Chapter 7

Two-layer film flow on a rotating disk : A numerical study

7.1 Introduction

The work presented in this chapter\(^1\) is a continuation of the work presented in the previous chapter. Here, we have solved the same problem numerically using the finite difference technique for any Reynolds number. Most of the basic considerations are same as the previous chapter except non-dimensionalization scheme for the variables. Effect of different physical parameters viz. Reynolds number, density, viscosity and thickness ratios of two fluids are analyzed.

7.2 Mathematical formulation

We consider a stable configuration of two horizontal layers of immiscible liquids with different physical properties having different uniform thickness over a disk of large diameter. The disk is made to rotate in its own plane with constant angular velocity \(\Omega\). Following Higgins [83] and Dandapat [89], we restrict our attention to the flows that maintains a planar film thickness throughout the entire process. In fact, earlier works of Emslie et al. [77] and Kitamura [92] have shown that

the initial non-uniformity of the film thickness dies out in course of time and a uniform planar final film thickness is the neat outcome of the process. Further, due to axisymmetric flow assumption, the dependent variables are independent of azimuthal co-ordinate \( \theta \). The initial film thickness of the lower layer is considered to be \( h_{10} \) and that of the upper layer is \( h_{20} - h_{10} \). In other words, total film thickness is \( h_{20} \). A schematic diagram given in Figure 7.1 describes the details of the system. All the variables corresponding to the bottom layer are marked by subscript 1 and those corresponding to the top layer are marked by 2. \( V_i, p_i, \rho_i, \nu_i, \mu_i \) denote respectively the velocity, pressure, density, kinematic and dynamic viscosity, of the \( i \) th (\( i = 1, 2 \)) fluid layer. Under this convention, we represent the velocity field as:

\[
V_i = u_i(r,z,t)e_r + v_i(r,z,t)e_\theta + w_i(r,z,t)e_z \tag{7.1}
\]

where, \( e_r, e_\theta, e_z \) are the unit vectors along the co-ordinates axes. The governing equations of continuity and motion for incompressible Newtonian fluid are:

\[
\frac{\partial u_i}{\partial r} + \frac{u_i}{r} + \frac{\partial w_i}{\partial z} = 0, \tag{7.2}
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The boundary conditions at the surface of the disk are:

\[ u_i(r, 0, t) = 0, \quad v_i(r, 0, t) = f_i r, \quad w_i(r, 0, t) = 0. \] (7.6)

The boundary conditions at the liquid-liquid interface \( z = h_i(t) \), are the following:

- the continuity of the velocity field \( V_i = V_2 \); (7.7)
- the balance of shear stress \( \mu_1 \left( \frac{\partial u_1}{\partial r} + \frac{\partial v_1}{\partial z} \right) = \mu_2 \left( \frac{\partial w_2}{\partial r} - \frac{\partial u_2}{\partial z} \right) \); (7.8)
- the balance of normal stress \( p_1 - p_2 = 2 \left( \mu_1 \frac{\partial u_1}{\partial z} - \mu_2 \frac{\partial u_2}{\partial z} \right) \); (7.10)
- and the kinematic condition at the liquid-liquid interface \( z = h_i(t) \):

\[ \frac{dh_i}{dt} = u_i(r, h_i, t). \] (7.11)

The boundary conditions at the free surface \( z = h_2(t) \) the normal and the shear stress must vanish and these are given by:

\[ -p_2 + 2\mu_2 \frac{\partial w_2}{\partial z} = 0; \] (7.12)
\[ \frac{\partial w_2}{\partial r} + \frac{\partial v_2}{\partial z} = 0; \] (7.13)
\[ \frac{\partial v_2}{\partial z} = 0. \] (7.14)
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In addition, the kinematic condition at the free surface \( z = h_2(t) \) reads as:

\[
\frac{dh_2}{dt} = w_2(r, h_2, t). \tag{7.15}
\]

The initial conditions for the velocity field and the film thickness satisfy:

\[
V_i = 0, \quad h_i = h_{i0}, \quad \frac{dh_i}{dt} = 0. \tag{7.16}
\]

Following von Kármán [122], the solution of the above system may be assumed in the following well-known similarity form:

\[
\begin{align*}
&u_i = r f_i(z, t), \\
v_i = r g_i(z, t), \\
w_i = w_i(z, t), \\
&\rho_i(r, z, t) = -\rho_i \left( \frac{r^2}{2} \right) A_i(z, t) + B_i(z, t),
\end{align*}
\tag{7.17}
\]

