In this section, the governing equations of classical elasticity, classical thermoelasticity, classical coupled thermoelasticity, generalized thermoelasticity and magneto-thermoelasticity are presented.

Let $V$ and $S$ be respectively the volume and surface of the body under consideration and $T_0$ be the reference temperature (measured in terms of absolute unit) in its undeformed state. Let $x_i$ ($i=1,2,3$) denote the co-ordinates of any of its points referred to a fixed rectangular Cartesian co-ordinate system.

### 2.1 Classical Elasticity

**Strain-displacement relations**

\[ e_{ij} = \frac{1}{2}(u_{ij} + u_{ji}), \quad i, j = 1, 2, 3. \]  \hspace{1cm} (2.1.1)

where $e_{ij}$ is the strain tensor and $u_i$ is the displacement component.

**Strain compatibility equations**

\[ e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0, \quad i, j, k, l = 1, 2, 3. \]  \hspace{1cm} (2.1.2)
Basic Equations

Stress-strain relations

\[
\sigma_{ij} = C_{ijkl}e_{kl}, \quad i, j, k, l = 1, 2, 3.
\]

or

\[
e_{ij} = S_{ijkl}\sigma_{kl}, \quad i, j, k, l = 1, 2, 3.
\]

where

\[
C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}
\]

\[
S_{ijkl} = S_{jikl} = S_{ijlk} = S_{klij}
\]

\[
S_{ijkl} = (C_{ijkl})^{-1}
\]

and \(\sigma_{ij}, C_{ijkl}, S_{ijkl}\) are stress tensor, elasticity tensor and elastic compliance tensor respectively.

Equations of motion

\[
\sigma_{ij,j} + \rho F_i = \rho \ddot{u}_i, \quad i, j = 1, 2, 3.
\]

where \(F_i\) is the component of external body force per unit mass and \(\rho > 0\) is the mass density.

2.2 Classical Thermoelasticity (CTE)

Stress-strain temperature relations

\[
\sigma_{ij} = C_{ijkl}e_{kl} - \beta_{ij}(T - T_0), \quad i, j, k, l = 1, 2, 3.
\]

or

\[
e_{ij} = S_{ijkl}\sigma_{kl} + \alpha_{ij}(T - T_0), \quad i, j, k, l = 1, 2, 3.
\]

where \(\beta_{ij} = C_{ijkl}\alpha_{kl}\), \(\alpha_{ij} = S_{ijkl}\beta_{kl}\) and \(\beta_{ij}, \alpha_{ij}\) are thermal moduli, thermal expansion tensor respectively, and \(T\) is the thermodynamic temperature.
Basic Equations

Fourier law of heat conduction

\[ q_i = -K_{ij}T_{j,i}, \quad i, j = 1, 2, 3, \]  \hfill (2.2.3)

where \( q_i \) is the component of the heat flux vector \( q \) and \( K_{ij} \) is the thermal conductivity tensor.

**Energy equation**

\[ -q_{i,i} + \rho Q = \rho c_v \dot{T}, \quad i = 1, 2, 3, \]  \hfill (2.2.4)

where \( Q \) is the heat source acting per unit mass per second and \( c_v \) is the specific heat at constant strain.

**Heat equation**

Eqs. (2.2.3) and (2.2.4) together give the parabolic type heat transport equation

\[ (K_{ij}T_{j,i})_{,i} + \rho Q = \rho c_v \dot{T}, \quad i, j = 1, 2, 3. \]  \hfill (2.2.5)

Eqs. (2.1.5), (2.2.1) and (2.2.5) constitute the complete mathematical model of the theory of classical thermoelasticity (CTE).

### 2.3 Classical Coupled Thermoelasticity (CCTE)

**Energy equation**

\[ -q_{i,i} + \rho Q = \rho c_v \dot{T} + T_0 \beta_{ij} \dot{e}_{ij}, \quad i, j = 1, 2, 3, \]  \hfill (2.3.1)

where the term \( T_0 \beta_{ij} \dot{e}_{ij} \) brings about coupling between temperature and strain field.

**Heat equation**

Eqs. (2.2.3) and (2.3.1), on elimination of \( q_i \), lead to the parabolic type heat transport
Eqs. (2.1.5), (2.2.1) and (2.3.2) together constitute the complete mathematical model of the theory of classical coupled thermoelasticity (CCTE).

