CHAPTER 2

THE THEORY OF CONSUMER BEHAVIOUR —
THE INDIFFERENCE PREFERENCE THEORY.
1. Introduction

The purpose of this chapter is to provide a summary picture at a glance of the salient features of the received theory of consumer behaviour based on indifference-preference hypothesis, together with relevant concepts and definitions.

The indifference-preference theory formulated by Pareto and Johnson, and developed by Slutsky and Hicks and Allen, appears to have culminated in Hicks' Value and Capital.

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1. For a succinct criticism and references see:
   (ii) Hicks, J.R., A Revaluation of Demand Theory, Oxford, (1956), particularly chapter 2.
   (iii) ... Value and Capital, London, (1948), Part I.


6. Hicks, J.R., Value and Capital, op. cit.
This theory is based on the assumption that a consumer has an ordered scale of preferences described by an ordinal utility function and that his market behaviour as reflected by the allocation of his expenditure on various commodities which command prices, is the outcome of an attempt by him to maximize that function subject to constraints imposed by fixed money income and the given market prices he has to pay.

2. **Ordinal Utility Function**

The indifference-preference theory assumes the existence of an ordinal utility function. The existence of this function has the following implications. It is assumed that there is a fixed number of different commodities (including services), say \( n \), each of which has a well-defined unit of measurement. A bundle of commodities, \( q = (q_1, q_2, \ldots, q_n) \) is composed of \( q_1 \) units of the 1st commodity, \( \ldots \), and \( q_n \) units of the \( n \)th commodity. It may be conceived as a point in the \( n \)-dimensional Euclidean space (usually called the budget space), with coordinates \( q_1, q_2, \ldots, q_n \) respectively. If \( q^0 \) and \( q^1 \) represent any two bundles of commodities under consideration of a consumer then it is postulated that he will be able to place them in one of the mutually exclusive categories:

- a. \( q^0 \) preferred to \( q^1 \)
- b. \( q^0 \) disfavoured to \( q^1 \)
- c. \( q^0 \) and \( q^1 \) equally preferred or indifferent
To each bundle of commodities, \( q = (q_1, q_2, \ldots, q_n) \), we may attach a number which is assumed to be a continuous differentiable function of components \( q_i \) \((i=1,2,\ldots,n)\). This function (or rule of numbering) may be written as

\[
(2.2.2) \quad u = u(q) = u(q_1, q_2, \ldots, q_n)
\]

\( u(q) \) is the utility index or an ordinal utility function, if it has the following three properties corresponding to the above three conditions respectively:

1. \( u^1 \cdot u(q^1) < u(q^0) \)
2. \( u^0 \cdot u(q^0) < u(q^1) \)
3. \( u^1 \cdot u(q^0) = u(q^1) \)

In order that these conditions may hold or that the ordinal utility function may exist, it may be proved that the following additional assumptions are needed:

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1. Samuelson, op. cit., p.94.
(i) The order of preferences is logically consistent in the following sense: For any three bundles of commodities, \( q^0, q^1 \) and \( q^2 \), if \( q^0 \) is indifferent (preferred, disfavored) to \( q^1 \), and \( q^1 \) is indifferent (preferred, disfavored) to \( q^2 \), then \( q^0 \) is indifferent (preferred, disfavored) to \( q^2 \).

This is known as the axiom of transitivity.

(ii) A larger bundle of commodities is always preferred to a smaller one in the sense that if \( q^0 < q^1 \), then \( q^1 \) is preferred to \( q^0 \). This postulate is called the axiom of non-satiety.

(iii) Let \( q^0, q^1, q^2 \) be any three bundles of commodities such that \( q^0 \) is preferred to \( q^1 \), and \( q^1 \) is preferred to \( q^2 \). Then on any curve joining the points \( q^0 \) and \( q^2 \) in the budget space there exists at least one point \( q \) such that \( q \) is indifferent to \( q^1 \). This assumption is known as the axiom of continuity.

It is evident that any function

\[
\tilde{u} = F(u) \quad \text{such that} \quad F'(u) = \frac{dF}{du} > 0
\]

is also a utility index or an ordinal utility function.

For \( u(q^1) \tilde{<} u(q^0) \) is equivalent to \( \tilde{u}(q^1) \tilde{<} \tilde{u}(q^0) \) respectively.
This implies that many utility indices exist for a given scale of preferences. If \( u(q) \) is one utility index then every monotonically increasing function \( \tilde{u} = F(u) \) is also a utility index. It is evident that for a given value of the utility index, \( c \) say, the one parameter locus or equivalently the indifference surface

\[ u(q) = c \]

(2.2.4)\(^1\)

does not depend on the choice of utility index but is uniquely determined by the scale of preferences.

5. 

**Consumer Equilibrium**

The individual consumer, with an ordinal utility function (2.2.2) has a given money income \( E \) and can purchase the \( n \) commodities \( q = (q_1, q_2, \ldots, q_n) \) in a market at given prices \( p = (p_1, p_2, \ldots, p_n) \). The problem is to determine the allocation of his income or to determine his demands \( q = (q_1, q_2, \ldots, q_n) \) for maximum utility. The solution is that of conditional maximization; the variables \( q_1, q_2, \ldots, q_n \) are to be determined for:

\[
\max u \text{ subject to } \sum_{r=1}^{n} p_r q_r = E
\]

\(^1\) For different values of \( c \), (2.2.4) represents a set of indifference surfaces called an indifference map.
The Lagrangean technique for solving this problem is to maximize the Lagrange function \( s = u - m \left( \sum_{r=1}^{n} p_r q_r - E \right) \).

This leads to the system of \((n+1)\) equations:

\[ (2.5.1) \quad u_r = m p_r \quad (r = 1, 2, \ldots, n) \]

where \( u_r = \frac{\partial u}{\partial q_r} \) and \( m \) is Lagrangean multiplier.

\[ (2.5.2) \quad \sum_{r=1}^{n} p_r q_r = E \]

In the \((n+1)\) equations of \((2.5.1)\) and \((2.5.2)\), there are \((n+1)\) unknowns, \( q_1, q_2, \ldots, q_n \) and \( m \). We may solve them for \( n \) commodities demands, \( q_r (r = 1, 2, \ldots, n) \) in terms of parameters comprising prices \( p_r \) and income \( E \) and obtain the following demand system:

\[ (2.5.3) \quad q_r = q_r (p_1, p_2, \ldots, p_n, E) \quad (r = 1, 2, \ldots, n) \]

Equations \((2.5.1)\) and \((2.5.2)\) are the necessary conditions for maximum of \( u \). The sufficient conditions for a true maximum of \( u \) are that:

1. \( m > 0 \) because \( u_r > 0 \), \( p_r > 0 \)
\[(2.3.4) \quad d^2 u = \sum_{r=1}^{n} \sum_{s=1}^{n} u_{rs} \, dq_r \, dq_s < 0\]

for all values of \(dq_1, dq_2, \ldots, dq_n\), subject to

\[(2.3.5) \quad du = \sum_{r=1}^{n} u_r \, dq_r = 0\]

or equivalently, making use of (2.3.1)

\[du = \sum_{r=1}^{n} p_r \, dq_r = 0\]

where \(u_{rs} = \frac{\partial u}{\partial q_r}\) and \(u_{rs} = u_{sr}\)

