2} &+ \sum_0^\infty d_{-j} \log \left\{ x^2 + (2j+1 \ h-\eta)^2 \right\}^{1/2} \\
+ \int B \cos kx \, dk &+ \int \left[ \sum_1^\infty c_j \exp \left\{ -k (2j-1 \ h-\eta) \right\} \right] \\
- \sum_0^\infty c_{-j} \exp \left\{ -k (2j+1 \ h+\eta) \right\} &+ \sum_1^\infty d_{-j} \exp \left\{ -k (2j-1 \ h+\eta) \right\} + kC \cos kx \, dk \\
- \sum_0^\infty d_{-j} \exp \left\{ -k (2j+1 \ h-\eta) \right\} &+ kC \cos kx \, dk \\
= s_2 K \left[ \sum_0^\infty p_{-j} \log \left\{ x^2 + (2j+1 \ h+\eta)^2 \right\}^{1/2} \right] + q_{-j} \log \left\{ x^2 + (2j+1 \ h-\eta)^2 \right\}^{1/2} \\
+ \int D \exp(-kh) \cos kx \, dk &+ s_2 \int \left[ \sum_0^\infty p_{-j} \exp \left\{ -k (2j+1 \ h+\eta) \right\} \right] \\
- \sum_0^\infty p_{-j} \exp \left\{ -k (2j+1 \ h+\eta) \right\} &+ s_2 \int 
\end{align*}
\]
from which we obtain similarly
\[ c_{j+1} - d_j = s_2 q_j \quad \text{and} \quad d_j + c_j = s_2 p_j, \quad j = 0, 1, 2 \ldots \] (3.8)

so that \( c_1 + 1 = s_2 q_0 \) as \( d_0 = 1 \). Condition (2.3) gives

\[ \sum_{j=0}^{\infty} (f_j - 2c_j + \frac{f_j}{s_1}) \exp \{-k(2j + \eta)\} + \sum_{j=0}^{\infty} (g_j - 2d_j + \frac{g_j}{s_1}) \exp \{-k(2j + \eta)\} = k (A + B \sinh kh + C \cosh kh). \] (3.9)

For convergence of the integrals in (3.2), (3.3) and (3.4), the expression in the left side of (3.9) must vanish for \( k = 0 \) so that

\[ \sum_{j=1}^{\infty} (f_j - 2c_j + \frac{f_j}{s_1}) + \sum_{j=0}^{\infty} (g_j - 2d_j + \frac{g_j}{s_1}) = 0. \]

This is satisfied by choosing
Finally the condition (2.4) gives

\[
\sum_{j=1}^{\infty} \left[ c_{j+1} - \frac{f_j}{s_1} + \frac{1}{s_2} (c_{j+1} + \frac{f_j}{s_1} - c_j) \right] \exp \left[ -k \{ (2j+1)h - \eta \} \right] + \sum_{j=0}^{\infty} \left[ -\frac{g_j}{s_1} + d_j + d_{j+1} + \frac{1}{s_2} (d_{j+1} + \frac{g_j}{s_1} - g_j) \right] \exp \left[ -k \{ (2j+1)h + \eta \} \right] + \{ c_1 - 1 + \frac{1}{s_2} (c_1 + 1) \} \exp \{-k(h-\eta)\} = k \{ D \exp(-kh) - C \} \quad (3.11)
\]

The left side of (3.11) must vanish for \( k = 0 \) from the convergence consideration so that

\[
\sum_{j=1}^{\infty} \left[ (1 + \frac{1}{s_2})c_{j+1} + (1 - \frac{1}{s_2})c_j + (\frac{1}{s_2} - 1) \frac{f_j}{s_1} \right] + \sum_{j=0}^{\infty} \left[ -\frac{g_j}{s_1} + d_j + d_{j+1} + \frac{1}{s_2} (d_{j+1} + \frac{g_j}{s_1} - g_j) \right] = k \{ D \exp(-h) - C \} .
\]
This is satisfied by choosing

\[ c_{j+1} - v c_j + \frac{v}{s_1} f_j = 0 \quad j = 1, 2, \ldots \]

\[ d_{j+1} - v d_j + \frac{v}{s_1} g_j = 0 \quad j = 0, 1, 2, \ldots \] \hspace{1cm} (3.12)

\[ c_1 = -v \text{ where } v = \frac{1 - s_2}{1 + s_2} \]

Then from (3.6), (3.8), (3.10) and (3.12) we can obtain

\[ f_j = \frac{2s_1}{1 + s_1} (-1)^j \nu^j \mu^{j-1} \quad j = 1, 2, 3, \ldots \] \hspace{1cm} (3.13 contd.)

\[ g_j = \frac{2s_1}{1 + s_1} (-1)^j (\mu \nu)^j \quad j = 0, 1, 2, \ldots \]

\[ c_j = (-1)^j \nu^j \mu^{j-1} \quad j = 1, 2, 3, \ldots \]
\[
d_j = (-1)^j (\mu \nu)^j \quad \text{for } j = 0, 1, 2, \ldots
\]
\[
c_{-j} = (-1)^j \nu^j \mu^{j+1} \quad \text{for } j = 0, 1, 2, \ldots
\]
\[
d_{-j} = (-1)^j (\mu \nu)^j \quad \text{for } j = 1, 2, 3, \ldots
\]
\[
p_{-j} = \frac{2}{1+s_2} (-1)^j \nu^j \mu^{j+1} \quad \text{for } j = 0, 1, 2, \ldots
\]
\[
a_{-j} = \frac{2}{1+s_2} (-1)^j (\mu \nu)^j \quad \text{for } j = 0, 1, 2, \ldots
\]

where \( \mu = \frac{1-s_1}{1+s_1} \).