2.4 Lord-Shulman Model (ETE)

Modified Fourier law of heat conduction

\[ q_i + \tau_0 \dot{q}_i = -K_{ij}T_j, \quad i, j = 1, 2, 3, \quad (2.4.1) \]

Heat equation

Elimination of \( q_i \) from Eqs. (2.3.1) and (2.4.1) results in generalized heat transport equation

\[ (K_{ij}T_j)_i + \rho Q = \rho c_v \dot{T} + T_0 \beta_{ij} \dot{\varepsilon}_{ij}, \quad i, j = 1, 2, 3. \quad (2.4.2) \]

This model gives hyperbolic type heat transport equation with finite speed of thermal wave. This theory is also known as extended thermoelasticity (ETE). As \( \tau_0 \to 0 \) Lord-Shulman model (ETE) reduces to classical coupled thermoelasticity (CCTE). \( \tau_0 \) is called relaxation time, which is the time required to maintain steady state heat conduction in an element of volume of an elastic body when a sudden temperature gradient is imposed on that volume element.

Eq. (2.4.2) together with Eqs. (2.1.5), (2.2.1) constitute the field equations of Lord-Shulman model.
2.5 Green-Lindsay Model (TRDTE)

Stress-strain temperature relations

\[ \sigma_{ij} = C_{ijkl} e_{kl} - \beta_{ij} [(T - T_0) + t_1 \dot{T}], \quad i, j = 1, 2, 3. \]  \hspace{1cm} (2.5.1)

Fourier law of heat conduction

\[ q_i = -(C_i \dot{T} + K_{ij} T_j), \quad i, j = 1, 2, 3, \] \hspace{1cm} (2.5.2)

with \( K_{ij} = K_{ji} \). \[81\].

Energy equation

\[-q_{i,i} + \rho Q = \rho c_v (\dot{T} + t_2 \ddot{T}) - C_i \dot{T}_i + T_0 \beta_{ij} \dot{e}_{ij}, \quad i, j = 1, 2, 3. \] \hspace{1cm} (2.5.3)

where \( t_1, t_2, C_i \) are new material constants for GL model.

Heat equation

Elimination of \( q_i \) from (2.5.2) and (2.5.3) gives rise to the generalized heat transport equation

\[ (K_{ij} T_j) ,i + \rho Q = \rho c_v (\dot{T} + t_2 \ddot{T}) - 2C_i \dot{T}_i + T_0 \beta_{ij} \dot{e}_{ij}, \quad i, j = 1, 2, 3. \] \hspace{1cm} (2.5.4)

Here \( t_1 \) and \( t_2 \) (\( t_1 \geq t_2 \geq 0 \)) are two constitutive constants having the dimension of time. This theory gives hyperbolic type heat transport equation with finite speed of thermal wave. This theory is often referred to as temperature-rate dependent thermoelasticity (TRDTE). For a material having a center of symmetry at every point Eq. (2.5.2) reduces to classical Fourier's law. Accordingly, GL theory admits the second sound without violating the classical Fourier's law [81].

The system of Eqs. (2.1.5), (2.5.1) and (2.5.4) constitute the complete mathematical
2.6 Green-Naghdi Model II (Thermoelasticity of type II)

Law of heat conduction

\[ q_i = -K_{ij}^\nu \nu_j \text{ where } \nu = T, \ i, j = 1, 2, 3. \]  

(2.6.1)

Here \( \nu \) is the thermal displacement and \( K_{ij}^\nu \) is the tensor of additional material constants associated with the GN theory.

Heat equation

Eqs. (2.3.1) and (2.6.1) together give the generalized heat transport equation

\[ (K_{ij}^\nu \nu_j)_j + \rho \dot{Q} = \rho c_v \dot{T} + T_0 \beta_{ij} \dot{\epsilon}_{ij}, \ i, j = 1, 2, 3. \]  

(2.6.2)

Eqs. (2.1.5), (2.2.1) and (2.6.2) constitute complete mathematical model of the Green-Naghdi model II.

Reconsideration of Eqs. (2.2.1) and (2.6.2) for the GN theory type II reveals that no damping term is appeared in the system of equations and therefore the GN theory type II is known as the thermoelasticity without energy dissipation.