As \(d^2 u = \sum_{r=1}^{n} \sum_{s=1}^{n} u_{rs} \, dq_r \, dq_s\) is a quadratic form in \(dq_1, dq_2, \ldots, dq_n\), the required conditions that it should be negative definite for all values of \(dq_1, dq_2, \ldots, dq_n\) subject to the linear constraint (2.3.5), are that the bordered Hessians

\[\begin{vmatrix}
  u_{11} & u_{12} & u_1 \\
  u_{21} & u_{22} & u_2 \\
  u_1 & u_2 & 0
\end{vmatrix},
\begin{vmatrix}
  u_{11} & u_{12} & u_{13} & u_1 \\
  u_{21} & u_{22} & u_{23} & u_2 \\
  u_{31} & u_{32} & u_{33} & u_3 \\
  u_1 & u_2 & u_3 & 0
\end{vmatrix},
\begin{vmatrix}
  u_{11} & u_{12} & \cdots & u_{1n} & u_1 \\
  u_{21} & u_{22} & \cdots & u_{2n} & u_2 \\
  \ddots & \ddots & \ddots & \ddots & \ddots \\
  \cdots & \cdots & \ddots & \ddots & \ddots \\
  u_{n1} & u_{n2} & \cdots & u_{nn} & u_n \\
  u_1 & u_2 & \cdots & u_n & 0
\end{vmatrix}\]

1 Hessians are determinants of matrices composed of second order partial derivatives, \(u_{rs}\).
or equivalently

\[
\begin{pmatrix} u_{11} & u_{12} & p_1 \\ u_{21} & u_{22} & p_2 \\ \vdots & \vdots & \vdots \\ u_{n1} & u_{n2} & p_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \cdots & u_{1n} & p_1 \\ u_{21} & u_{22} \cdots & u_{2n} & p_2 \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1} & u_{n2} \cdots & u_{nn} & p_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix}
\]

must alternate in signs, starting with plus.\(^1\)

Let \( D = \begin{pmatrix} U & F \\ p' & 0 \end{pmatrix} \)

where \( U = (u_{rs})_{n \times n} \)

and \( p' = (p_1 \ p_2 \ \cdots \ p_n) \), a row vector containing \( n \) elements.

Denoting the cofactor of \((r,s)\) th element of \( D \) by \( |D_{rs}| \),

it directly follows from the last two conditions of (2.3.6)

that \( |D_{nn}| / |D| < 0 \) \( (\because \) the signs of \(|D_{nn}|\) and \(|D|\) are given

by \((-1)^{n-1}\) and \((-1)^n\) respectively). Since the \( n \) commodities

may be taken in any order, it follows from above that

\(|D_{rr}| / |D| < 0 \) \( (r=1,2,\ldots,n) \).

This means that first \( n \) diagonal elements of inverse of

\( D \) i.e., \( D^{-1} \) are all negative.

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\(^1\) Hicks, Value and Capital, op. cit., pp.304-305.
4. **Comparative Statics of Consumer Demand System**

Comparative statics deals with changes in the equilibrium values of the variables resulting from given changes in the values of the parameters. Now we shall examine the comparative statics properties of the demand system (2.3.3). More specifically, here we shall attempt to answer the followings:

What are the effects on the equilibrium demands \(q_r\)

(a) of an equal proportionate change in the parameters \(p_r\) s and \(E_s\), i.e., when the price-income parametric situation changes from \((p_1, p_2, \ldots, p_n, E)\) to \((kp_1, kp_2, \ldots, kp_n, kE)\) say,

(b) of change in parameter \(E_s\), parameters \(p_r\) s remaining constant, and

(c) of change in parameter \(p_r\), other prices and income parameters remaining constant?

**No Money Illusion**

Eliminating \(u_r\) among the equations of (2.3.1), it may be rewritten as

\[(2.4.1) \quad u_r/u_s = p_r/p_s \quad (r=1, 2, 3, \ldots, n-1, n+1, \ldots, n)\]

and (2.3.2) may be put in the form

\[(2.4.2) \quad \sum_{r=1}^{R} p_r q_r/p_s = \bar{X}/p_s\]

\(^1\) Comparative statics is described as "...the investigation of changes in statical system from one position of equilibrium to another without regard to the transitional process involved in adjustment", (Cf. Samuelson, op. cit., p.3).
Thus the \((n+1)\) parameters viz. \(n\) prices \(p_x\) and \(E\) may be expressed as \(n\) parameters comprising \((n-1)\) price ratios \(\frac{p_x}{p_0} \quad (x=1, 2, \ldots, n-1, n+1, \ldots, n)\) and \(E/p_0\). Evidently, the equilibrium values \(q_x\) of the demand system \((2.3.3)\) whether obtained from \((2.3.1)\) and \((2.3.2)\) or \((2.4.1)\) and \((2.4.2)\) are the same. In view of the forms of \((2.4.1)\) and \((2.4.2)\), it easily follows that the change in the price-income situation as envisaged under \((a)\) above, does not affect the quantity demanded of each commodity. In other words, this means that the demand equations of \((2.3.3)\) are homogeneous of order zero in all prices and income parameters, i.e.,

\[
q_x(kp_1, kp_2, \ldots, kp_n, kE) = q_x(p_1, p_2, \ldots, p_n, E) \quad (x=1, 2, \ldots, n)
\]

This implies that there is no money illusion.

**Income Effect**

Here we analyse the demand system \((2.3.3)\) under price-income situation \((b)\) above. Differentiating partially both the sides of each equation of \((2.3.1)\) end of equation \((2.3.2)\), with respect to \(E\), the resulting equations are

\[
\begin{align*}
\frac{\partial q_1}{\partial E} + u_{12} \frac{\partial q_2}{\partial E} + \cdots + u_{1n} \frac{\partial q_n}{\partial E} &= \frac{\partial m}{\partial E} = 0 \\
& \vdots \\
& \vdots
\end{align*}
\]
\[ \begin{align*}
\frac{\partial q_1}{\partial E} &= u_{n1} \frac{\partial q_1}{\partial E} + u_{n2} \frac{\partial q_2}{\partial E} + \ldots + u_{nn} \frac{\partial q_n}{\partial E} = p_n \frac{\partial m}{\partial E} = 0 \\
p_1 \frac{\partial q_1}{\partial E} + p_2 \frac{\partial q_2}{\partial E} + \ldots + p_n \frac{\partial q_n}{\partial E} &= 1
\end{align*} \]

The above equations may be rewritten compactly as

\[ (2.4.3) \quad \sum_{s=1}^{n} u_{rs} \frac{\partial q_s}{\partial E} = p_r \frac{\partial m}{\partial E} = 0 \quad (r=1, 2, \ldots, n) \]

\[ (2.4.4) \quad \sum_{s=1}^{n} p_s \frac{\partial q_s}{\partial E} = 1 \]

The solution of the above system, using matrix notation, is given by

\[ (2.4.5) \quad \begin{bmatrix}
\frac{\partial q_1}{\partial E} \\
\frac{\partial q_2}{\partial E} \\
\vdots \\
\frac{\partial q_n}{\partial E} \\
- \frac{\partial m}{\partial E}
\end{bmatrix} =
\begin{bmatrix}
u_{11} & u_{12} & \cdots & u_{1n} & p_1 \\
u_{21} & u_{22} & \cdots & u_{2n} & p_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
u_{n1} & u_{n2} & \cdots & u_{nn} & p_n \\
- p_1 & - p_2 & \cdots & - p_n & 0
\end{bmatrix}^{-1}
\begin{bmatrix}0 \\
0 \\
\vdots \\
0 \\
1\end{bmatrix}
\]

Equivalently (2.4.5) may be expressed as

\[ (2.4.6) \quad \frac{\partial q_r}{\partial E} = \left| \frac{D_{n+1,r}}{D} \right| \quad (r=1, 2, \ldots, n) \]

where, \( D_{n+1,r} \) and \( D \) have the same connotation as under (2.3.6).
Since the sufficient conditions (2.3.6) above say nothing about the sign of $E_{b+1,r}$ it follows that the income effect given by (2.4.6) may be positive or negative. It may also be noted that under price-income situation (b) complete demand system (2.3.3) reduces to a partial one parameter system (2.4.7) which represents a curve in n-dimensional quantity space, which is known as an Engel curve.

