CHAPTER VI

An Unified methodological approach
6.1 A LITERATURE REVIEW AND MOTIVATION

In the earlier four chapters, we have attempted to deal with the massive and live congestion problem of pilgrim traffic at TIRUMALA using well known tools and results in queueing theory, originated by Erlang (1909-1920) and ever since developing fast into a popular research area for applied probabilists, operations researchers, industrial and management workers. With a variety of workers taking keen interest in waiting line models, gaps kept on being increasingly identified, stimulating newer methods and techniques being developed to handle the problems. Notable contributions to the storage theory (slightly more general than queueing theory) were from Moran (1959 - Dam Theory), Gani (1957), Prabhu (1965), Takacs (1967) and Cohen (1969).
These researches, while highly original and epochal, are based on different methods to handle more or less the same types of problems. For example, Prabhu used 'ladder indices', Takacs leaned heavily on 'combinatorics' while Cohen made a generous use of 'Pollaczek's integral representation' technique. Without venturing to detract any merit whatsoever from these efforts, it may be generally observed that such a cumbersome array of tools and techniques would be rather too unwieldy for researchers, specially interested in applicational needs.

The two vital gaps thus may be identified namely:

(i) need for more direct, simple and application oriented methodology and (ii) need for evolving some kind of 'unified methodological approach' to handle a large number of (atleast) cognate areas, for example, inventory, dam, queueing and reliability problems. The recent thinking on these lines picked up momentum with the crucial suggestion of Kendall (1964) for effective use of 'point processes' in stochastic storage theory (in particular, in queueing systems) which was ably followed by Srinivasan (1969 and later) who, along with others made significant contributions to storage and reliability problems using product density technique in point processes. Lewis (1972) further highlighted the role of point processes as a powerful modern tool for
varied problems belonging to a wide spectrum of areas like storage, management, hydrological, reliability, forests, biostochastics and so on.

While, again these landmarks of research efforts admirably narrow the gap indicated in (i) above, a large scope still remains for further work in connection with, (ii) cited along with. In fact, it was Srinivasan (1976) who made a strong suggestion for development of 'unified method'.

Most recently efforts are initiated in this direction by Venugopal and others (1979 and later) who while generously using techniques in point processes, concentrated on the vital 'dual systems' (Venugopal and Sarma (1979-1983)) and later considered crucial ramifications of these 'dual models' (Venugopal and Meenakshi (1980-84)) to extract an 'unified method' to handle storage and reliability problems through the same body of manipulative labour. These workers while obtaining more general and useful explicit results for storage and reliability models, also obtained agreement with some well-known results particularly in Cohen (1969) and Gaver (1963).

Some added contributions in this Chapter are based particularly on the works of Venugopal et al., cited above by exploiting some scopes, still to be covered.
It is therefore appropriate that we should briefly indicate the developments leading to our work in this chapter. This task we propose to undertake on a 3-tier level. The usual type of model considered in 'classical storage' theory is based on a random manifestation of storage level as a resultant of inter-play of the two random processes: (a) inter-event (event being input or arrival) times and (b) quantums produced by these events (inputs or service times), together with the other deterministic parameters like queue discipline (release rule), waiting room (Dam Capacity or height) and number of channels (servers). A typical model, in terms of realisation of the general random process of storage content (load on the counter or water height in a dam) is depicted in Fig. 21. This is a typical model extensively studied in classical storage theory which in the form of discrete states is employed by us in the earlier four chapters (queueing results). Its 'dual', when the 'stochastic' and the 'deterministic' natures of 'built-up' (raise) and 'releases' are interchanged, (that is: built-up is deterministic while release is random) has many interesting and crucial implications. Though this model is mentioned in literature (e.g. Prabhu (1964)) referred to it in terms of departure processes while Cohen (1969) indicated its use in some duality relations), explicit
results in depth or detail (specially covering 'transient' cases) are lacking and this was, in a somewhat comprehensive manner, answered in Venugopal and Sarma (1979-83). A visual appeal of the dual model is provided through the depiction in Fig. 22. The value of the work, comprises in that, it led to the 'crucial ramification', if introduced, could produce an unified method capable of tackling storage and reliability systems through the same body of manipulative techniques. This powerful idea is proposed and well exploited by Venugopal and Meenakshi (1980-84) who interpreted reliability systems in storage language basing on the condition that the 'random drop' (pull) at an epoch is sharper than at the previous epoch. An effective schematic depiction of this 'ramification' is shown in Fig. 23.