2.7 Green-Naghdi Model III (Thermoelasticity of type III)

Law of heat conduction

\[ q_i = -(K_{ij} T_j + K_{ij}^\nu \nu_j) \text{ where } \nu = T, \ i, j = 1, 2, 3. \]  

(2.7.1)
Heat equation

Eqs. (2.3.1) and (2.7.1) together give the generalized heat transport equation

\[(K_{ij}T_j)_j + (K^*_{ij}T_j)_j + \rho \dot{Q} = \rho c_v \dot{T} + T_0 \beta_{ij} \ddot{e}_{ij}, \quad i, j = 1, 2, 3. \] (2.7.2)

Eqs. (2.1.5), (2.2.1) and (2.7.2) constitute complete mathematical model of the Green-Naghdi model III. This theory is also known as \textit{thermoelasticity with energy dissipation}.

Two special cases of the GN theory, namely type II and I, may be obtained from the equations of the GN theory type III by setting \(K_{ij} \to 0\) and \(K^*_{ij} \to 0\) respectively, where 0 is the zero tensor. To obtain the GN theory type II, from the equations of the GN theory type III, we set \(K^*_{ij} \to 0\). When \(K^*_{ij} \to 0\) the equations of the GN theory type III reduce to the GN theory type I, which is identical with the classical theory of thermoelasticity.

2.8 Three-phase-lag Model

Law of heat conduction

\[q_i + \tau_q \dot{q}_i + \frac{\tau^2}{2} \ddot{q}_i = -(\tau^*_q)_{ij}T_j + \tau_T K_{ij} \dot{T}_j + K^*_{ij} \nu_j, \quad i, j = 1, 2, 3, \] (2.8.1)

where \(\dot{v} = T\) and \((\tau^*_q)_{ij} = K_{ij} + \tau_v K^*_{ij}\). The delay time \(\tau_T\) is called the phase-lag of the temperature gradient and the other delay time \(\tau_q\), the phase-lag of the heat flux. The delay time \(\tau_T\) is caused by the microstructural interactions (small scale effects of heat transport in space such as phonon-electron interaction or phonon scattering). The second delay time \(\tau_q\) is caused due to the fast-transient effects of thermal inertia (or small scale effects of heat transport in time). The phase lags \(\tau_q\) and \(\tau_T\) are small, positive and assumed to be intrinsic properties of the medium. The third delay time \(\tau_v\) may be
interpreted, following Tzou[221], as the phase-lag of the thermal displacement gradient.

**Heat equation**

Elimination of \( q_i \) from (2.3.1) and (2.8.1) lead to the following hyperbolic type heat transport equation

\[
((\tau^*_i)_{ij} \dot{T}_{ij})_t + \tau_T (K_{ij} \dot{T}_{ij})_t + (K^*_{ij} T_{ij})_t = (1 + \tau_q \frac{\partial}{\partial t} + \frac{1}{2} \tau_\nu \frac{\partial^2}{\partial t^2}) \left( \rho \epsilon_{ij} \dot{T} + T_0 B_{ij} \ddot{e}_{ij} - \rho \dot{Q} \right).
\]  

(2.8.2)

Eqs. (2.1.5), (2.2.1) along with (2.8.2) constitutes complete mathematical model of hyperbolic thermoelasticity theory that includes three phase lag effects.

For \( \tau_q = 0, \tau_T = 0 \) and \( \tau_\nu = 0 \), this theory reduces to (for much low thermal conductivity) Green-Naghdi's second model of generalized thermoelasticity without energy dissipation. For \( \tau_q = 0, \tau_T = 0, \tau_\nu = 0 \) and hence \( (\tau^*_i)_{ij} = K_{ij} \), this theory becomes third model of Green-Naghdi. In the case when \( \tau_T = 0 = \tau^*_q, K^*_{ij} = 0, \tau_\nu = 0 \) and hence \( (\tau^*_i)_{ij} = K_{ij} \), this theory clearly reduces to L-S theory.

2.9 Two Temperature Thermoelasticity (2TT)

**Law of heat conduction**

\[
q_i = -K_{ij} \phi_{ij}, i, j = 1, 2, 3
\]

(2.9.1)

where \( \phi \) is the conductive temperature.

**Relation Between Thermodynamic Temperature and Conductive Temperature**

\[
\phi - T = (A_{ij} \phi_{ij})_t,
\]

(2.9.2)

where \( A_{ij} \) is the temperature discrepancy tensor.
Heat equation

Eqs. (2.2.3) and (2.9.1) together give the parabolic type heat transport equation

\[(K_{ij}\phi_{,j})_{,i} + \rho Q = \rho c_v T_{,i}, \quad i, j = 1, 2, 3.\] (2.9.3)

Eqs. (2.1.5), (2.2.1), (2.9.2) and (2.9.3) together constitute the complete mathematical model of the theory of two temperature thermoelasticity (2TT).