(2.4.7) \[ q_x = q_x(E) \quad (x=1,2,\ldots,n). \]

This means that demand is a function of income, prices being held constant.

**Price Effect**

Here we analyse the complete demand system (2.3.3) under the price-income situation (c) above. Differentiating partially both the sides of each equation of (2.3.1) and of equation (2.3.2) with respect to $p_x$, the resulting equations in a compact form are:

(2.4.8) \[ -p_x \frac{\partial m}{\partial p_x} + \sum_{j=1}^{m} u_{xj} \frac{\partial q_j}{\partial p_x} = 0 \quad (s=1,2,\ldots,n) \]

This would be positive unless the commodity in question is an inferior good.
\[(2.4.9) \quad -p_j \frac{\partial m}{\partial p_j} + \frac{n}{j=1} u_j \frac{\partial q_j}{\partial p_j} = m \quad (s = r) \]

\[(2.4.10) \quad \sum_{j=1}^{n} p_j \frac{\partial q_j}{\partial p_j} = -q_x \]

The solution of the above system, using matrix notation, is given by

\[
\begin{bmatrix}
\frac{\partial q_1}{\partial p_1} \\
\frac{\partial q_2}{\partial p_1} \\
\vdots \\
\frac{\partial q_n}{\partial p_1} \\
\frac{\partial m}{\partial p_1} \\
\end{bmatrix} = 
\begin{bmatrix}
u_{11} & u_{12} & \cdots & u_{1r} & u_{1n} & p_1 \\
u_{21} & u_{22} & \cdots & u_{2r} & u_{2n} & p_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
u_{n1} & u_{n2} & \cdots & u_{nr} & u_{nn} & p_n \\
p_1 & p_2 & \cdots & p_r & p_n & 0 \\
\end{bmatrix}^{-1} 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
-m_p \\
\end{bmatrix} 
\]

Equations of (2.4.5) and of (2.4.11) respectively, may be written compactly as

\[
\begin{bmatrix}
q_R \\
q_p \\
\end{bmatrix} = 
\begin{bmatrix}
U & p \\
-p' & 0 \\
\end{bmatrix}^{-1} 
\begin{bmatrix}
0 \\
-m_p \\
\end{bmatrix}, \quad \text{where} 
\]

\[
q'_R = \begin{bmatrix}
\frac{\partial q_1}{\partial s} & \frac{\partial q_2}{\partial s} & \cdots & \frac{\partial q_n}{\partial s} \\
\end{bmatrix}, \quad q_p = \begin{bmatrix}
\frac{\partial q_1}{\partial p_1} & \frac{\partial q_2}{\partial p_1} & \cdots & \frac{\partial q_n}{\partial p_1} \\
\frac{\partial q_1}{\partial p_2} & \frac{\partial q_2}{\partial p_2} & \cdots & \frac{\partial q_n}{\partial p_2} \\
\frac{\partial q_1}{\partial p_n} & \frac{\partial q_2}{\partial p_n} & \cdots & \frac{\partial q_n}{\partial p_n} \\
\end{bmatrix}_{n \times n} 
\]

\[
m_p = \begin{bmatrix}
\frac{\partial m}{\partial p_1} & \frac{\partial m}{\partial p_2} & \cdots & \frac{\partial m}{\partial p_n} \\
\end{bmatrix}, \quad \text{and} \quad m_p = \begin{bmatrix}
\frac{\partial m}{\partial s} \\
\end{bmatrix}, \quad I \text{ is an identity matrix of order} \ n, \ \text{and} \ 0 \ \text{is a null column vector of order} \ n \ \text{in the second matrix on the right hand of the above system of equations.} \]
From (2.4.11), it follows that the effect of the change of the
price on $q_8$, other prices and income remaining unchanged, is

$$\frac{\partial q_8}{\partial p_r} = -q_8 \frac{D_{n+1,s}}{D} + m \frac{D_{rs}}{D}$$

Using (2.4.6), this becomes

$$\frac{\partial q_8}{\partial p_r} = -q_8 \frac{\partial q_8}{\partial E} + m \frac{D_{rs}}{D}$$

and as a particular case of this,

$$\frac{\partial q_r}{\partial p_r} = -q_r \frac{\partial q_r}{\partial E} + m \frac{D_{rr}}{D}$$

For the sake of brevity, let $k_{sr} = m \frac{D_{rs}}{D}$

Thus corresponding to (2.4.12) and (2.4.13), we have respectively:

$$\frac{\partial q_8}{\partial p_r} = -q_8 \frac{\partial q_8}{\partial E} + k_{sr}$$

$$\frac{\partial q_r}{\partial p_r} = -q_r \frac{\partial q_r}{\partial E} + k_{rr}$$

**Substitution Effect**

Equation (2.4.14), originally due to Slutsky, is known as the
Fundamental Equation of Value Theory. It gives us the effect of a
change in $p_r$ on a consumer's demand for $q_8$, split into two terms.
These two terms may be given the following economic interpretations:
with the increase (decrease) in the price \( p_r \), the original consumption \( (q_1, q_2, \ldots, q_n) \) costs \( q_r dp_r \) units of money more (less). But equation (2.4.6) shows that the effect on consumption of \( q_n \) of a reduction in income of \( q_r dp_r \) units of money is

\[
\frac{\partial q_n}{\partial E} q_r dp_r.
\]

The first of the right hand terms of equation (2.4.14) may be conceived as the income effect of change in \( p_r \). The second term in the above equation is known as the substitution effect which is defined as response of quantity demanded to a compensated variation of price, that is, to a variation in price accompanied by a compensating change in money income. It has the following two interpretations which are shown to be equivalent in the limit:

(a) Effect on equilibrium consumption of \( q_n \) of a compensated change in \( p_r \) (other prices remaining constant) so as to leave the consumer, if he chose, on his original highest indifference surface. This interpretation is due to Hicks.\(^2\)

(b) Effect on equilibrium consumption of \( q_n \) of a compensated change in \( p_r \) (other prices remaining constant) which would assure the consumer, if he chose, his original consumption \( (q_1, q_2, \ldots, q_n) \). This interpretation is due to Slutsky.\(^3\)

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1. Appendix to this chapter.
Following are some of the important properties of the demand system with particular reference to substitution effects:

**Symmetry**

\[(2.4.16) \quad k_{sr} = k_{rs} \]

That is, \( k_{sr} \) is \textit{symmetrical} between \( s \) and \( r \). This follows from the fact that \( k_{sr} = \frac{D_{rs}}{|D|} \) and \( D_{rs} \) is \textit{symmetrical} between \( r \) and \( s \). Further from (2.4.14) and (2.4.16) it follows that

\[
\frac{\partial q_s}{\partial p_r} = \frac{\partial q_r}{\partial p_s}
\]

if and only if,

\[
q_r \frac{\partial q_s}{\partial q_r} = q_s \frac{\partial q_r}{\partial q_s}
\]

or \( (\partial^2 q) \frac{\partial q_s}{\partial q_r} = (\partial^2 q) \frac{\partial q_r}{\partial q_s} \)

i.e. if the income elasticities of the two commodities are the same. Thus, the cross-price effects of two commodities are the same if and only if their income elasticities are the same.