Substituting (7.17) into the system of Eqs. (7.2)-(7.16) and collecting the different orders of \( r \), we get

\[
\begin{align*}
2 f_i + w_{i,z} &= 0, \tag{7.18} \\
f_{i,z} + f_i^2 - g_i^2 + w_i f_{i,z} - \nu_i f_{i,zz} &= \rho_i A_i(z, t), \tag{7.19} \\
g_{i,z} + 2 f_i g_i + w_i g_{i,z} &= \nu_i g_{i,zz}, \tag{7.20} \\
\rho_i (w_i + w_{i,z} - \nu_i w_{i,zz}) &= -B_i, \tag{7.21} \\
A_{i,z} &= 0. \tag{7.22}
\end{align*}
\]

The suffixes except \( i \) indicate derivatives with respect to that variable. The viscous term in the normal stress boundary conditions (7.10) and (7.12) are independent of \( r \), implying \( A_i = 0 \). We therefore conclude that \( A_i = 0 \) are valid for the entire depth of the liquid. Now, \( B_i(z, t) \) can be found after integrating Eq (7.21) with respect to \( z \), from \( z \) to \( z = h(t) \), by using the conditions on \( z = h_2(t) \)

\[
B_2 = 2\mu_2 \frac{\partial w_2}{\partial z},
\]

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and on \( z = h_1(t) \)

\[
B_1 - B_2 = 2 \left( \mu_1 \frac{\partial u_1}{\partial z} - \mu_2 \frac{\partial u_2}{\partial z} \right),
\]

and thus we can evaluate the pressure from Eq. (7.17). Using Eq. (7.17) in the boundary conditions (7.6) - (7.15), we get

On the surface of the disk: at \( z = 0 \)

\[
f_1(t, 0) = 0, \quad g_1(t, 0) = 0, \quad w_1(t, 0) = 0; \quad (7.23)
\]

On the interface between the two liquids, at \( z = h_1(t) \)

\[
\begin{align*}
&f_1(t, h_1) = f_2(t, h_1), \\
g_1(t, h_1) = g_2(t, h_1), \\
w_1(t, h_1) = w_2(t, h_1), \\
&\mu_1 f_{1,z}(t, h_1) = \mu_2 f_{2,z}(t, h_1), \\
&\mu_1 g_{1,z}(t, h_1) = \mu_2 g_{2,z}(t, h_1), \\
&\mu_1 w_{1,z}(t, h_1) = \mu_2 w_{2,z}(t, h_1);
\end{align*}
\]

(7.24)

On the free surface: \( z = h_2(t) \)

\[
f_{2,x}(t, h_2) = 0, \quad g_{2,x}(t, h_2) = 0; \quad (7.25)
\]

The kinematic conditions for the evolution of the surfaces:

\[
h_{t} = w_{1}(t, h_1). \quad (7.26)
\]

The initial conditions at \( t = 0 \)

\[
\begin{align*}
f_1(0, z) &= 0, \quad g_1(0, z) = 0, \quad w_1(0, z) = 0, \\
h_t &= h_{40}, \quad h_{t,t} = 0.
\end{align*}
\]

(7.27)
Using the following dimensionless variables

\[ \tau = t\Omega, \quad \eta = \frac{z}{h_{10}}, \quad m = \frac{\rho_1}{\rho_2}, \quad n = \frac{\mu_1}{\mu_2}, \quad \delta = \frac{h_{20} - h_{10}}{h_{30}} \]

\[ H_1 = \frac{h_1}{h_{10}}, \quad H_2 = \frac{h_2}{h_{30} - h_{10}} = \frac{h_2}{\delta h_{10}}, \quad Re_1 = \frac{\rho_1 h_{10}^2 \Omega}{\mu_1} \]

\[ Re_2 = \frac{\rho_2 (h_{20} - h_{10})^2 \Omega}{m}, \quad W_1 = \frac{w_1}{h_{10} \Omega}, \quad W_2 = \frac{w_2}{(h_{20} - h_{10}) \Omega} = \frac{w_2}{\delta h_{10} \Omega} \]

\[ F_i = \frac{f_i}{\Omega}, \quad G_i = \frac{g_i}{\Omega} \]

\[ (7.28) \]