2.10 Two Temperature Green-Naghdi Model II (2TG-NII)

Law of heat conduction

\[q_i = -K_{ij}^* \nu_{j}, \text{ where } \nu = \phi, i, j = 1, 2, 3\] (2.10.1)

Heat equation

Eqs. (2.3.1) and (2.10.1) together give the generalized heat transport equation

\[(K_{ij}\phi_{,j})_{,i} + \rho \dot{Q} = \rho c_v \ddot{T} + T_0 \beta_{ij} \epsilon_{ij}, \quad i, j = 1, 2, 3.\] (2.10.2)

Eqs. (2.1.5), (2.2.1), (2.9.2) and (2.10.2) constitute complete mathematical model of two temperature Green-Naghdi model II.

2.11 Two Temperature Green-Naghdi Model III (2TG-NIII)

Law of heat conduction

\[q_i = -(K_{ij}\phi_{,j} + K_{ij}^* \nu_{j}), \text{ where } \nu = \phi, i, j = 1, 2, 3\] (2.11.1)
Heat equation

Eqs. (2.3.1) and (2.11.1) together give the generalized heat transport equation

\[
(K_{ij}\dot{\phi}_j)_i + (K_{ij}^* \dot{\phi}_j)_i + \rho \dot{Q} = \rho c_v \ddot{T} + T_0 \beta_y \ddot{e}_{ij}, \quad i, j = 1, 2, 3.
\]  
(2.11.2)

Eqs. (2.1.5), (2.2.1), (2.9.2) and (2.11.2) constitute complete mathematical model of the two temperature Green-Naghdi model III.

2.12 Two Temperature Three-phase-lag Model (2T3P)

Law of heat conduction

\[
q_i + \tau_q \dot{q}_i + \frac{\tau^2_q}{2} \ddot{q}_i = -((\tau^*_q)_{ij} \dot{\phi}_j + \tau_T K_{ij}^* \dot{\phi}_j + K_{ij}^* \dot{\nu}_j), \quad i, j = 1, 2, 3,
\]  
(2.12.1)

where \( \dot{\nu} = \dot{\phi}_i, (\tau^*_q)_{ij} = K_{ij}^* + \tau_\nu K_{ij}^* \).

Heat equation

Elimination of \( q_i \) from Eqs. (2.3.1) and (2.12.1) leads to the following hyperbolic type heat transport equation

\[
((\tau^*_q)_{ij} \dot{\phi}_j)_i + (K_{ij}^* \dot{\phi}_j)_i = (1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau^2_q}{2} \frac{\partial^2}{\partial t^2}) (\rho c_v \ddot{T} + T_0 \beta_y \ddot{e}_{ij} - \rho \dot{Q})
\]  
(2.12.2)

Eqs. (2.1.5), (2.2.1), (2.9.2) along with (2.12.2) constitute complete mathematical model of two temperature hyperbolic thermoelasticity theory that includes three phase lag effects.

For \( \tau_q = 0, \tau_T = 0 \) and \( \tau_\nu = 0 \), this theory reduces to (for much low thermal conductivity) two temperature Green-Naghdi model II. For \( \tau_q = 0, \tau_T = 0, \tau_\nu = 0 \) and hence \((\tau^*_q)_{ij} = K_{ij}\), this theory becomes two temperature Green-Naghdi model III. In the case when \( \tau_T = \tau^2_q = 0, K_{ij}^* = 0, \tau_\nu = 0 \) and hence \((\tau^*_q)_{ij} = K_{ij}\), this theory clearly reduces to
two temperature Lord-Shulman theory.

2.13 Magneto-Thermoelasticity

Maxwell’s Equations
Maxwell's electro-magnetic equations governing the variations of magnetic and electric fields as

\[
\text{curl } \mathbf{H} = \mathbf{J}, \quad \text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{D} = \rho_e, \quad \mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu_e \mathbf{H} \tag{2.13.1}
\]

where \( \mathbf{H} \) total magnetic field vector at any time, \( \mathbf{J} \) is the induced current density vector, \( \mathbf{B} \) is the magnetic induction vector, \( \mathbf{E} \) is the electric current field vector, \( \rho_e \) is the electric charge density, \( \varepsilon \) is the electric permittivity and \( \mu_e \) is the magnetic permeability.