**Negativity**

\[(2.4.17) \quad k_{rr} < 0 \]

\[
e_{rr} = m \frac{|D_{rr}|}{|D|} \quad \text{and} \quad |D_{rr}| < 0 \quad (\text{Cf. (2.3.6), and } m > 0)
\]
(2.4.17) shows that the substitution effect of an own-price change, briefly termed as direct-price substitution effect\(^1\) is negative. This implies that the demand of a commodity always falls as the compensated price of the commodity rises. But, the same is not always true in the case of uncompensated price rise. Although very stringent conditions need to be fulfilled before there could be an exception to this. Not only should the good be an inferior good i.e.

\[
\frac{\partial q_x}{\partial E} < 0
\]

but also a Giffen good so that the negative income effect swamps the substitution effect there by resulting in

\[
\frac{\partial q_x}{\partial p_x} > 0
\]

**Substitutes and Complements**

It may be noted that while \( k_{xx} \) is always negative,

nothing definite can be said of \( k_{xs} = k_{sx} \) except for a two commodities case when it can be shown to be positive\(^2\). In

Hicks-Allen terminology\(^3\) two commodities \( r \) and \( s \) are substitutes or complements according as

\(^1\) analogously \( k_{sr} = k_{rs} \) is termed cross-price substitution effect.

\(^2\) Follows from \( (2.4.21) \).

(2.4.18) \[ k_{sr} = k_{rs} \geq 0 \]

The two properties viz.,

(2.4.19) \[ \sum_{s=1}^{n} p_s k_{sr} = 0 \]

(2.4.20) \[ \sum_{s=1}^{n} p_s \left| \frac{b_{sr}}{D} \right| = 1 \]

are known as the homogeneity and the additivity conditions respectively. These two conditions may be derived as a special case of the two general properties of a determinant that its expansion by alien cofactors yields a value of zero and that its expansion by corresponding cofactors yields its actual value. In the present case the determinant is \[ D \]

and it is expanded by alien cofactors of elements of \( p \) in the case of (2.4.19) and by corresponding cofactors of the elements of \( p \) in the case of (2.4.20). From (2.4.19) and (2.4.17) it follows that

(2.4.21) \[ \sum_{s(r)}^{n} p_s k_{sr} > 0 \]

This shows that there must be at least one commodity (not commodity \( r \)) such that

\[ k_{rs} > 0 \]

This means that at least one pair of commodities must be substitutes.

1 Alternatively (2.4.19) may be derived by applying Euler's theorem to (2.5.5) and making use of (2.4.14). As regards (2.4.20), it follows from (2.4.4) and (2.4.6).
Composite Commodities

\[(2.4.22) \sum_{s=1}^{m} \sum_{r=1}^{m} l_{s} l_{r} k_{sr} < 0 \quad \text{(for all values of } m < n)\]

for all values of arbitrary \(l\) coefficients. This may be derived
from (2.3.6) by using the property of reciprocal determinants\(^1\).

In particular, let \(l_{s} = p_{s}\), then (2.4.22) becomes

\[(2.4.25) \sum_{s=1}^{m} \sum_{r=1}^{m} p_{r} p_{s} k_{sr} < 0 \quad \text{(for all values of } m < n)\]

Using (2.4.25) and (2.4.14), Hicks proves the composite
commodities theorem which states that if the prices of a
group of commodities change in the same proportion, then
for the purposes of demand analysis such a group of commodities
may be treated logically as a single elementary commodity
and consequently the various results applicable to a single
elementary commodity, hold true in this case as well.\(^2\)

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\(^1\) Hicks (1948), op. cit., pp.519-511.

\(^2\) (a) ibid., pp.513-515, and
(b) Leontief, W., "Composite Commodities and the Problem
5. The Choice of Utility Function

Results and derivations of sections 3 and 4 above are invariant against a replacement of the utility function, $u$ by $\bar{u} = F(u)$, provided $\bar{u}$ is a strictly increasing function i.e. $F'(u) > 0$, and has a continuous second order derivative.

More exactly, for given values of prices $p_r$s and income $E$, the quantities $q_r$, $\frac{\partial q_r}{\partial E}$, $k_r$s, and also $\frac{\partial q_s}{\partial p_r}$ ($s, r = 1, 2, \ldots, n$) are unaffected when the utility function $u$ is replaced by $\bar{u}$.

This may be proved by using the following results:

\begin{align*}
(2.5.1) & \quad \bar{u}_r = F'(u) u_r \\
(2.5.2) & \quad \bar{u}_{rs} = F'(u) u_{rs} + F''(u) u_r u_s \\
(2.5.3) & \quad \bar{m} = F'(u) m
\end{align*}

where, $\bar{m}$ is the Lagrange's multiplier corresponding to the utility function $\bar{u}$. (2.5.1) and (2.5.2) follow from the definition of $\bar{u}$. To derive (2.5.3), we first substitute the value of $u_r$ from (2.5.1), in the necessary conditions for the maximum of $\bar{u}$ vis. $u_r = m p_r$ and obtain

\begin{equation}
(2.5.4) \quad p_r = F'(u) u_r / \bar{m}
\end{equation}

Now substituting $u_r/p_r = m$ from (2.5.1) in (2.5.4) and simplifying, we obtain (2.5.3).