Into the system of Eqs (7.18)-(7.20) and the corresponding boundary and initial conditions (7.23)-(7.27) we obtain the dimensionless governing equations as well as their boundary and initial conditions as:

\[ 2F_1 + \lambda_1 W_{1,\eta} = 0, \]

\[ Re_1 (F_{1,\tau} + F_1^2 + \lambda_1 W_{1,\tau} - G_1^2) = \alpha_1 F_{1,\eta\eta}, \]

\[ Re_1 (G_{1,\tau} + 2G_1 F_1 + \lambda_1 W_{1,\eta} G_1) = \alpha_1 G_{1,\eta\eta}, \]

where, \( \lambda_* = (1, \delta) \) and \( \alpha_* = (1, \frac{m}{\Omega}). \)

The boundary conditions are:

\[ F_1(\tau, 0) = 0, \quad G_1(\tau, 0) = 1, \quad W_1(\tau, 0) = 0, \]

\[ F_1(\tau, H_1) = F_2(\tau, H_1), \]

\[ G_1(\tau, H_1) = G_2(\tau, H_1), \]

\[ W_1(\tau, H_1) = \delta W_2(\tau, H_1), \]

\[ F_{1,\eta}(\tau, H_1) = \frac{1}{\alpha} F_{2,\eta}(\tau, H_1), \]

\[ G_{1,\eta}(\tau, H_1) = \frac{1}{\alpha} G_{2,\eta}(\tau, H_1), \]

\[ F_{2,\eta}(\tau, \delta H_2) = 0, \quad G_{2,\eta}(\tau, \delta H_2) = 0. \]

The initial conditions become.

\[ F_1(0, \eta) = 0, \quad G_1(0, \eta) = 0, \quad W_1(0, \eta) = 0, \quad H_1(0) = H_* \]
and kinematic conditions read as:

\[ H_1' = W_1(\tau, H_1), \quad H_2' = W_2(\tau, 5H_2). \]  \hspace{1cm} (7.36)

\section*{7.3 Method of solution}

The above coupled non-linear system of Eqs. (7.29) - (7.36) with the corresponding boundary conditions can be solved by using the finite-difference technique. It is to be noted here that the conventional finite-difference method cannot be applied in this problem as thickness of both the layers are continuously decreasing with time. Considering this fact, the time-dependent physical domain is transformed to a fixed computational domain \([0, 1+\delta]\) such that the film thickness will always remain in a fixed computational domain for all times. Further, care has been taken through a fine grid distribution for the large velocity gradients that may be present near the disk surface. It should be pointed out here that the same transformation will be useful for fine as well as uniform grid distribution.

We choose the transformation as

\[ \xi = 1 - A_1 \log \left( \frac{A_2 - \frac{\eta}{H_1(\tau)}}{B_2 + \frac{\eta}{H_1(\tau)}} \right) + \left[ 1 - u(\eta, \tau) \right] \left\{ \delta - A_1 \delta \log \left( \frac{A_2 - \frac{\eta-H_1(\tau)}{\delta H_2(\tau)-H_1(\tau)}}{B_2 + \frac{\eta-H_1(\tau)}{\delta H_2(\tau)-H_1(\tau)}} \right) + A_1 \log \left( \frac{A_2 - \frac{\eta}{H_1(\tau)}}{B_2 + \frac{\eta}{H_1(\tau)}} \right) \right\}, \]  \hspace{1cm} (7.37)

where

\[ u(\eta, \tau) = \begin{cases} 1 & \text{for } 0 \leq \eta < H_1(\tau), \\ 0 & \text{for } H_1(\tau) \leq \eta \leq \delta H_2(\tau) \end{cases} \]  \hspace{1cm} (7.38)

and \( A_1 = (\log(\frac{\delta\eta}{B_2}))^{-1} \), with \( A_2 = (c + 1), \quad B_2 = (c - 1) \). Here, \( c \) is the grid controlling parameter in the physical domain. Small values of \( c \) cluster grid points at the disk surface where as large values make grid spacing uniform throughout the two-layer film. The Crank-Nicholson scheme is used to solve the transformed non-linear system of Eqs. (7.29) - (7.36) after approximating the nonlinear terms according to the Newton’s linearization technique (Fletcher [123]). Computation
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is carried out in each time level on the following linear tridiagonal system of algebraic equations:

\[
P(G_i)^{k+1}_{j-1} + Q(G_i)^{k+1}_{j} + R(G_i)^{k+1}_{j+1} = (S_1)^{k}_j \quad (7.39)
\]

\[
P(F_i)^{k+1}_{j-1} + Q(F_i)^{k+1}_{j} + R(F_i)^{k+1}_{j+1} = (S_2)^{k}_j \quad (7.40)
\]