Ohm’s law

generalized Ohm's law in a moving continuum in the form

\[
\mathbf{J} = \kappa \left[ \mathbf{E} + \left( \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} \right) \right] - K_0 \nabla T + \rho_e \mathbf{u} \tag{2.13.2}
\]

where \( \kappa \) electric conductivity of the medium, \( K_0 \) coefficient connecting the temperature gradient and the induced current density vector \( \mathbf{J} \).

Displacement equations of motion

\[
\sigma_{ij,j} + \rho_e E_i + (\mathbf{J} \times \mathbf{B})_i + \rho \ddot{u}_i = \mathbf{F}_i, \quad i,j = 1,2,3, \tag{2.13.3}
\]

where \( \sigma_{ij} \) are given by (2.2.1) and the terms \( \rho_e E_i + (\mathbf{J} \times \mathbf{B})_i \) modifying the body force term, represents the Lorentz's ponderomotive force.
2.14 Lord-Shulman Model of Magneto-Thermoelasticity (EMTE)

Generalized Fourier law of heat conduction

\[ q_i + \tau_0 \dot{q}_i = -K_{ij} T_j + \pi_{ij} J_i \quad i, j = 1, 2, 3 \quad (2.14.1) \]

where \( \pi_{ij} \) is the tensor describing the action of current intensity on the heat flux.

Heat equation

Elimination of \( q_i \) from Eqs. (2.3.1) and (2.14.1) results in generalized heat transport equation

\[ (K_{ij} T_j)_i + \rho (Q + \tau_0 \dot{Q}) - \pi_{ij} J_i = \rho c_v (\ddot{T} + \tau_0 \ddot{T}) + T_0 \beta_{ij} (\ddot{\epsilon}_{ij} + \tau_0 \ddot{\epsilon}_{ij}) \quad i, j = 1, 2, 3 \quad (2.14.2) \]

Eqs. (2.2.1), (2.13.3), (2.14.2), (2.13.1) and (2.13.2) form the basic equations of extended magneto-thermoelasticity with thermal relaxation proposed by Lord and Shulman.

2.15 Green-Naghdi Model II of Magneto-Thermoelasticity

Law of heat conduction

\[ q_i = -K_{ij}^* \nu_j \quad \text{where} \quad \dot{\nu} = T, \quad i, j = 1, 2, 3. \quad (2.15.1) \]

Here the effects of current density on the heat flux are neglected.
Heat equation

Eqs. (2.3.1) and (2.15.1) together give the generalized heat transport equation

\[(K_{ij}^*T_j)_i + \rho \dot{Q} = \rho c_v \ddot{T} + T_0 \beta_{ij} \ddot{e}_{ij}, \quad i, j = 1, 2, 3. \quad (2.15.2)\]

Eqs. (2.2.1), (2.13.3), (2.15.2), (2.13.1) and (2.13.2) constitute complete mathematical model of the Green-Naghdi model II of magneto-thermoelasticity.

2.16 Green-Naghdi Model III of Magneto-Thermoelasticity

Law of heat conduction

\[q_i = -(K_{ij}T_j + K_{ij}^*v_j) \quad \text{where} \quad \dot{v} = T, \quad i, j = 1, 2, 3. \quad (2.16.1)\]

Here the effects of current density on the heat flux are neglected.

Heat equation

Eqs. (2.3.1) and (2.16.1) together give the generalized heat transport equation

\[(K_{ij}^*\dot{T}_j)_i + (K_{ij}^*T_j)_i + \rho \dot{Q} = \rho c_v \ddot{T} + T_0 \beta_{ij} \ddot{e}_{ij}, \quad i, j = 1, 2, 3. \quad (2.16.2)\]

Eqs. (2.2.1), (2.13.3), (2.16.2), (2.13.1) and (2.13.2) constitute complete mathematical model of the Green-Naghdi model III of magneto-thermoelasticity.

2.17 Three-phase-lag Model of Magneto-Thermoelasticity

Law of heat conduction

\[q_i + \tau_\nu \dot{q}_i + \tau^2_\nu \ddot{q}_i = -((\tau^\ast_\nu)_jT_j + \nu T K_{ij}^*\dot{T}_j + K_{ij}^*v_j), \quad i, j = 1, 2, 3. \quad (2.17.1)\]
Here the effects of current density on the heat flux are neglected.