From (2.5.1) it may be seen that $\frac{\bar{u}_r}{\bar{u}_s} = \frac{u_r}{u_s}$ and hence from (2.4.1) and (2.4.2) it follows that the demand system (2.5.5) is independent of the choice of the utility
function. Turning to the sufficient conditions for maximum of \( \bar{u} \), we shall examine whether or not sign of a typical \( i \)th bordered hessian

\[
\begin{array}{cccc}
\bar{u}_{11} & \bar{u}_{12} & \cdots & \bar{u}_{1,i+1} \\
\bar{u}_{21} & \bar{u}_{22} & \cdots & \bar{u}_{2,i+1} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{u}_{i+1,1} & \bar{u}_{i+1,2} & \cdots & \bar{u}_{i+1,i+1} \\
p_1 & p_2 & \cdots & p_{i+1} \\
\end{array}
\]

\( i=1,2,\ldots,n-1 \)

is affected by the replacement of \( u \) by \( \bar{u} \), since only the signs of bordered Hessians are indicative of the stability or otherwise of consumer equilibrium. \(^1\) Substituting the values of \( \bar{u}_x \) and \( p_x \) from (2.5.2) and (2.5.4) respectively in (2.5.5), and subtracting \( \frac{\bar{F}^\prime(u)u_x}{F^\prime(u)} \) times the \( s \)th element of the last row from the corresponding element of each of the \( r \)th row \( (r=1,2,\ldots,i+1) \) of the bordered hessian. Then, (2.5.5) after some simplification becomes

\(^1\) Cf. (2.3.6).
(2.5.6) \((F'(u))^i\) shows that the bordered hessian corresponding to
\(\bar{u}\) is obtained by multiplying the \(i\)th bordered hessian cor-
responding to \(u\) with \((F'(u))^i\). Since \(F'(u)\) is assumed to be positive,
none of the bordered hessians corresponding to \(\bar{u}\) have their signs
changed by the introduction of such a factor, and since only the
signs of the bordered hessians, (2.5.5) indicate the stability or
otherwise of consumer equilibrium, the sufficient conditions are
invariant against the substitution of \(\bar{u}\) for \(u\).

Analogous to (2.4.6), the income effect corresponding
to the utility function \(\bar{u}\) can be written as

\[
\frac{\partial u_{x}}{\partial p} = \frac{[\bar{D}^n+1_{x}]}{[\bar{D}]} \quad (r=1,2, \ldots, n)
\]

where,

\[
(2.5.8) \quad |\bar{D}| =
\]

\[
\begin{bmatrix}
\bar{u}_{11} & \bar{u}_{12} & \cdots & \bar{u}_{1n} & p_1 \\
\bar{u}_{21} & \bar{u}_{22} & \cdots & \bar{u}_{2n} & p_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\bar{u}_{n1} & \bar{u}_{n2} & \cdots & \bar{u}_{nn} & p_n \\
p_1 & p_2 & \cdots & p_n & 0
\end{bmatrix}
\]
and \( \left| \tilde{D}_{n+1,r} \right| \), the cofactor of \((r,n+1)\) th element of \( \tilde{D} \) is given by

\[
\begin{vmatrix}
  \tilde{u}_{11} & \tilde{u}_{12} & \cdots & \tilde{u}_{1n} \\
  \tilde{u}_{21} & \tilde{u}_{22} & \cdots & \tilde{u}_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  \tilde{u}_{n-1,1} & \tilde{u}_{n-1,2} & \cdots & \tilde{u}_{n-1,n} \\
  \tilde{u}_{n,1} & \tilde{u}_{n,2} & \cdots & \tilde{u}_{nn} \\
  \tilde{p}_{1} & \tilde{p}_{2} & \cdots & \tilde{p}_{n}
\end{vmatrix}^{n+1}
\]

(2.5.9) \( \left| \tilde{D}_{n+1,r} \right| = (-1)^{n+1} \)

From (2.5.6) it follows that

(2.5.10) \( \tilde{D}_r = \left( P'(u) \right)^{n-1} \tilde{D} \)

Similarly, by using (2.5.2) and (2.5.4) in (2.5.9), and after some simplification we obtain

(2.5.11) \( \left| \tilde{D}_{n+1,r} \right| = \left( P'(u) \right)^{n-1} \left| D_{n+1,r} \right| \)

From (2.5.10) and (2.5.11), it clearly emerges that

\[
\frac{\left| \tilde{D}_{n+1,r} \right|}{\left| \tilde{D} \right|} = \frac{\left| D_{n+1,r} \right|}{\left| D \right|}
\]

and as such the income effect \( \frac{\partial q_r}{\partial \bar{E}} \) is invariant against the substitution of \( u \) for \( \bar{u} \).
Let analogous to $k_{sr}$, the substitution effect corresponding to the utility function $u$ be written as

$$\begin{align*}
(2.5.12) \quad & \frac{k_{sr}}{n} = m \frac{|D_{rs}|}{|D|} \\
\end{align*}$$

where $n$ and $|D|$ have already been defined above and $|D_{rs}|$ being the cofactor of the $(s,r)$th element of $|D|$ is given by

$$\begin{align*}
\begin{array}{cccccccc}
\tilde{u}_{11} & \tilde{u}_{12} & \ldots & \tilde{u}_{1,r-1} & \tilde{u}_{1,r+1} & \ldots & \tilde{u}_{1n} & p_1 \\
\tilde{u}_{21} & \tilde{u}_{22} & \ldots & \tilde{u}_{2,r-1} & \tilde{u}_{2,r+1} & \ldots & \tilde{u}_{2n} & p_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{u}_{s-1,1} & \tilde{u}_{s-1,2} & \ldots & \tilde{u}_{s-1,r-1} & \tilde{u}_{s-1,r+1} & \ldots & \tilde{u}_{s-1,n} & p_{r-1} \\
\tilde{u}_{s+1,1} & \tilde{u}_{s+1,2} & \ldots & \tilde{u}_{s+1,r-1} & \tilde{u}_{s+1,r+1} & \ldots & \tilde{u}_{s+1,n} & p_{r+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{u}_{n1} & \tilde{u}_{n2} & \ldots & \tilde{u}_{n,r-1} & \tilde{u}_{n,r+1} & \ldots & \tilde{u}_{nn} & p_n \\
p_1 & p_2 & \ldots & p_{r-1} & p_{r+1} & \ldots & p_n & 0 \\
\end{array}
\end{align*}$$

(2.5.13) $|D_{rs}| = (-1)^{r+s}$

By using (2.5.2) and (2.5.4) in (2.5.13) and after some simplification we obtain

$$\begin{align*}
(2.5.14) \quad & |D_{rs}| = (F'(u))^{n-2} |D_{rs}| \\
\end{align*}$$

By using (2.5.3) together with (2.5.10) and (2.5.14) in (2.5.12) we obtain,

$$\begin{align*}
(2.5.15) \quad & \frac{k_{sr}}{m} = k_{sr} = m \frac{|D_{sr}|}{|D|} \\
\end{align*}$$
This shows that substitution effect is invariant against the replacement of $u$ by $\tilde{u}$. Since the $n$ commodities demands $q_x$, the income effect $\partial q_x / \partial \tilde{u}$ and substitution effect $k_{\text{gr}}$ have been shown to be invariant, it follows from (2.4.14) that uncompensated cross-price effect $\partial q_x / \partial p_x$ is also invariant and so also as a particular case of this, uncompensated direct-price effect $\partial q_x / \partial p_x$. The upshot of this is that results derived in the sections 3 and 4 above are invariant against a substitution of $\tilde{u}$ for $u$, provided any arbitrary function $\tilde{u} = F(u)$ is a monotonic increasing function and also possesses a second order derivative.

6. Integrability Conditions

From family-budget data one can deduce the slopes of indifference surfaces at various points representing different bundles of commodities in the budget space. This is because the slopes of the budget hyperplanes are equal to the slopes of indifference surfaces at the points of contact or equivalently the equilibrium points, as clearly emerges from (2.5.1). Thus, a differential equation of an indifference surface can be determined empirically from family-budget data. From the operational angle, the question is: given the differential equation of an indifference surface, can it be integrated to yield the equation of the indifference surface? Equivalently the question is: will the choices
that a consumer makes between bundles of commodities differing by infinitesimal amounts be consistent with the choices he makes between bundles of commodities differing by finite amounts? To elucidate it further, let us suppose that a consumer chooses initially a bundle \( q^0 \). By infinitesimal steps, examining an infinite number of bundles each equivalent to the preceding one, he chooses finally a bundle \( q^1 \). The question is: will the consumer consider \( q^1 \) equivalent to \( q^0 \) or not?