A noteworthy point in the last mentioned work above, is that the condition (of sleeper drop) through appears rather 'stringent', is really a clever way of manipulation of the methodology to facilitate extraction of explicit results in storage and reliability contexts.

We shall present more details of the unified method in the next section, but the scope that stimulated the present work in this thesis is again the content in the work which is analytic - biased. While the gaps
FIG. 22  THE STORAGE MODEL (infinite capacity)
Fig. 23 A typical realisation of \( \{X(t)\} \) of basic model.
indicated in (i) and (ii) above are thus being strongly attempted by workers in the field, we recall the case made out earlier, for evolving a methodology or theory facile enough for ready use of an applied worker interested in management problems. Specifically the methodology may be such that whenever the assumption (or postulates) are satisfied, closed form (explicit) results are neatly delivered and otherwise the methodology should be capable of leading to numerical (possibly computer-aided) research, yielding respectably accurate results sufficient to answer real life problems (specially when they are complex and massive).

Elaborating, for the work in the next section, we briefly indicate some more details of the 'unified method' and results in Venugopal and Meenakshi (1983) and suggest further lines in terms of evolving a 'unified methodological approach' suited for attacking problems in the absence of knowledge of probabilistic structures governing the variates that participate in model building. The efforts, are based on the simple 'recurrence relations' (with suitable 'initial equations'), results for 'power-series summations', Laplace transforms (particularly in 'convolute' forms) and other simple probabilistic arguments. The methodological approach is envisaged to be particularly useful for computer programming, in developing 'package programmes' and 'ready reckoner
sheets' for management worker interested in reliability engineering and reliability systems.

An aim in this chapter is to provide a companion research view point (reliability being an intimately connected area to storage, particularly queueing situations) with a desirable bias to numerical work-scopes for applicational purposes.

6.2 AN UNIFIED APPROACH

We first explain, in general technical jargon, the basic content of the method (facilitating use of point processes in storage and reliability models) employed by Srinivasan et al., (1969 and later) and Venugopal et al., (1974 and later). The following are the basic techniques:

**T (6.2.1):** A general non-negative and continuous stochastic process \( \{ X(t) \} \) in storage and reliability contexts is not locked at in terms of its continuous states, but rather, is looked at as an 'event process', as a given 'x' - level crossings from 'below' or 'above' (depending upon the physics of the problem, for example, refer the typical realisations depicted in Fig. 22 or 23 and Fig. 21 respectively).
This sort of perception of a general random process is facilitated by the relabelling device, accomplished with the help of suitable first order product densities (vide Chapter I) introduced, for the purpose. In general terms, a typical first order product density of the above type is depicted in the general definition:

\[ A(x,y,t) = \lim_{dx \to 0} \Pr \left[ \circ < x < X(t) \leq x + dx \right], \]

where \( I_1 \) denotes 'intermediary' conditions in \((0,t)\), \( E_t \) denotes the 'end' condition at 't' and \( I_0 \) denotes the 'initial' condition at the 'start of observation', viz., 'origin' and is intimately related to 'y'.

With the interpretation described above, it is easy to recognise \( \Lambda(x,y,t) \) as a first order product density of the x-crossing (from below or above, as the case may be) at the arbitrary epoch 't', so that \( \Lambda(x,y,t) \, dx \) is the probability of the 'crossing event' at 't', and using the property of 'product density' explained in Chapter I (see (1.4.1.f) and (1.4.1.g)) integration of \( \Lambda(x,y,t) \), with respect to (w.r.t) 'x' yields the average number of entities (events, i.e., x-crossings) at 't' in the range \((0, a)\) and \((0, b)\), \( a < b \), \((a, b)\) being the range of integration, while
a further integration w.r.t. \('t'\) yields the average number of such events in the range of integration of the time domain. Now suitably integrating \(A(x,y,t)\) w.r.t. \('x'\) and \('y'\), the unconditional average number of occurrence of events of interest is extracted (see also Bartlett (1955) for an alternative explanation of the concept).

\[ T(6.2.4): \text{Backward integral equations are formulated for product densities of the type } A(x,y,t) \text{ using the technique due to Bellman and Harris (1948) and } A(x,y,t)'s \text{ are generally solved in terms of their Laplace transforms (L.T.'s), under suitable assumptions. The formulation of the integral equations for the product densities (involving 'Schwarizian measures') is based on the identification of suitable regenerative processes: } \{ X(t_n+) \} \text{, embedded in the general random process } \{ X(t) \}. \text{ Note that } X(t_n+) \text{'s are regenerative epochs (see Figs. 21, 22, 23) in the sense that } X(t_n+) \text{ is known after occurrence of the event of interest at } t_n. \]

\[ T(6.2.5): \text{The required average numbers are now extracted by setting the L.T. arguments } = 0 \text{ (see also (1.5.16) of chapter I) whereby many questions of interest are answered.} \]

With the description of the method, as narrated above, it is easy to see the directness, simplicity and
comprehensiveness of the product density and point processes techniques in stochastic modelling and the consequent prominence they enjoy and the vital role they play in many areas of modelling applications.