Using these in (3.5), we can obtain

\[
(k-K)A + s_1(K \cosh kh + k \sinh kh)B + s_1(K \sinh kh + k \cosh kh)C
\]
\[
= \frac{2s_1 \mu \left[ \exp(-k\eta) - \nu \exp \{ k(\eta-2h) \} \right]}{1 + \mu \nu \exp(-2kh)} = E(k), \text{ say,} \quad (3.14)
\]
(3.7) gives

\[ KB + KC - s_2 (k + K) \exp(-kh)D \]

\[ = \frac{2v \exp(-kh) \{ \mu \exp(-k\eta) + \exp(k\eta) \}}{1 + \mu \exp(-2kh)} = F(k), \text{ say} \]  

(3.15)

and from (3.9) and (3.11) we can obtain

\[ A + B \sinh kh + C \cosh kh = 0 \]  

(3.16)

\[ D \exp(-kh) - C = 0 \]  

(3.17)

solving for A, B, C and D from (3.14), (3.15), (3.16) and (3.17) we can obtain

\[ A = -\frac{F}{K} \sinh kh + \left[ \frac{1}{K} \left\{ k - s_2(k+K) \right\} \sinh kh - \cosh kh \right] \frac{W}{\Delta} \]

\[ B = \frac{F}{K} + \frac{1}{K} \left\{ s_2(k + K) - k \right\} \frac{W}{\Delta} \]  

(3.18 contd.)
\[ G W A D = \frac{\exp (\alpha)}{\Delta} \]  
\( (3.18) \)

\[ W(k) = E \left\{ (K-k+s_1k) \sinh kh + s_1 K \cosh kh \right\} \]  
\( (3.19) \)

and

\[ \Delta(k) = \left\{ \frac{1}{K} (K-k+s_1k)(s_2k+s_2K-k) + s_1K \right\} \sinh kh \]  
\[ + \left\{ s_1s_2(k+K) + K-k \right\} \cosh kh \]  
\( (3.20) \)

Now \( \Delta(k) \) has three zeros at \( k = K_1, k_0, -k_0 \), say, all on the real axis and complex zeros at \( k = k_n, \) say, \( (n \geq 1) \), where \( k_n = \alpha_n + i\beta_n \), say. It may be noted that when \( s_2 = 0, K_1 \) becomes \( K \). Thus \( \Delta(k), B(k), C(k) \) and \( D(k) \) have simple poles at \( k = K_1 \) and \( k = k_0 \) on the positive real axis. In the line integrals from 0 to \( \infty \) we make indentations below these poles which account for the behaviour of the potential functions at infinity particularly as \( |x| \to \infty \). This will be evident later. Thus using the
above results we obtain

\[ \varphi_1 = \frac{2s_1}{1+s_1} \sum_{j=0}^{\infty} (-1)^j (\nu^j) \log R_j' + \frac{2s_1}{1+s_1} \sum_{j=0}^{\infty} (-1)^j (\mu^j) \log R_j' \]

\[- \int_0^F \frac{f}{K} \sinh kh \exp(-ky) \cos kx \, dk \]

\[ + \int_0^F \frac{L}{K} \left[ \{ k-s (K+k) \} \sinh kh \cosh kh \right] \frac{W}{\Delta} \exp(-ky) \cos kx \, dk \]

\[ (3.21) \]

\[ \varphi_2 = \sum_{j=0}^{\infty} (-1)^j (\nu^j) \mu^{j-1} \log R_j + \sum_{j=0}^{\infty} (-1)^j (\nu^j) \mu^{j+1} \log R_{-j} \]

\[ + \sum_{j=0}^{\infty} (-1)^j (\mu^j) \log R_j' + \sum_{j=0}^{\infty} (-1)^j (\mu^j) \log R_{-j}' \]

\[ + \int_0^F \frac{F}{K} \cosh(k(h+y)) \cos kx \, dk + \]

\[ + \int_0^F \left[ \frac{1}{K} \left\{ s_2 (k+K) - k \right\} \cosh(k(h+y)) + \sinh(k(h+y)) \right] \frac{W}{\Delta} \cos kx \, dk \]

\[ (3.22) \]
Putting $s_2 = 0$ we find that the expressions for $\psi_1$ and $\psi_2$ agree with the corresponding results in the case of a two-fluid medium with upper fluid of finite depth and the lower fluid of infinite depth obtained in section 1.1, and further letting $h \rightarrow \infty$ (the case of a two-fluid medium when both the fluids are unbounded) the results given by Gorgui and Kassem (1978) are recovered. Also if we put $\rho_1 = \rho_2 = \rho_3$, then the three-layered medium reduces to a single fluid medium of infinite extent, and in that case $s_1 = 1$, $s_2 = 1$ so that $\mu = 0$, $\nu = 0$. Then it is easily seen that (3.21), (3.22) and (3.23) readily give $\psi_1 = \psi_2 = \psi_3 = \log R_0$ which is in fact the potential function in an infinite fluid due to a line singularity of logarithmic type at $(0, -\eta)$.