**Heat equation**

Elimination of $q_i$ from (2.3.1) and (2.17.1) lead to the following hyperbolic type heat transport equation

$$(\tau_{ij})(T_j)_i + T_T(K_{ij}T_j)_i + = (1 + \tau_\omega \frac{\partial}{\partial t} + \frac{1}{2} \tau_\omega^2 \frac{\partial^2}{\partial t^2})(\rho c_T T + T_0 \beta_0 \varepsilon_{ij} - \rho \dot{Q}).$$

(2.17.2)

Eqs. (2.2.1), (2.13.3), (2.17.2), (2.13.1) and (2.13.2) constitutes complete mathematical model of hyperbolic magneto-thermoelasticity theory that includes three phase lag effects.

For $\tau_q = 0, \tau_T = 0$ and $\tau_\nu = 0$, this theory reduces to (for much low thermal conductivity) Green-Naghdi's second model of magneto-thermoelasticity without energy dissipation. For $\tau_q = 0, \tau_T = 0, \tau_\nu = 0$ and hence $(\tau_{*})_{ij} = K_{ij}$, this theory becomes Green-Naghdi's third model of magneto-thermoelasticity. In the case when $\tau_T = \tau_\omega^2 = 0, K_{ij}^* = 0, \tau_\nu = 0$ and hence $(\tau_{*})_{ij} = K_{ij}$, this theory clearly reduces to Lord-Shulman theory of magneto-thermoelasticity.

The relations given above reduce to corresponding relations of thermoelasticity in absence of the externally applied magnetic field.

**2.18 Two Temperature Magneto-Thermoelasticity**

**Law of heat conduction**

$$q_i = -K_{ij}\phi_{,j} + \pi_{ij}J_i, \quad i, j = 1, 2, 3$$

(2.18.1)

where $\pi_{ij}$ is the tensor describing the action of current intensity on the heat flux.
Basic Equations

Heat equation

Eqs. (2.2.4) and (2.18.1) together give the parabolic type heat transport equation

\[(K_{ij} \phi_j)_i + \rho Q - \pi_{ij} J_i = \rho c_v \dot{T}, i, j = 1, 2, 3. \] (2.18.2)

Eqs. (2.13.1), (2.13.2), (2.2.1), (2.13.3), (2.9.2), and (2.18.2) constitute the complete mathematical model of the theory of two temperature magneto-thermoelasticity.

2.19 Two Temperature Magneto-Thermoelasticity Based on Green-Naghdi Model II

Law of heat conduction

\[q_i = -K_{ij} \nu_j, \text{ where } \nu = \dot{\phi}, i, j = 1, 2, 3 \] (2.19.1)

Here the effects of current density on the heat flux are neglected.

Heat equation

Eqs. (2.3.1) and (2.19.1) together give the generalized heat transport equation

\[(K_{ij} \phi_j)_i + \rho \dot{Q} = \rho c_v \ddot{T} + T_0 \dot{\beta}_{ij} \dot{e}_{ij}, i, j = 1, 2, 3. \] (2.19.2)

Eqs. (2.13.1), (2.13.2), (2.2.1), (2.13.3), (2.9.2), and (2.19.2) constitute complete mathematical model of two temperature magneto-thermoelasticity based on Green-Naghdi model II.
2.20 Two Temperature Magneto-Thermoelasticity Based on Green-Naghdi Model III

Law of heat conduction

\[ q_i = -(K_{ij}\phi_j + K_{ij}^*\nu_j), \text{where } \phi_j = \phi, i, j = 1, 2, 3 \] (2.20.1)

Here the effects of current density on the heat flux are neglected.

Heat equation

Eqs. (2.3.1) and (2.20.1) together give the generalized heat transport equation

\[ (K_{ij}\dot{\phi}_j)_{,i} + (K_{ij}^*\phi_j)_{,i} + \rho\dot{Q} = \rho c_v\ddot{T} + T_0\beta_{ij}\dot{\varepsilon}_{ij}, i, j = 1, 2, 3. \] (2.20.2)

Eqs. (2.13.1), (2.13.2), (2.2.1), (2.13.3), (2.9.2), and (2.20.2) constitute complete mathematical model of two temperature magneto-thermoelasticity based on Green-Naghdi model III.

2.21 Two Temperature Magneto-Thermoelasticity Based on Three-phase-lag Model

Law of heat conduction

\[ q_i + \tau_u\dot{q}_i + \frac{\tau_q}{2}\dddot{q}_i = -((\tau_{\nu}^*)_{ij}\phi_j + \tau_T K_{ij}\dot{\phi}_j + K_{ij}^*\nu_j), i, j = 1, 2, 3, \] (2.21.1)

Here the effects of current density on the heat flux are neglected.