Suppose the partial derivatives \( u_i \) of the function \( u \) are arbitrarily given. Then,

\[
(2.6.1) \quad \sum_{i=1}^{n} u_i \, dq_i = 0
\]

is the partial differential equation for determining the indifference surface \( u = c \). To begin with, let the consumer choose a bundle \( q^0 \). By infinitesimal steps, let him obtain from the differential equation (2.6.1), an infinite number of bundles, each equivalent to the preceding one, choosing finally a bundle \( q^1 \). The consumer will regard the bundles \( q^0 \) and \( q^1 \) as equivalent if (2.6.1) satisfies the integrability conditions. Since it violates consistency in preference if \( q^1 \) and \( q^0 \) are not considered equivalent to each other by the consumer, the integrability conditions must be fulfilled.

We now introduce briefly the integrability conditions. Equilibrium condition (2.3.1) states that
\[
\frac{u_1}{p_1} = \frac{u_2}{p_2} = \ldots = \frac{u_n}{p_n}
\]

For \( n \geq 3 \), \( u_1 \) cannot be replaced by any \( n \) arbitrary functions, \( R^i(q_1, q_2, \ldots, q_n) \) say; \( R^i = \frac{\partial R}{\partial q_i} \). Instead, it should be

\[
(2.6.2) \quad \dot{u}_1 = 1R^i, \quad \text{where}
\]

\[
(2.6.3) \quad 1 = l(q_1, q_2, \ldots, q_n)
\]

is an appropriate integrating factor, and so \( R^i \) must fulfill the following conditions:

\[
(2.6.4) \quad u_{ij} = \frac{\partial}{\partial q_j} (1R^i) = \frac{\partial}{\partial q_i} (1R^j) = u_{ji} \quad (i, j = 1, 2, \ldots, n)
\]

It can be shown how for an integral utility function, \( u_1 \), the conditions \( u_{ij} = u_{ji} \) are equivalent to the following \( \binom{R}{3} \) integrability conditions:

\[
(2.6.5) \quad R^i (R^j_k - R^j_k) + R^j (R^k_i - R^k_i) + R^k (R^i_j - R^i_j) = 0
\]

where \( \frac{\partial R}{\partial q_j} \), etc.

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7. **Quadratic Form of Utility Function**

Let the quadratic form of utility function be given by

\[ u = \frac{1}{2} q' A q + B' q + C \]

where,

\[ A = A^n = (A_{rs})_{n \times n} \text{ is a symmetric matrix of order } n, \]

\[ B^n = (B_1, B_2, ..., B_n) \text{ is a row vector of order } n, \text{ and} \]

\[ C = \text{a scalar constant}. \]

It may be shown that in this case corresponding to the necessary conditions (2.5.1) and (2.5.2), we have

\[ u_r = \sum_{s=1}^{n} A_{rs} q_s + B_r = m P_r \quad (r = 1, 2, ..., n) \]

or more compactly the above system may be expressed as

\[ A q + B = m p \]

where, \( p \) is a column vector of prices and \( q \) is a column vector of quantities, each of order \( n \), and

\[ p' q = E \]

Solving (2.7.3) and (2.7.4) for \( q \) and \( m \), we obtain:

\[
\begin{pmatrix}
q \\
-m
\end{pmatrix}
= \begin{pmatrix}
I & p \\
p' & 0
\end{pmatrix}^{-1}
\begin{pmatrix}
-B \\
E
\end{pmatrix}
\]

From (2.7.2) it follows that

\[ u_{rs} = A_{rs} \text{ or } U = A \]
and hence the sufficient condition of equilibrium or the stability condition corresponding to (2.3.6) is that the first \( n \) diagonal elements of

\[
p^{-1} = \begin{bmatrix}
A & P \\
p' & 0
\end{bmatrix}^{-1}
\]

should be all negative.

Substituting \( A \) for \( U \) in (2.4.5) and (2.4.11), we may obtain the income effects, price effects and hence substitution effects from (2.4.14) within the framework of quadratic form of utility function. Similar remarks apply to the various other results of sections 3, 4 and 5 above; in particular the homogeneity and additivity conditions vis-a-vis (2.4.19) and (2.4.20).

**Separable Quadratic Form**

Following Corman and Usawa, three separability concepts may be distinguished. Given a set of commodities

\[ q = (q_1, q_2, \ldots, q_k) \]

which may be grouped into \( G \) mutually exclusive subsets:

\[ (q_1^{(1)}, q_2^{(1)}, \ldots, q_k^{(1)}) \]

A grouping of commodities is said to be strongly, weakly and piecewise separable according as the marginal rate of substitution \( u_i/u_j \) between the two commodities
\( i \) and \( j \) fulfills the following three criteria respectively:

\[
(2.7.7) \quad \frac{\partial}{\partial q_k} \left( \frac{u_i}{u_j} \right) = 0 \quad \text{for all} \quad q_i \subseteq q(s), \quad q_j \subseteq q(h) \quad \text{and} \\
q_k \not\subseteq q(s) \subseteq q(h) \quad (s \not= h)
\]

\[
(2.7.8) \quad \frac{\partial}{\partial q_k} \left( \frac{u_i}{u_j} \right) = 0 \quad \text{for all} \quad (q_i, q_j) \subseteq q(s) \quad \text{and} \quad q_k \not\subseteq q(s)
\]

\[
(2.7.9) \quad \frac{\partial}{\partial q_k} \left( \frac{u_i}{u_j} \right) = 0 \quad \text{for all} \quad (q_i, q_j) \subseteq q(s) \quad \text{and} \quad q_k \not= q_i \text{ or } q_j
\]

where, \( u_i = \frac{\partial u}{\partial q_i} \) and \( \subseteq, \not\subseteq \) denote respectively contained in, not contained in and union of.

---

Corresponding to the above three criteria of separability, we have the following three characterizations of the form of the underlying utility function:

\[(2.7.10) \quad u(q) = u(q^{(1)}, q^{(2)}, \ldots, q^{(G)}) = F_1(u^{(1)}(q^{(1)}) + u^{(2)}(q^{(2)}) + \cdots + u^{(G)}(q^{(G)}))\]

where, \(F_1\) is an additive function of \(G\) variables \(u^{(g)}(g=1,2,\ldots,G)\) and for each \(g\), \(u^{(g)}\) is a function of the sub-vector \(q^{(g)}\) only.

\[(2.7.11) \quad u(q) = F_2(u^{(1)}(q^{(1)}), u^{(2)}(q^{(2)}), \ldots, u^{(G)}(q^{(G)}))\]

where, \(F_2\) is a function of \(G\) variables \(u^{(g)}(g=1,2,\ldots,G)\) and for each \(g\), \(u^{(g)}\) is a function of sub-vector \(q^{(g)}\) only.

\[(2.7.12) \quad u(q) = F_3(u^{(1)}(q^{(1)}), u^{(2)}(q^{(2)}), \ldots, u^{(G)}(q^{(G)}))\]

where, \(F_3\) is a function of \(G\) variables \(u^{(g)}(g=1,2,\ldots,G)\) and for each \(g\), \(u^{(g)}\) is an additive function of sub-vector \(q^{(g)}\), whenever the number of commodities in the sub-vector is more than two.