Now to investigate the behaviour of $\psi_1$, $\psi_2$, and $\psi_3$ for large $|x|$ we note that we have to consider only the behaviour of the last integral in each expression. We put $2 \cos kx = \exp(ik|x|) + \exp(-ik|x|)$ in these integrals so that
\[ \int_0^\infty \left[ \frac{1}{k} \{k-s(k+K)\} \sinh kh \cos kh \right] \exp(-ky) \frac{W}{\Delta} \cos kx \, dk \]

\[ = \int_0^\infty e^{ik|x|} \, dk + \int_0^\infty e^{-ik|x|} \, dk, \text{ say} . \quad (3.24) \]

For the first integral of (3.24) we consider in the complex k-plane a contour in the first quadrant bounded by a portion of the real axis of large length \( X_1 \) with indentations below the poles at \( k = K_1, k = k_0 \), a circular arc \( \Gamma' \) of radius \( X_1 \) with centre at the origin and the line joining the origin with point \( X_1 e^{i\alpha} \) where \( 0 < \alpha < \pi/2 \). Now the integrals along the arc \( \Gamma' \) and this line become exponentially small for large \( |x| \). The contribution from the poles \( \sigma_m + i\beta_m \), say, in the first quadrant which lie inside the contour has also a factor \( \exp(-\beta_m|x|) \) which becomes exponentially small for large \( |x| \). The line may cross some complex zeros of \( \Delta(k) \) in the first quadrant. To account for this, if it crosses a zero of \( \Delta(k) \) we indent the line about it so that it lies outside the region bounded by these contours, and the contribution for this indentation will also contain a factor which becomes exponentially small for large \( |x| \). Thus for considering the behaviour as \( |x| \to \infty \), we only need to
consider the behaviour of the integral arising from the 
residues at \( k = K_1 \) and \( k = k_0 \). Hence making \( X_1 \to \infty \)
we find that, as \( |x| \to \infty \).

\[
\int_0^{\infty} \frac{\exp(ik|x|)dk}{|x|} \to 2\pi i \left\{ \text{sum of the residues of}
\int_0^{\infty} \exp(ik|x|) \text{ at } k = K_1 \text{ and } k = k_0 \right\}.
\]

For the second integral of (3.24) we consider in 
the complex \( k \)-plane a contour in the fourth quadrant bounded 
by the real axis from 0 to \( X_1 \) with indentations below 
the poles at \( k = K_1 \) and \( k = k_0 \), a circular arc \( \gamma' \) of 
radius \( X_1 \) with centre at the origin and the line joining 
the origin with the point \( X_1 \exp(-i\alpha) \) where \( 0 < \alpha < \pi/2 \).
Since now the singularities on the positive real axis are 
taken to be outside this contour, following a similar 
argument as above we obtain as \( |x| \to \infty \),

\[
\int_0^{\infty} \frac{\exp(-ik|x|)dk}{|x|} \to 0. \text{ Hence we find that as } |x| \to \infty,
\]

\[
\varphi_1 \to \pi i \left[ + \frac{1}{K} \{K_1 - s_2(K_1 + K)\} \sinh K_1 h - \cosh K_1 h \right] \left( \frac{M}{A} \right)^{\frac{1}{2}} \]
\[ \exp \left\{ K_1(i|x| - y) \right\} + \pi i \left[ \frac{1}{k} \left\{ s_2(k_0 + k) \right\} \sinh k_0 h - \cosh k_0 h \right] \]

\[ \therefore \left( \frac{W}{\Delta'} \right)_{k = k_0} \exp \left\{ k_0(i|x| - y) \right\} \]

where \( \Delta' = \frac{d}{dk} \Delta \).

Similarly we can obtain as \( |x| \to \infty \),

\[ \varphi_2 \to \pi i \left[ \frac{1}{k} \left\{ s_2(K_1 + K) - K_1 \right\} \cosh K_1(h + y) + \sinh K_1(h + y) \right] \]

\[ \left( \frac{W}{\Delta'} \right)_{k = K_1} \exp(iK|x|) + \pi i \left[ \frac{1}{k} \left\{ s_2(k_0 + K) - k_0 \right\} \cosh k_0(h + y) + \sinh k_0(h + y) \right] \]

\[ \left( \frac{W}{\Delta'} \right)_{k = k_0} \exp(ik_0|x|) \]

\[ \varphi_3 \to \pi i \left( \frac{W}{\Delta'} \right)_{k = K_1} \exp \left\{ K_1(h + y + i|x|) \right\} + \pi i \left( \frac{W}{\Delta'} \right)_{k = k_0} \exp \left\{ k_0(h + y + i|x|) \right\} \]

Thus \( \varphi_1, \varphi_2, \varphi_3 \) satisfy the radiation condition as \( |x| \to \infty \).
Putting $s_2 = 0$, the far field behaviour of $\varphi_1$ and $\varphi_2$ agrees with results obtained in section 1.1.