Heat equation

Elimination of \( q_i \) from Eqs. (2.3.1) and (2.21.1) leads to the following hyperbolic type
heat transport equation

\[(\tau^*_{ij})_i + \tau_T (K_{ij})_i + (K^*_{ij})_i = \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{1}{2} \tau_{ij} \frac{\partial^2}{\partial t^2}\right)(\rho c_v T + T_0 \beta_{ij} \gamma_{ij} - \rho \dot{Q}) \] (2.21.2)

Eqs. (2.13.1), (2.13.2), (2.2.1), (2.13.3), (2.9.2), and (2.21.2) constitute complete mathematical model of two temperature hyperbolic magneto-thermoelasticity theory that includes three phase lag effects.

For \( \tau_q = 0, \tau_T = 0 \) and \( \tau_{ij}^* = 0 \), this theory reduces to (for much low thermal conductivity) two temperature magneto-thermoelasticity based on Green-Naghdi model II.

For \( \tau_q = 0, \tau_T = 0, \tau_{ij}^* = 0 \) and hence \( (\tau^*_{ij})_i = K_{ij} \), this theory becomes two temperature magneto-thermoelasticity based on Green-Naghdi model III. In the case when \( \tau_T = \tau_q^2 = 0, K_{ij}^* = 0, \tau_{ij} = 0 \) and hence \( (\tau^*_{ij})_i = K_{ij} \), this theory clearly reduces to two temperature Lord-Shulman theory of magneto-thermoelasticity.

The relations given above reduce to corresponding relations of two temperature thermoelasticity in absence of the externally applied magnetic field.

### 2.22 Rotating Medium

For a medium rotating uniformly with an angular velocity \( \Omega = \Omega n \), where \( n \) is the unit vector representing the direction of axis of rotation, Eq. (2.1.5) can be modified as

\[\sigma_{ij} + \rho F_i = \rho \left[ (\ddot{u})_i + (\Omega \times (\Omega \times u))_i + (2\Omega \times \dot{u})_i \right], \quad i, j = 1, 2, 3, \] (2.22.1)

and as

\[\sigma_{ij} + \rho s E_i + (J \times B)_i + \rho F_i = \rho \left[ (\ddot{u})_i + (\Omega \times (\Omega \times u))_i + (2\Omega \times \dot{u})_i \right], \quad i, j = 1, 2, 3, \] (2.22.2)

for magneto-thermoelasticity.
Here \( \mathbf{u} \) is the dynamic displacement vector measured from a steady state deformed position and the deformation is assumed to be small. The terms \( \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}), 2\mathbf{\Omega} \times \dot{\mathbf{u}} \) on the right hand side of the above equation arise due to the centripetal acceleration and coriolis acceleration respectively.

Eq. (2.22.1) together with suitable stress-strain relations and suitable heat transport equation constitute complete mathematical model for rotating medium and Eq. (2.22.2) together with suitable stress-strain relations and suitable heat transport equation constitute complete mathematical model for magneto-thermoelastic rotating medium.

The relations given above, valid for an anisotropic body, readily reduce to the corresponding relations for an isotropic body by means of the relations

\[
C_{ijkl} = \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \lambda \delta_{ij} \delta_{kl},
\]

\[
S_{ijkl} = \mu' (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \lambda' \delta_{ij} \delta_{kl},
\]

\[
\beta_{ij} = \beta \delta_{ij}, \alpha_{ij} = \alpha_t \delta_{ij}, K_{ij} = K^* \delta_{ij}, A_{ij} = a \delta_{ij}, C_i = 0,
\]

where \( \mu' = \frac{1}{4\mu}, \lambda' = -\frac{\lambda}{2\mu}(3\lambda + 2\mu), \beta = 3k_1 \alpha_t, k_1 = \lambda + \frac{2}{3}\mu. \)

Here \( \delta_{ij} \) Kronecker delta, \( \lambda \) and \( \mu \) are Lamé constants, \( k_1 \) is the compressibility modulus, \( \alpha_t \) the coefficient of linear thermal expansion for an isotropic body and \( a(\geq 0) \) is the two temperature parameter called temperature discrepancy.