Referring in particular to the works of Venugopal and Meenakshi (as already indicated about the close connection of our present work), the first order product densities that they introduced are connected to the typical realisations of $X(t)$, schematically given in Fig. 23 where the two product densities $A(x,y,t)$ and $A_1(x,y,t)$, they introduced respectively correspond to the $E_t$'s that there is a pull at 't' or otherwise. By suitably integrating out 'x' in $A(x,y,t)$ and $A_1(x,y,t)$ corresponding respectively to the $E_t$'s referred to above and further setting $s = 0$ and $p = 0$ (the respective L.T. arguments w.r.t. 'y' and 't') in the double L.T.'s of $A(., ., .)$ and $A_1(., ., .)$, Venugopal and Meenakshi (1983) obtained a basic result for reliability of a system (employing storage models' language and the same body of manipulative tools), in terms of average time to failure (ATF) as:

$$E(t) = \text{ATF} = \frac{\lambda + \mu}{\mu^2}$$

(6.2.6)

where a storage (or reliability system) system is of the $M(\lambda)|M(\mu)|1|\infty|\text{FIFO type}$, the entries in the brackets
denoting the parameters of the Markovian laws. Agreement of (6.2.6) with the result in Gaver (1963) is also established.

We now come to the 'main point' of our contribution in this connection. We first observe that the result in (6.2.6) is based on the assumption of knowledge of the random laws governing the random processes that participate in the stochastic modelling. Further, the key expression in the manipulative body in deriving the result (6.2.6) is the crucial convolute term of the probability densities corresponding to the 'random pulls' and 'inter-random pulls' which employs the vital condition that: \( X(t_{n+}) < X(t_{n-1}+) \). The condition leads to the manipulations that facilitate the development of the 'unified method' proposed by Venugopal and Meenakshi (1983).

In the following, we give below (without derivation or proof) some important definitions and steps that yield the result (6.2.6), which body of method we improve to render it suitable for numerical work, referred to as 'main point', above.

The product densities are defined as :
\[
A(x,y,t) = \int \Pr \left[ \circ < x < X(t) < x + dx, \right.
\]
\[
dx \rightarrow 0 \left. \right| X(u) > \circ, u \in (\circ, t), \right.
\]
\[
X(t+) < X(t) \left| \right. \right.
\]
\[
X(\circ-) > X(\circ) = X(\circ+) = y]/dx,
\]
\[
(6.2.7)
\]
and \(A_1(x,y,t)\) is defined similarly, relaxing the end condition at \(t\), \(X(t+) < X(t)\), that is, there is no occurrence of random pull at \(t\) (see Fig. 23).

A \((x,y,t)\), then is shown to satisfy the integral equation:

\[
A(x,y,t) = U(x-y) \delta(t - (x-y)) h(t)
\]
\[
+ \int h(u) du \int A(x,y+u-v, t-u) g(v) dv,
\]
\[
(6.2.8)
\]
where: \(U(.)\), \(\delta(.)\), \(h(.)\) and \(g(.)\) denote respectively the Heaviside unit-step function, the Dirac-Delta function (Schwarzian measure), the law governing inter-pull times and the law governing the quantums of pulls.

\(A_1(x,y,t)\) satisfies a similar equation with the change in the I term on the KHS of (6.2.8) when \(h'(t)\) is replaced by the survivor function \(\int^\infty_t h(u) du\), because of the change of end condition at \(t\).
Let $A^*(x,s,p)$ and $A_1^*(x,s,p)$ denote the double L.T. of $A(x,s,p)$ and $A_1(x,s,p)$ respectively, defined as:

$$A^*(x,s,p) = \int_0^\infty e^{sy} dy \int_0^{\infty} e^{-pt} A(x,y,t) dt,$$

Re. s, Re. p > 0, (6.2.9)

and $A_1^*(x,s,p)$ being defined similarly as above. Then the A.T.F., denoted by $F(t)$ after some calculations is derived as:

$$F(t) = \int_0^\infty dx \left[ A^*(x,0,0) \int_0^{\infty} g(y) dy + A_1^*(x,0,0) g(x) \right], \quad (6.2.10)$$

where $X(t)$ is interpreted as the 'operating level' of the system. For the Markovian cases, (6.2.10) delivers the formula given in (6.2.6).