4. Line singularity submerged in lower fluid

Let there be a logarithmic type singularity at the point $(0, \eta)$, then

$$\varphi_1 \rightarrow \log R_0 \text{ as } R_0 \rightarrow 0 . \quad (4.1)$$

Proceeding similarly as in sub-section 3 above we can obtain

$$\varphi_1 = \log R_0 - \mu \log R_0' + \frac{4s_1}{(1+s_1)^2} \sum_{j=0}^{\infty} (-1)^j \nu^j \mu^{j-1} \log R_j'$$

$$- \int_0^\infty \frac{F_1}{K} \sinh kh \exp(-ky) \cos kx \, dk$$

$$+ \int_0^\infty \left[ \frac{1}{K} \left\{ k \cosh kh - \sinh kh \right\} \right] \frac{W_1}{\Delta} \exp(-ky) \cos kx \, dk$$

$$\varphi_2 = \frac{2}{1+s_1} \left[ \log R_0 + \sum_{j=0}^{\infty} (-1)^j \mu^j \nu^j \log R_{-j}' + \sum_{j=0}^{\infty} (-1)^j \mu^{j-1} \nu^j \log R_{-j}' \right] +$$
\[ + \frac{F_1}{K} \int_0^\infty \cosh k(h+y) \cos kx \, dk \]

\[ + \int_0^\infty \left[ \frac{1}{K} \left( s_2(k+k) - k \right) \cosh k(h+y) + \sinh k(h+y) \right] \frac{W_1}{\Delta} \cos kx \, dk \]

\[ \phi_3 = \frac{4}{(1+s_1)(1+s_2)} \sum_{j=1}^\infty (-1)^j \mu_j^4 \log R_j + \int_0^\infty \frac{W_1}{\Delta} \exp{k(h+y)} \cos kx \, dk \]

\[ \phi_3 = \frac{4}{(1+s_1)(1+s_2)} \sum_{j=1}^\infty (-1)^j \mu_j^4 \log R_j + \int_0^\infty \frac{W_1}{\Delta} \exp{k(h+y)} \cos kx \, dk \]

(4.4)

where \( \Delta \) is given by (3.20), and

\[ W_1 = E_1 - \frac{F_1}{K} \left\{ (K-k+s_1) \sinh kh + s_1 K \cosh kh \right\} \]

\[ E_1 = -2\mu \exp(-k) \left[ 1 + \frac{2s_1 \nu \exp(-2kh)}{(1+s_1)(1+\nu \exp(-2kh))} \right] \]

(4.5)

\[ F_1 = \frac{4\nu \exp{-k(h+n)}}{(1+s_1)(1+\mu \nu \exp(-2kh))} \]

If we now put \( \rho_1 = \rho_2 = \rho_3 \) so that \( \mu = \nu = 0 \) in (4.3),
(4.4), (4.5) then we obtain \( \varphi_1 = \varphi_2 = \varphi_3 = \log R_0 \) which is the potential function for a line source at \( (0, \eta) \) in an infinite fluid.

As \( |x| \to \infty \) we can show that

\[
\varphi_1 \sim \pi i \left[ \frac{1}{k} \left\{ K_1 - s_2(K_1 + k) \right\} \sinh K_1 h - \cosh K_1 h \right] \left( \frac{W_1}{\Delta'} \right)_{k = K_1}
\]

\[
\times \exp \left\{ K (|x| - y) \right\} + \pi i \left[ \frac{1}{k} \left\{ k_0 - s_2(k_0 + k) \right\} \sinh k_0 h - \cosh k_0 h \right]
\]

\[
\times \left( \frac{W_1}{\Delta'} \right)_{k = k_0} \exp \left\{ k_0 (|x| - y) \right\}
\]

\[
\varphi_2 \sim \pi i \left[ \frac{1}{k} \left\{ s_2(K_1 + k) - K_1 \right\} \cosh K_1 (h + y) + \sinh K_1 (h + y) \right] \left( \frac{W_1}{\Delta'} \right)_{k = K_1}
\]

\[
\times \exp (iK_1 |x|) + \pi i \left[ \frac{1}{k} \left\{ s_2(k_0 + K) - k_0 \right\} \cosh k_0 (h + y) + \sinh k_0 (h + y) \right]
\]

\[
\times \left( \frac{W_1}{\Delta'} \right)_{k = k_0} \exp (iK_0 |x|)
\]

\[
\varphi_3 \sim \pi i \left( \frac{W_1}{\Delta'} \right)_{k = K_1} \exp \left\{ K_1 (h + y + i|x|) \right\} + \pi i \left( \frac{W_1}{\Delta'} \right)_{k = k_0} \exp \left\{ k_0 (h + y + i|x|) \right\}
\]
where $\Delta' = \frac{d}{dk} \Delta$ \being given by (3.20).