Here, \(i, j\) and \(k\) denote the position of the layer, spatial level of discretization and time level, respectively. For \(i = 1\) or \(2\), \(j\) takes values from 1 to \(N_1+1\) or \(N_1+1\) to \(N_1+N_2+1\), respectively. Here, \(N_1\) and \(N_2\) denote the number of partitions in the lower and upper layer of the fluid. The expressions for \(P, Q, R, (S_1)^{k}_j\) and \((S_2)^{k}_j\) are quite lengthy but their derivations are straightforward. (we have provided exact expressions for these terms in Appendix C.) At each and every time level \((G_i)^{k+1}_{j}\) and \((F_i)^{k+1}_{j}\) are computed using Eqs. (7.39) - (7.40) and then the axial velocity \((W_i)^{k+1}_{j}\) is obtained from finite difference representation of continuity equation by using the values of \((F_i)^{k+1}_{j}\) at that time level. The iteration process continues until it attains the desired level of thickness.

Numerical Computation is carried out on 101 grids in the vertical direction with \(c = 10^4\). This gives uniform grid distribution in physical and as well as in the computational domain. However, for higher Reynolds number one needs to increase the number of grid points in the vertical direction. Under this situation, the value of \(c\) must decrease depending on the rotational speed of the disk.

The time step has been calculated by using

\[
\Delta \tau \leq 0.25 \times \Delta \xi^2 \quad (7.41)
\]

This relation comes from the Courant - Friedrichs - Lewy condition of numerical stability (Fletcher [123]). The domain of \(\Delta \tau\) has been chosen smaller than the stability domain for linear equation due to the coupled non-linear system.