From (2.7.10) and (2.7.11), it follows that a quadratic form of preference indicator (2.7.1), if separable — whether strongly or weakly—may be expressed as

\[(2.7.13) \quad \frac{1}{2} \sum_{g=1}^{G} (q^{(g)})^2 A_g(q^{(g)}) + B^t q + C\]
where $A$ is a symmetric matrix of order $n$, such that $\sum_{g=1}^{G} n_g = n$ and corresponding to (2.7.5) we have,

$$\begin{bmatrix} q \\ -m \end{bmatrix} = \begin{bmatrix} A & p^{-1} \\ p & 0 \end{bmatrix} \begin{bmatrix} \delta \\ 0 \end{bmatrix}$$

where,

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_G \end{bmatrix}$$

and that 0 and $A$ are matrices of appropriate order.

Substituting the value of $A$ for $U$ from (2.7.15) in (2.4.5) and (2.4.11), we obtain the income effects, price effects and hence substitution effects from (2.4.14) in the case of separable quadratic form of preference indicator.

By virtue of (2.7.15) and making the further assumption that $A^{-1}$ exists, it can be shown that the effect of a change in the price $p_1$ of a commodity belonging to the sub-set $g$ on commodities $r$ and $s$ not belonging to this sub-set is given by
\[ \frac{\partial q_x}{\partial p_i} = \left( \frac{m}{a} \frac{\partial q_i}{\partial \lambda} - q_i \right) \frac{\partial q_x}{\partial \lambda}, \text{ and} \]
\[ \frac{\partial q_s}{\partial p_i} = \left( \frac{m}{a} \frac{\partial q_i}{\partial \lambda} - q_i \right) \frac{\partial q_s}{\partial \lambda} \]

where, \( \frac{1}{a} = -p' \Lambda^{-1} p \)

From (2.7.16) and (2.7.17), it follows that
\[ \frac{\partial q_x}{\partial p_i} / \frac{\partial q_s}{\partial p_i} = \frac{\partial q_x}{\partial \lambda} / \frac{\partial q_s}{\partial \lambda} \]

Thus, in the case of a separable quadratic preference function, the effect of the change in the price of a commodity on commodities belonging to other sub-set/sub-sets is proportional to their income effect. The implication of this is that once the budget allocation for a sub-set has been made by the consumer, intra-sub-set distribution of the expenditure is made solely with reference to the price-structure of the commodities comprising the sub-set. This brings forth the idea of two-stage budgeting whereunder it may be conceived that the consumer first decides how to allocate income to the separable sub-sets and then, for each sub-set makes independent decisions regarding the disbursement of the first-stage budget allotments.
A THEOREM ON EQUIVALENCE OF SLUTSKY'S AND HICKS' INTERPRETATIONS OF SUBSTITUTION EFFECT

1. Introduction

Substitution effect is defined as response of quantity demanded to a "compensated variation of price" that is, to a variation in price accompanied by a compensating change in money income. Compensating change in money income has been conceptualised in two different ways. Slutsky has expressed it in terms of observable phenomenon, taking it equal to the change in price times the quantity demanded at the initial price. In other words, according to Slutsky, the compensating change in money income is such as would enable a consumer, if he chose, to purchase his initial basket of commodities. Hicks on the other hand has regarded it as identical with change in money income required to keep the consumer, if he chose, on his initial level of utility or his highest initial indifference surface. Corresponding to these two definitions of the compensating change of money income, the substitution effect has been given two interpretations in economic literature.

* This paper was prepared when the author was working as Senior Research Fellow at Gokhale Institute of Politics and Economics, Poona. (1970-71).
In this paper an attempt has been made to show how, in the limit, the two interpretations of substitution effect tend to equality. To be more explicit, here we have proved the Equivalence Theorem for a general n-commodities case, within the framework of differential calculus (Cf. sections 2 and 3). It may be noted that earlier attempts by Mosak [3] and by Yokohama [4] in this regard are special cases of ours.

The structure of this paper, in bare outline, is as follows. The paper contains three sections. Section 1 tersely introduces what this paper is about. Section 2 states the Equivalence Theorem. Section 3 dilates on the proof of the theorem.

2. The Equivalence Theorem

Before we proceed to the Equivalence Theorem, we shall briefly introduce the relevant concepts and definitions.

Given a fixed number of n different commodities (including services), \( q_x (x = 1, 2, \ldots, n) \) which a consumer can purchase in a market at given prices, \( p_x (x = 1, 2, \ldots, n) \), his money income, \( e \) and his utility function,

\[
(2.1) \quad u = u (q_1, q_2, \ldots, q_n),
\]

which is assumed to be a continuous differentiable function of \( q_x (x = 1, 2, \ldots, n) \), \( q_x \) being the quantity of \( q_x \).
in the static theory of consumer behaviour, the problem is, inter alia, to determine the equilibrium consumption,
\[ q = (q_1, q_2, \ldots, q_n) \]
of the consumer such that his utility is maximised. The solution is that of conditional maximisation; the vector \( q = (q_1, q_2, \ldots, q_n) \) is to be determined for:

\[(2.2) \quad \text{max. } u \text{ subject to } \sum_{r=1}^{n} p_r q_r = e.\]

Lagrangean technique for solving this problem is to maximize the Lagrangean function,

\[(2.5) \quad s = u - m(\sum_{r=1}^{n} p_r q_r - e).\]

This leads to the system of \((n+1)\) equations:

\[(2.4a) \quad u_r = m p_r \quad (r=1, 2, \ldots, n), \quad \text{and} \]

\[(2.4b) \quad \sum_{r=1}^{n} p_r q_r = e \]

where, \( u_r = \frac{\partial u}{\partial q_r} \) and \( m \) is Lagrangean multiplier.

In (2.4), there are \((n+1)\) equations and \((n+1)\) unknowns, namely, \( q_1, q_2, \ldots, q_n \) and \( m \). We may solve it for the equilibrium consumption \( q = (q_1, q_2, \ldots, q_n) \) as well as for \( m \).
We now turn to the Equivalence Theorem which may be stated as follows. The limiting value of the effect on the equilibrium consumption, \( q_s \) of the \( s \)th commodity of a change in the price, \( p_r \) of the \( r \)th commodity (prices of remaining commodities being held constant) combined with such a change in his money income as would enable the consumer to purchase, if he chose, his equilibrium consumption, \( q = (q_1, q_2, \ldots, q_n) \) is equal to the limiting value of the effect on the equilibrium consumption, \( q_s \) of the \( s \)th commodity of a change in the price, \( p_r \) of the \( r \)th commodity (prices of remaining commodities being held constant) combined with such a change in his money income as would keep the consumer, if he chose, on his initial level of utility or his initial indifference surface. In short this means that, in the limit, the substitution effect in Slutsky's sense tends to be equal to that in Hicks' sense. Equivalently this may be written as

\[
(2.5) \quad \left( \frac{\partial q_s}{\partial p_r} \right)_S = \left( \frac{\partial q_s}{\partial p_r} \right)_H
\]

where, the subscripts 'S' and 'H' are short-hand notations to indicate the fact that the expression on the left hand side of (2.5) stands for the partial derivative of the quantity \( q_s \) with respect to the price \( p_r \); the price change being compensated in Slutsky sense and the expression
on the right hand side of (2,5) stands for the partial derivative of the quantity \( q \) with respect to the price \( p_r \); the price change being compensated in Hicks' sense.