5. Line singularity submerged in upper fluid

In this case the singularity is situated at the point $(0, -2h + \eta)$, say, so that

$$\varphi_3 \sim \log R_1 \text{ as } R_1 \to 0 \quad (5.1)$$

Proceeding as in sub section 3 it can be shown that

$$\varphi_1 = \frac{4s_1s_2}{(1+s_1)(1+s_2)} \sum \frac{(-1)^j \mu^j}{j} \log R_j$$

$$- \int_0^\infty \frac{F_2}{k} \sinh kh \exp(-ky) \cos kx \, dk$$

$$+ \int_0^\infty \left[ \frac{1}{k} \{k-s(k+k) \} \sinh kh-cosh kh \right] \frac{W_2}{\Delta} \exp(-ky) \cos kx \, dk$$

$$\varphi_2 = \frac{2s_2}{1+s_2} \sum \frac{(-1)^j \mu^j}{j} \log R_j + \frac{2s_2}{1+s_2} \sum \frac{(-1)^j j \mu^j}{j} \log R_j$$
\[ + \int_0^\infty \frac{F_2}{K} \cosh (h+y) \cos kx \, dk \]

\[ + \int_0^\infty \left[ \frac{1}{k} s_2 (k-K-k) \cosh (h+y) + \sinh (h+y) \right] \frac{W_2}{\Delta} \cos kx \, dk \quad (5.3) \]

\[ \varphi_3 = \log R_1 + \nu \log R'_0 + \frac{4s_2}{(1+s_2)^2} \sum_{j=1}^\infty (-1)^j \mu^j \nu^{j-1} \log R'_{-j} \]

\[ + \int_0^\infty \frac{W_2}{\Delta} \exp \{k(h+y)\} \cos kx \, dk \]

where

\[ W_2 = E_2 - \frac{F_2}{K} \left\{ (K-k) \sinh kh + s_1 (K \cosh kh + k \sinh kh) \right\} \]

\[ E_2 = \frac{4s_1 s_2 \mu \exp \{k(\eta-2h)\}}{(1+s_2)(1+\mu \exp (-2kh))} \]

\[ F_2 = \frac{2s_2}{1+s_2} \exp \{-k(h-\eta)\} \left[ s_2 + \frac{\mu \exp (-2kh) - 1}{1 + \mu \exp \{-2kh\}} \right] \]
As before by putting \( p_1 = p_2 = p_3 \) it is easily verified that \( \varphi_1 = \varphi_2 = \varphi_3 = \log R_1 \) which is the potential function in an infinite fluid due to a line singularity at the point \( (0, -2\eta) \).

The behaviours of \( \varphi_1, \varphi_2, \) and \( \varphi_3 \) as \( |x| \to \infty \) can be shown as the outgoing waves

\[
\varphi_1 \sim \pi i \left[ \frac{1}{K} \left\{ K_1 - s_2(K_1 + K) \right\} \sinh K_1 h \right.
\]
\[
- \cosh K_1 h \exp(-K_1 y) \left( \frac{W_2}{\Delta'} \right)_{k=K_1} \exp(iK_1 |x|) + \left. \pi i \left[ \frac{1}{K} \left\{ k_0 - s_2(k_0 + K) \right\} \sinh K_0 h \right. \right.
\]
\[
- \cosh K_0 h \exp(-K_0 y) \left( \frac{W_2}{\Delta'} \right)_{k=k_0} \exp(iK_0 |x|) \right]
\]

\[
\varphi_2 \sim \pi i \left[ \frac{s_2(K_1 + K) - K_1}{K} \cosh K_1 (h+y) + \sinh K_1 (h+y) \right] \left( \frac{W_2}{\Delta'} \right)_{k=K_1} \exp(iK_1 |x|) + \right]
\]
We consider only point singularities for which the y-axis is an axis of symmetry, so that \( \phi_1, \phi_2, \phi_3 \) are independent of the azimuthal angle, and satisfy the same set of equations of sub-section 2.

In the present case let there be a point source at \((0,-\eta)\), then

\[
\phi \sim \frac{P_n(\cos \theta)}{R_0^{n+1}} \quad \text{as} \quad R_0' = \left\{r^2(y+\eta)^2\right\}^{1/2} \to 0, \quad n = 0,1,2,\ldots \tag{6.1}
\]

where \( r \) is the distance from the y-axis and \( \theta = \tan^{-1}\left(\frac{r}{y+\eta}\right) \).

Let us assume
\[ \phi_1 = \int_0^\infty A(k) \exp(-ky) J_\nu(kr) \, dk \]  

(6.2)

\[ \phi_2 = \frac{P_n(\cos \theta)}{R^n+1} + \int_0^\infty \left\{ B(k) \cosh k(n+y) + C(k) \sinh k(h+y) \right\} J_\nu(kr) \, dk \]  

(6.3)

\[ \phi_3 = \int_0^\infty D(k) \exp(ky) J_\nu(kr) \, dk \]  

(6.4)

The following integral representation is necessary,

\[ \frac{P_n(\cos \theta)}{R^n+1} = \frac{1}{n!} \int_0^\infty k^n \exp\{-k(y+n)\} J_\nu(kr) \, dk, \quad y > -n \]  

(6.5)

\[ = \frac{(-1)^n}{n!} \int_0^\infty k^n \exp\{k(y+n)\} J_\nu(kr) \, dk, \quad y < -n . \]