2.23 Boundary Conditions

(A) Elastic boundary and initial conditions

(i) Stress boundary conditions
In the case of first fundamental boundary value problems of elasticity, the stress vector \( \mathbf{T} \) (with components \( T_i \)) is prescribed on the bounding surface \( S \) of the body, where \( n \)
(with components \( n_i \)) is the unit outward drawn normal at any point on \( S \). In this case boundary conditions are

\[
\begin{align*}
\dot{T}_i^\mathbf{n} & = \sigma_{ij} n_j, i, j = 1, 2, 3, \text{ where} \\
\dot{T}_i^\mathbf{n} & = f_i(x_1, x_2, x_3, t), x_i \in S, t > 0, i = 1, 2, 3,
\end{align*}
\]

are three prescribed functions of boundary point \( x_i \) and time \( t \). In the static case \( f_i \) do not depend upon \( t \).

(ii) Displacement boundary conditions

In the case of second fundamental boundary value problems, the displacement vector \( u \) is prescribed on \( S \). Thus in this case we have

\[
\begin{align*}
\dot{u}_i & = g_i(x_1, x_2, x_3, t), x_i \in S, t > 0, i = 1, 2, 3,
\end{align*}
\]

where \( u_i \) is the component of the displacement vector and \( g_i \) are prescribed functions of the boundary point \( x_i \) and time \( t \).

(iii) Mixed boundary conditions

The mixed boundary conditions may be written in the following form

\[
\begin{align*}
\dot{T}_i^\mathbf{n} & = f_i(x_1, x_2, x_3, t), x_i \in S, t > 0, i = 1, 2, 3, \\
\dot{u}_i & = g_i(x_1, x_2, x_3, t), x_i \in S, t > 0, i = 1, 2, 3,
\end{align*}
\]

where \( S = S_1 \cup S_2 \) and the functions \( f_i, g_i \) are both given at \( x_i \) and time \( t \).

(iv) Initial conditions

In the case of dynamical problems, it is necessary to prescribe initial conditions as well.
These may be written in the form
\begin{align*}
u_i(x_1, x_2, x_3, 0) &= 0 \ u_i(x_1, x_2, x_3), x_i \in V \tag{2.23.4} \\
\dot{u}_i(x_1, x_2, x_3, 0) &= 0 \ v_i(x_1, x_2, x_3), x_i \in V
\end{align*}

where the functions \(0 \ u_i, 0 \ v_i\) are prescribed throughout the volume \(V\).

(B) **Thermal boundary and initial conditions**

Along with elastic boundary conditions, the thermal boundary conditions are required to be satisfied in solving boundary value problems of thermoelasticity. They can be one of the following types:

(i) **Prescribed surface temperature**

In this case the temperature \(T\) may be prescribed over the whole of the boundary \(S\), that is
\[T(x_i, t) = F(x_i, t), x_i \in S, t > 0, i = 1, 2, 3,\] (2.23.5)
where \(F\) is given at the boundary point \(x_i\) and time \(t\).

(ii) **Radiation boundary conditions**

When radiation into a medium at temperature \(T_0\) takes place at the boundary \(S\), the linearized boundary conditions can be written as
\[K \frac{\partial T}{\partial n} + H_1 T = H_1 T_0, x_i \in S, t > 0, i = 1, 2, 3,\] (2.23.6)
where \(n\) is the outward drawn unit normal at any point on \(S\), \(\frac{\partial}{\partial n}\) is the normal derivative and \(H_1\) the surface heat transfer coefficient. Here \(T\) is function of the boundary point \(x_i \in S\) and \(t\), and \(T_0\) is generally constant.

(iii) **Prescribed heat input**

In this case normal temperature gradient or the heat flux in the normal direction is
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prescribed on $S$. Therefore, in this case, we have

$$K \frac{\partial T}{\partial n} = q(x_1, x_2, x_3, t), x_i \in S, t > 0, i = 1, 2, 3,$$ \hspace{1cm} (2.23.7)

where $q(x_1, x_2, x_3, t)$ is given at the boundary point $x_i$ and time $t$.

For perfectly insulated surface above condition becomes

$$\frac{\partial T}{\partial n} = 0, x_i \in S, t > 0, i = 1, 2, 3.$$ \hspace{1cm} (2.23.8)

(iv) Initial conditions

The initial condition here consists of merely prescribing the temperature $T$ at $t = 0$ at all points $x_i \in V$, viz.

$$T(x_i, 0) = g(x_i), x_i \in V, i = 1, 2, 3,$$ \hspace{1cm} (2.23.9)

where $g(x_i)$ is a prescribed function.

In the case of steady state, no initial condition is required and the time $t$ will not appear in the equation of heat conduction and the boundary conditions.