3. Proof

In this section we shall prove the Equivalence Theorem namely, that the left hand side of (2,5) is equal to its right hand side.

First, we shall evaluate the expression on the left hand side of (2,5). Differentiating partially with respect to \( p_r \) both the sides of each equation of (2,4a) and of (2,4b) and noting that in this case the change in price \( p_r \) is accompanied by compensatory change in money income \( e \) such that \( \frac{de}{dp_r} = q_r \)

---

1 To prove how \( \frac{de}{dp_r} = q_r \) at the equilibrium consumption point, \( q = (q_1, q_2, \ldots, q_n) \) corresponding to the given price-income situation, that is, \( p_1 = p_1(i=1,2,\ldots,n) \) and \( E = q_e \), where \( p_1 \) and \( E \) can vary from one price-income situation to another. Let \( p_1 \) be held fixed at \( p_1 \) \( (i \neq r) \) and \( p_r \) change about \( p_r \). Now in conformity with Slutsky's interpretation of the compensating change in money income, let \( E \) change with \( p_r \) in such a way that the initial equilibrium consumption point \( q = (q_1, q_2, \ldots, q_n) \) lies on budget plane determined by new price-income situation:

\[ E = E(p_r) = p_1 q_1 + p_2 q_2 + \cdots + p_{r-1} q_{r-1} + p_r q_r + p_{r+1} q_{r+1} + \cdots + p_n q_n \]

From this, by differentiating and evaluating the derivative at the equilibrium consumption point we obtain

\[ \frac{de}{dp_r} = q_r \] (Also see [2] p.309).
We obtain the following system of equations:

\[(3.1a) \quad -p_s \frac{\partial m}{\partial p_s} + \sum_{j=1}^{n} u_{sj} \frac{\partial q_j}{\partial p_s} = 0 \]

\[(s=1,2,\ldots,r-1, r+1,\ldots,n).\]

\[(3.1b) \quad -p_{r} \frac{\partial m}{\partial p_{r}} + \sum_{j=1}^{n} u_{rj} \frac{\partial q_j}{\partial p_{r}} = m \quad (s = r)\]

where, \(u_{ij} = \frac{\partial q_i}{\partial p_j} \cdot \frac{\partial q_j}{\partial p_i}\)

\[(3.1c) \quad \sum_{j=1}^{n} p_j \frac{\partial q_j}{\partial p_r} = -q_r + \frac{de}{dp_r}\]

\[(3.1d) \quad \sum_{j=1}^{n} p_j = 0 \quad \text{(using (3.2))}\]

Solving the above system of equations (3.1a), (3.1b) and (3.1d), we get

\[(3.2) \quad \begin{bmatrix} \frac{\partial q_1}{\partial p_s} & \frac{\partial q_2}{\partial p_s} & \cdots & \frac{\partial q_n}{\partial p_s} \\ \frac{\partial q_1}{\partial p_r} & \frac{\partial q_2}{\partial p_r} & \cdots & \frac{\partial q_n}{\partial p_r} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q_1}{\partial p_n} & \frac{\partial q_2}{\partial p_n} & \cdots & \frac{\partial q_n}{\partial p_n} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} & p_1 \\ u_{21} & u_{22} & \cdots & u_{2n} & p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{r1} & u_{r2} & \cdots & u_{rn} & p_r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} & p_n \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}\]
From (3.2), we have

\[(3.3) \quad \left( \frac{\partial q_s}{\partial p_r} \right) = \text{m} \frac{|D_{rs}|}{|D|} \quad (s=1,2,\ldots,n)\]

where, \(|D|\) is the determinant of the matrix

\[
\begin{bmatrix}
u_{11} & v_{12} & \cdots & v_{1n} & p_1 \\
v_{21} & v_{22} & \cdots & v_{2n} & p_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
v_{n1} & v_{n2} & \cdots & v_{nn} & p_n \\
p_1 & p_2 & \cdots & p_n & 0
\end{bmatrix}
\]

and \(|D_{rs}|\) is the cofactor of \((r,i)\)th element of \(|D|\).

As a next step we shall evaluate the right hand side of (2.5). Differentiating partially with respect to \(p_r\) both sides of each equation of (2.4a) and of

\[(3.4) \quad u(q_1,q_2,\ldots,q_n) = c\]

where \(c\) is a constant.

---

1 This implies that the compensating change in money income in such as would keep the consumer, if he chose, on his initial level of utility or initial indifference surface represented by \(u(q_1,q_2,\ldots,q_n) = c\).
We obtain the following system of equations:

\[(5.5a)\quad -p_s \frac{\partial m}{\partial p_r} + \sum_{j=1}^{n} u_{bj} \frac{\partial q_j}{\partial p_r} = 0\]

\[(s=1,2,\ldots,n-1,n+1,\ldots,n)\]

\[(5.5b)\quad -p_j \frac{\partial m}{\partial p_r} + \sum_{j=1}^{n} u_{rj} \frac{\partial q_j}{\partial p_r} = m \quad (s = r)\]

\[(5.5c)\quad \sum_{j=1}^{n} u_j \frac{\partial q_j}{\partial p_r} = 0\]

or equivalently

\[(5.5d)\quad \sum_{j=1}^{n} p_j \frac{\partial q_j}{\partial p_r} = 0 \quad (\text{using (2.4a)})\]

---

1 This follows from the partial differentiation of (5.4).
Solving the above system of equations, (3.5a), (3.5d) we get

\[
\begin{pmatrix}
\frac{\partial q_1}{\partial p_r} & \frac{\partial q_2}{\partial p_r} & \cdots & \frac{\partial q_n}{\partial p_r} \\
1 & 2 & \cdots & n
\end{pmatrix} \begin{pmatrix}
u_{11} & u_{12} & \cdots & u_{1n} & p_1 \\
u_{21} & u_{22} & \cdots & u_{2n} & p_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \vdots \\
u_{n1} & u_{n2} & \cdots & u_{nn} & p_n \\
1 & p_2 & \cdots & p_n & 0
\end{pmatrix}^{-1} = \begin{pmatrix}0 \\
0 \\
0 \\
0 \\
m \end{pmatrix}
\]

(3.6)

From (3.6) we have

\[
(3.7) \quad \left( \frac{\partial q_s}{\partial p_r} \right)_s = m \left| \frac{D_{rs}}{\partial} \right| (s=1,2,\ldots,n)
\]

where \( |\partial | \) and \( |D_{rs}| \) have the same connotations as for (3.3).

From (3.3) and (3.7) it is evident that

\[
(3.8) \quad \left( \frac{\partial q_s}{\partial p_r} \right)_s = \left( \frac{\partial q_s}{\partial p_r} \right)_H = m \left| \frac{D_{rs}}{\partial} \right| (s=1,2,\ldots,n)
\]

Hence the Equivalence Theorem is proved.
References