Using this integral representation and proceeding somewhat similar to sub-section 3, we can obtain
\[ \varphi_1 = \sum_{n=0}^{\infty} \frac{k^n}{n!} \left[ \exp(-kn) + (-1)^n \left(1 - s_2\right) \frac{1+K}{K} \sinh kh \exp\{-k(h-n)\} \right] \]

\[ = \exp(-ky) \int J_0(kr) \, dk \]

\[ + \sum_{n=0}^{\infty} \frac{k^n}{n!} \left(1 + s_2\right) \frac{1+K}{K} \sinh kh \exp(-ky) \int \frac{J_n(kr)}{\Delta} \, dk \]

\[ \varphi_2 = \frac{P_n(\cos \theta)}{R^{\prime n+1}_0} + (-1)^n \frac{s^2-1}{n!} \frac{k^n}{K} \int_0^\infty (k+K)^{-k} \exp\{-k(h-n)\} \]

\[ \times \cosh k(h+y) \int J_0(kr) \, dk \]

\[ + \sum_{n=0}^{\infty} \frac{k^n}{n!} \left(1 + s_2\right) \frac{1+K}{K} \sinh kh \exp(-ky) \int \frac{J_n(kr)}{\Delta} \, dk \]

\[ \varphi_3 = \frac{(-1)^n}{n!} \sum_{n=0}^{\infty} \frac{k^n}{\Delta} \exp\{k(n+y)\} \int J_0(kr) \, dk + \frac{\varphi}{\Delta} \sum_{n=0}^{\infty} \frac{k^n}{\Delta} \exp\{k(n+y)\} \int J_0(kr) \, dk \]

where \( \Delta \) is given by (3.20) and
\[ V = \frac{(1-s_1)}{n!} (k-K) k^n \exp(-kn) + \frac{(1-s_2)}{K} \frac{(-1)^n}{n!} k^n (k+K) \xi (K-k+s_1 k) \sinh kh \]
\[ + s_1 k \cosh kh \exp \{ -k(h-n) \} \]

By substituting \( \rho_1 = \rho_2 = \rho_3 \) it is verified that

\[ \varphi_1 = \varphi_2 = \varphi_3 = \frac{P_n(\cos \theta)}{R^{'n+1}} \]

which is obviously the potential in an infinite fluid due to a point source at \((0,-\eta)\).

Now putting \( 2j_0(kr) = H_0^{(1)}(kr) + H_0^{(2)}(kr) \) and rotating the contour in the integrals involving \( H_0^{(1)}(kr) \) in the first quadrant and in the integrals involving \( H_0^{(2)}(kr) \) in the fourth quadrant, we can reduce the integrals into suitable forms from which the far field behaviours of \( \varphi_-, \varphi_2, \varphi_3 \) as \( r \to \infty \) have the following forms

\[
\varphi_1 \left[ \pi i \left( \frac{1}{k} \{ K_1 - s_2 (K_1 + K) \} \sinh K_1 h - \cosh K_1 h \right) \left( \frac{V}{\Delta} \right)_{k=K_1} \exp(-k_1 y) H_0^{(1)}(K_1 r) \right]
\]
\[
+ \pi i \left[ \frac{1}{k} \{ K_0 - s_2 (K_0 + K) \} \sinh K_0 h - \cosh K_0 h \right] \left( \frac{V}{\Delta} \right)_{k=K_0} \exp(-k_0 y) H_0^{(1)}(K_0 r) \]

\[ \omega_2 \sim \pi i \left[ \frac{1}{K} \{ s_2(K_1+K) - K_1 \} \cosh K_1(h+y) + \sinh K_1(h+y) \right] \frac{V}{\Delta} k = K_1 H_0 (K_1 r) \]

\[ + \pi i \left[ \frac{1}{K} \{ s_2(K_0+K) - K_0 \} \cosh K_0(h+y) + \sinh K_0(h+y) \right] \frac{V}{\Delta} k = K_0 H_0 (K_0 r) \]

\[ \omega_3 \sim \pi i \exp \left\{ K_1(h+y) \right\} \frac{V}{\Delta} k = K_1 H_0 (K_1 r) + \pi i \exp \left\{ K_0(h+y) \right\} \frac{V}{\Delta} k = K_0 H_0 (K_0 r). \]

7. **Multipoles submerged in the lower fluid**

In this case

\[ \varphi_1 \sim \frac{P_n(\cos \psi)}{R_0^{n+1}} \quad \text{as} \quad R_0 = \left\{ \frac{r^2 + (y-\eta)^2}{2} \right\}^{1/2} \to 0, \quad n = 0, 1, 2, \]

where \( \psi = \tan^{-1} (r/y-\eta). \) It can be shown that

\[ \varphi_1 = \frac{P_n(\cos \psi)}{R_0^{n+1}} + \frac{(-1)^n}{n!} \int_0^\infty k^n \exp \left\{ -k(y+\eta) \right\} J_0(kr) \, dk \]

\[ - \frac{1}{K} \int_0^\infty \left[ s_2(k-K) \sinh kh + \cosh kh \right] \frac{V}{\Delta} \exp(-ky) J_0(kr) \, dk \]
\[ \varphi_2 = \frac{\infty}{\int_0^\infty} \left[ \frac{1}{k} \left\{ s_2(k+K) - k \right\} \cosh k(h+y) + \sinh k(h+y) \right] \frac{V_1}{\Delta} j_0(kr) \, dk \]