It may be seen from (6.2.8) and (6.2.10) that $h(.)$ and $g(.)$ participate as 'convolutive terms with $A(x,s,p)$ and $A_1(x,s,p)$, after noticing that setting the concerned L.T. arguments as zero is equivalent to (indirectly) dealing with probabilities. We recall the earlier remark that the derivation of (6.2.10) requires the knowledge of suitable forms of $h(.)$ and $g(.)$ and further the complexion of (6.2.10) is more analytic and less suited to a numerical attack in the absence of the underlying assumptions about $h(.)$ and $g(.)$. 
We modify the procedure, retaining the general method to render it suitable for a numerical attack.

The product densities $A(., ., .)$ and $A_{1}(., ., .)$ are now modified as:

$A_{n}(., ., .)$ and $A_{1,n}(., ., .)$ by including in the R.H.S. probability given in (6.2.7), the additional condition that there are 'n' events (random pulls) occurring in $(0, t), n = 0, 1, 2, \ldots \ldots \cdot$

We now prove the following:

**THEOREM (6.2.1):** The functions $A_{n}(., ., .)$'s satisfy the recurrence relations:

\[
A_{0}(x,y,t) = U(x-y) \delta (t - (x-y)) h(t), \quad (6.2.11 \ a)
\]

and

\[
A_{n}(x,y,t) = \int_{0}^{t} h(u) du \int_{u}^{y+u} A_{n-1}(x,y+u-v, t-u) g(v) dv, \quad n = 1, 2, \ldots \ldots \cdot \quad (6.2.11 \ b)
\]

Further $A(x,y,t)$ is given by:

\[
A(x,y,t) = \sum_{n=0}^{\infty} A_{n}(x,y,t). \quad (6.2.11 \ c)
\]

**Proof:** The proof is straightforward when one considers the mutually exclusive and completely exhaustive cases.
number of events corresponding to the \( i \) being \( 0, 1, 2, \ldots \) and following the other details of manipulations given in Venugopal and Meenakshi (1983), to derive (6.2.8), using (6.2.7) and the model set up depicted in Fig. 23.

The main advantage of theorem (6.2.1) is that \( A_n \)'s are iteratively determined in terms of convolutes of \( h(.) \) and \( g(.) \). However if \( h(.) \) and \( g(.) \) are not specifically known the corresponding probabilities may be estimated from a given data and these would help determine \( A(., ., .) \) in terms of \( A_n(., ., .) \)’s.

Similar work for \( A_{1n}(., ., .) \) would yield \( A_1(., ., .) \). Once these functions are determined, using once again the estimated probabilities corresponding to \( g(.) \) in (6.2.10) the value of \( E(T) \) may be numerically computed.

For example, for the special case of Markovian laws for \( h(.) \) and \( g(.) \) with parameters \( \lambda \) and \( \mu \) respectively, theorem (6.2.1) directly yields,

\[
E(T) = \frac{1}{\mu} \left[ 1 + \frac{A}{\lambda + \mu} + \left( \frac{A}{\lambda + \mu} \right)^2 + \ldots \right],
\]

\[
= \frac{1}{\mu} \left( 1 - \frac{A}{\lambda + \mu} \right) - 1,
\]

\[
= \frac{\lambda + \mu}{\mu},
\]

agreeing with the result in (6.2.6)
For general cases when the forms of $h(.)$ and $g(.)$ are not known, theorem (6.2.1) could still be used resorting to numerical method with computer aid (by estimating the needed probabilities for a given data).

6.3 FURTHER SCOPE AND USES

Theorems of the above type (Th.6.2.1) prove useful in reliability analysis and storage problems, for example, if computer package programme or ready reckoner sheets (tables) are developed for system failures corresponding to different probabilities of inter-event occurrences and quantum of occurrence.

A large scope appears to exist for work in this direction. In addition when models with introduction of barriers are considered, probabilities corresponding to truncated laws may be estimated from the data and fed into the programme for extracting the results.

We conclude with the observation that we have only attempted to indicate a possible line of work suited to numerical methods with some modification of known and well-tried theoretical lines.
We accordingly choose to call it an unified methodological approach to signify the implication that the approach focused in theorem (6.2.1) is suitable both for theoretical and numerical work equally well.

Attempts in this direction are briefly dealt here. Further work in this direction is in progress.