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Figure 7.2 represents the effect of \(\delta\), the ratio of the initial film thickness deposition between the two layers, on the gradual development of the final film thickness with respect to time \(\tau\) for fixed values of \(m, n\) and the Reynolds number \(Re_1\)
It is clear from the figure that the change of $\delta$ affects the film thickness at the initial stage but it is more or less insensitive to the final film thickness at large time. Figure 7.3 shows the gradual variation of the film height in both the layers with time $\tau$ when the fluid viscosity ratio $n$ is changed. It is clear from the figure that variation of viscosity between the two layers has least effect on film thinning. Figure 7.4 depicts the gradual development of film thickness when the fluid density ratio varies. It is evident from the figure that the variation of density ratio affects just after the spin off stage but the final film thickness is independent on it. Figure 7.5 shows the effects of the variation of the Reynolds number on film thinning for a particular set of values of $\delta$, $n$, and $\mu$. It is further clear that the thinning rate for both the layers increase with decrease in the Reynolds number during the time interval $0 < \tau < \tau_e$, where $\tau_e$ is a point on the time scale. On the other hand film thinning rate in both the layers increase with increase of the Reynolds number in the interval $\tau_e < \tau$. This anomalous behavior of the variation of the film height in different time interval can be explained by considering the two time regions $\tau < \tau_e$ and $\tau > \tau_e$. The point $\tau_e$ in the time interval indicates that time, when total amount of fluid flowing out of a certain radial distance for two different Reynolds numbers become equal. In other words we can say that the $\tau_e$ depends on the competitive Reynolds numbers. In interval $\tau < \tau_e$, thinning rate for both layers increase with the decrease in the rotational speed. This seems to be inconsistent with common observation that film thins faster with faster rotation of the disk. A close observation of this physical process exposes that the azimuthal component of velocity for both layers develop along the entire depth of the liquid faster, with slower rotation of the disk. This occurred due to the fact that the Reynolds number is the ratio between the centrifugal and viscous forces. Increase in the Reynolds number reflects that centrifugal force dominates over viscous force. Finally, viscous action builds up azimuthal component of velocity field faster throughout the depth of the fluid for smaller Reynolds number. This result can be found in Figure 7.6. It is expected that the faster development of azimuthal velocity field will introduce the corresponding radial velocity faster for smaller Reynolds numbers only. Figure 7.7 depicts the radial component of velocity for both the layers that developed over the entire depth for smaller values of Reynolds number in the interval $0 < \tau < \tau_e$. As a result film thins quicker
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for the smaller value of the Reynolds number in this interval. In time interval \( \tau > \tau_c \), azimuthal velocities for both the layers have developed throughout the film depth for both large and small Reynolds numbers and the corresponding developed centrifugal force in both the layers drive away large amount of fluid when the value of the Reynolds number is large. In other words, fluid moves out of the disk quickly when rotation speed of the disk is high. This explains why films thin faster in the time interval \( \tau_c < \tau \) for the large Reynolds number. Figure 7.8 depicts the variation of the azimuthal components of velocity for both the layers when \( \delta \), the ratio of the initial film thickness deposition between the layers varies for fixed \( Re_1 \), \( m \) and \( n \). It is clear from the figure that thinner film deposition on the top layer in compared to the bottom layer, produces faster azimuthal velocity in both the layers and enhance film thinning rate via faster radial velocity development as shown in Figure 7.9.
Figure 7.2: Variation of top film’s height ($H_2\delta$) and the bottom film (interface height $H_1$) with $\tau$ for three different values of $\delta$. Solid, dashed and dashed-dotted lines represent for $\delta = 1.2$, 1.0 and 0.9, respectively. While $m = 1.6$, $n = 1.4$ and $Re_1 = 6.0$ are considered as fixed.
Figure 7.3: Variation of top film’s height \( H_2 \delta \) and the bottom film (interface height \( H_{1} \)) with \( \tau \) for two different values of \( n \). For total layer, dotted and solid lines represent film thickness for \( n = 3.5 \) and 6.0, respectively. Dashed and dashed-dotted lines representing the lower layer thickness for \( n = 3.5 \) and 6.0, respectively. While \( m = 2.0 \), \( Re_1 = 10.0 \) and \( \delta = 1.0 \) are considered as fixed.
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Figure 7.4: Variation of top film’s height ($H_2 \delta$) and the bottom film (interface height $H_1$) with $\tau$ for two different values of $m$. Here, dotted and solid lines represent total film thickness for $m = 1.4$ and 2.0, respectively. Dashed and dotted-dashed lines represent lower layer thickness for $m = 1.4$ and 2.0, respectively. While $n = 2.5$, $Re_1 = 10.0$ and $\delta = 1.0$ are considered as fixed.
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Figure 7.5: Variation of top film's height ($H_2\delta$) and the bottom film (interface height $H_1$) with $\tau$ for two different $Re_1$. Here, dotted and solid lines represent total film thickness for $Re_1 = 0.5$ and 0.9, respectively. Dashed and dotted-dashed lines represent lower layer thickness for $Re_1 = 0.5$ and 0.9, respectively. While $m = 2.0$, $n = 2.5$ and $\delta = 1$ are considered as fixed.
Figure 7.6: Variation of $G_2$ at free surface ($H_2\delta$) and $G_1$ at interface ($H_1$) with time $\tau$ for two different $Re_1$. Dashed and solid line represent $Re_1 = 0.5$ and $Re_1 = 0.9$, respectively. While $m = 2.0$, $n = 2.5$ and $\delta = 1.0$ are considered as fixed.
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Figure 7.7: Variation of $F_2$ at free surface ($H_2\delta$) and $F_1$ at interface ($H_1$) with time $\tau$ for two different $Re_1$. Dashed and solid line represent $Re_1 = 0.5$ and $Re_1 = 0.9$, respectively. While $m = 2.0$, $n = 2.5$ and $\delta = 1.0$ are considered as fixed.
Figure 7.8: Variation of $G_2$ at the free surface ($H_2\delta$) and $G_1$ at the interface ($H_1$) with time $\tau$ for three different $\delta$. Solid, dashed and dotted-dashed lines represent $\delta = 1.2$, 1.0 and 0.9, respectively. While $m = 1.6$, $n = 1.4$ and $Re_1 = 6.0$ are considered as fixed.
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Figure 7.9: Variation of $F_2$ at free surface ($H_2\delta$) and $F_1$ at interface ($H_1$) with time $\tau$ for three different $\delta$. Solid, dashed and dotted - dashed line represent $\delta = 1.2$, 1.0 and 0.9, respectively. While $m = 1.6$, $n = 1.4$ and $Re_1 = 6.0$ are considered as fixed.
7.5 Concluding remarks

Gradual development of film thickness over a rotating disk is studied numerically when two uniform layers of fluid are deposited over the disk. Under the action of the centrifugal force the film started thinning in both the layers. It is further found that at the initial stage, the rate of film thinning in both the layers are more under smaller rotation of the disk. However, film thins faster with strong rotation as the spinning continues for a long time. This apparently anomalous behaviour of film thinning is due to the crucial role played by the viscosity. Further it is observed that the film thinning rate at the initial stage of rotation depends on the ratio of the deposited film thickness between the layers. Smaller the thickness on the top layer, faster the initial film thinning rate in both the layers. It is also found that change of viscosity or density has very minor effect on initial film thinning.