\[ \varphi_3 = \frac{\infty}{\int_0^\infty} \frac{V_1}{\Delta} \exp \{ k(h+y) \} j_0(kr) \, dk \]

where \[ V_1 = 2k \frac{(-1)^n}{n!} k^n \exp(-kn). \] As \( r \to \infty \),

\[ \varphi_1 \sim -n_1 \frac{1}{k} \left[ \left\{ s_2(K_1+K) - K_1 \right\} \sinh K_1 h + \cosh K_1 h \right] \frac{V_1}{\Delta} k = K_1 \exp(-Ky) H_0^{(1)}(K_1r) \]

\[ \varphi_2 \sim -n_1 \frac{1}{k} \left[ \left\{ s_2(k_0+K) - k_0 \right\} \sinh k_0 h + \cosh k_0 h \right] \frac{V_1}{\Delta} k = k_0 \exp(-k_0 y) H_0^{(1)}(k_0r) \]

\[ \varphi_3 \sim n_1 \left( \frac{V_1}{\Delta} \right) k = K_1 \exp\{K_1(h+y)\} H_0^{(1)}(K_1r) \right. + n_1 \left( \frac{V_1}{\Delta} \right) k = k_0 \exp\{k_0(h+y)\} H_0^{(1)}(k_0r) \]

2. **Multipoles submerged in the upper fluid**

In this case \( \varphi_3 \sim \frac{P_n(cosx)}{R_1^{n+1}} \) as \( R_1 = \{ r^2 + (2h+y-n)^2 \}^{1/2} \to 0, \; n = 0,1,2... \)
where \( X = \tan^{-1} \frac{r}{(2h+y-n)} \). The velocity potentials are given by

\[
\varphi_1 = -\frac{2s_2}{n!} \int_0^\infty \frac{k^n}{k} \sinh kh \exp\{-k(h+y-n)\} J_0(kr) \, dk
\]

\[
+ \frac{\gamma}{\Delta} \left[ \frac{1}{k} \left\{ k - s_2(k+1) \right\} \sinh kh - \cosh kh \right] V_2 \exp(-ky) J_0(kr) \, dk
\]

\[
\varphi_2 = \frac{2s_2}{n!} \int_0^\infty k^n \exp\{-k(h-y)\} \cosh k(h+y) J_0(kr) \, dk
\]

\[
+ \frac{\gamma}{\Delta} \left[ \frac{1}{k} \left\{ k - s_2(k+1) \right\} \cosh k(h+y) + \sinh k(h+y) \right] V_2 J_0(kr) \, dk
\]

\[
\varphi_3 = \frac{P_n(\cos x)}{R_{n+1}} + \int_0^\infty \frac{k^n}{n!} \exp\{k(y+\eta)\} J_0(kr) \, dk
\]

\[
+ \frac{\gamma}{\Delta} V_2 \exp\{k(h+y)\} J_0(kr) \, dk
\]
where \( V_2 = -2s_2 \frac{k^n}{n!} \exp \left\{ -k(h-\eta) \right\} \left\{ (K-k+s_1 k) \sinh kh + s_1 k \cosh kh \right\} \).

As \( r \to \infty \) we can show that

\[
\varphi_1 \sim n! \left[ \frac{1}{K} \left\{ s_2 (K_1 + K) \right\} \sinh K_1 h - \cosh K_1 h \right]
\]

\[
\times \left( \frac{V_2}{\Lambda'} \right)_{k=K_1} \exp \left\{ -K_1 y \right\} H_0^{(1)} (K_1 r)
\]

+ a similar expression with \( K_1 \) replaced by \( k_0 \),

\[
\varphi_2 \sim n! \left[ \frac{1}{K} \left\{ s_2 (K_1 + K) - K_1 \right\} \cosh K_1 (h+y) + \sinh K_1 (h+y) \right]
\]

\[
\times \left( \frac{V_2}{\Lambda'} \right)_{k=K_1} H_0^{(1)} (K_1 r)
\]

+ a similar expression with \( K_1 \) replaced by \( k_0 \),

\[
\varphi_3 \sim n! \left( \frac{V_2}{\Lambda'} \right)_{k=K_1} \exp K_1 (h+y) \cdot H_0^{(1)} (K_1 r)
\]

+ a similar expression with \( K_1 \) replaced by \( k_0 \).
9. **Conclusion**

Integral representations of the potential functions in the different fluids of a three-layered fluid medium are obtained. When the upper medium is taken to be vacuo earlier results for the case of two-fluid medium are recovered (cf section 1.1). Again, when the three-fluid medium is reduced to an infinite one-fluid medium by making the densities equal, the corresponding results for the infinite one-fluid medium are readily recovered. Also the extension of the problem to the case where the lower fluid is of finite depth $H$, say, instead of infinity is not difficult, although the final result will be more complicated.

It may be noted that in the construction of the line source potentials in the present paper by the image method, an infinite set of image sources due to the two surfaces of separation has been introduced. Usefulness of this image method can be demonstrated as follows.

In the simple case of a line source at $(0,\eta)$ in a single layer of finite constant depth, there exists an infinite set of image sources due to the free surface and the bottom. Without using the whole set of images, Thorne (1953) used only the image source due to the free surface and constructed the potential function $\phi$ as
\[
\varphi = \log \frac{R_0}{R'_0} + 2 \sum_{j=1}^{\infty} (-1)^j \left( \log \frac{R_j}{R'_j} + \log \frac{R_{-j}}{R'_{-j}} \right) \\
+ 2 \int_0^\infty \left\{ \frac{\sinh k(h-n) \cosh k(h-y)}{k \cosh kh - k \sinh kh} \right\} \cos kx \, dk \tag{9.1}
\]

where \( R_0 \) is the distance from the source and \( R'_0 \) is the distance from the image source. However, if we introduce all the image sources then we obtain

\[
\varphi = \log \frac{R_0}{R'_0} + \sum_{j=1}^{\infty} (-1)^j \left( \log \frac{R_j}{R'_j} + \log \frac{R_{-j}}{R'_{-j}} \right) \\
+ 2 \int_0^\infty \left\{ \frac{\sinh k(h-n) \cosh k(h-y)}{k \cosh kh - k \sinh kh} \right\} \frac{\cos kx}{\cosh kh} \, dk , \tag{9.2}
\]

By using the representation

\[
\log \frac{x^2 + \alpha^2}{x^2 + \beta^2} = 2 \int_0^\infty \frac{1}{k} \left\{ \exp(-sk) - \exp(-\alpha k) \right\} \cos kx \, dk ,
\]

it can be shown that (9.2) reduces to (9.1). Thus the sum of
the image potentials (excepting the image at \((0,-\eta)\)) in (9.2) can be expressed as an integral and can be combined with the integral in (9.2) to give the integral in (9.1).

This naturally will motivate one to construct the potential functions in a layered medium by a similar technique used by Thorne (1953). However, this will lead to the appearance of some divergent integrals in the resulting expressions of the potential functions. To demonstrate this, we now take a simple case where we consider the construction of potentials in two superposed infinite fluids with a line source present in the lower fluid at \((0,\eta)\). By using the image method (there is only one image due to the surface of separation) Gorgui and Kassem (1978) obtained the following result

\[
\begin{align*}
\varphi_1 &= \log R_0 - \frac{1-s}{1+s} \log R'_0 - \frac{2(1-s)}{1+s} \int_0^\infty \frac{\exp \left\{ -k(y+\eta) \right\}}{\Delta} \cos kx \, dk, \\
& \quad y > 0, \\
\varphi_2 &= \frac{2}{1+s} \log R_0 + \frac{2(1-s)}{1+s} \int_0^\infty \frac{\exp \left\{ k(y-\eta) \right\}}{\Delta} \cos kx \, dk, \quad y < 0 ,
\end{align*}
\]

\[(9.3)\]

where \(\Delta = (1-s)k - (1+s)K\), \(s\) being the ratio of the densities of the upper and lower fluids respectively. One may note that
the integrals in (9.3) are convergent but \( \phi_1 \)'s become unbounded at infinity although \( \text{grad } \phi_1 \)'s remain bounded. We can also construct \( \phi_1, \phi_2 \) by the method used in (1) as

\[
\phi_1 = \log \frac{R_0}{R} + \int_0^\infty X \exp(-ky) \cos kx \, dk, \quad y > 0
\]

\[
\phi_2 = \int_0^\infty Y \exp(ky) \cos kx \, dk, \quad y < 0;
\]

where \( X, Y \) can be obtained from the two SS conditions. The resulting expressions for \( \phi_1, \phi_2 \) are

\[
\phi_1 = \log \frac{R_0}{R} + 2 \int_0^\infty \left\{ \frac{s(k+1)}{k} - 1 \right\} \frac{\exp(-k(y+\eta))}{\Delta} \cos kx \, dk, y > 0,
\]

\[
\phi_2 = -2 \int_0^\infty \frac{\exp[k(y-\eta)]}{k} \cos kx \, dk - 2 \int_0^\infty \left\{ \frac{s(k+1)}{k} - 1 \right\} \exp[k(y-\eta)] \frac{\cos kx \, dk, y < 0.}{\Delta}
\]

(9.4)
It is obvious that the integrals in (9.4) are divergent as the integrands have a pole at \( k = 0 \). However, the expressions in (9.4) can be identified with those in (9.3) if one is willing to replace the divergent integrals

\[
\int_{0}^{\infty} \frac{\exp\{-k(y+\eta)\}}{k} \cos kx \, dk
\]

and

\[
\int_{0}^{\infty} \frac{\exp\{-k(y-\eta)\}}{k} \cos kx \, dk
\]

appearing in (9.4) by the unbounded functions \( \log R' \) and \( \log R_0 \) respectively. In fact, the appearance of divergent integrals in (9.3) is not unexpected and this reflects the unbounded nature of the potential functions. We may point out here that in a single layer fluid, this unbounded nature of the potentials does not exist.

Thus to avoid the appearance of divergent integrals in the potentials, the image method used here seems to be convenient.