CHAPTER 7

AVERAGE HARMONIOUS LABELING OF GRAPHS

7.1. INTRODUCTION

In this chapter the researcher has introduced and developed the concept of Average harmonious labeling and Even Average harmonious labeling of some family of graphs. The researcher has proved the Y-tree graph, path $P_n$, the graph $C_3 \oplus pK_1$, the star graph, jelly fish, the jewel graph, the graph $C_3 \cup P_r^2$, the graph $P_r \odot K_1$, the graph $C_3 \cup P_r$, the bistar graph $B_{p,p}$, graph $C_3 \ast K_1,r$, the subdivision bistar graph, triangular snake graph $T_r$, alternative triangular snake graph and complete bipartite graph $K_{p,p}$ are the Average harmonious labeling graphs. In this chapter the researcher has developed the C++ program for finding Average harmonious labeling of the bistar graphs $B_{p,q}$.

Definition 7.1.1. A function $f$ is called Average harmonious labeling of a graph $G(V, E)$ with $n$ vertices and $m$ edges if $f : V \rightarrow \{0, 1, 2, \ldots, m + n\}$ is injective and the induced function $f^* : E \rightarrow \{0, 1, \ldots, (m - 1)\}$ is defined as

$$f^*(uv) = \begin{cases} \frac{f(u) + f(v) + 1}{2} \pmod{m} & \text{if } f(u) + f(v) \text{ is odd} \\ \frac{f(u) + f(v)}{2} \pmod{m} & \text{if } f(u) + f(v) \text{ is even} \end{cases}$$

is bijective, the resulting edge labels should be distinct. A graph which admits an average harmonious labeling is called Average harmonious graph.
Example 7.1.2. The Average harmonious graph is shown in Figure 7.1.1.

![Figure 7.1.1](image)

Theorem 7.1.3. The graph $Y$-tree is Average harmonious.

**Proof:** Consider the graph $Y$-tree with $n$, $(n \geq 3)$ vertices and $m = n - 1$ edges.

Let $V(Y) = \{v_1, v_2, \ldots, v_n\}$ and

Let $E(Y) = \{v_{n-1}v_n, v_{n-2}v_n, v_{n-3}v_n, \ldots, v_2v_1\}$

Define $f: V(Y) \to \{0, 1, 2, \ldots, m + n\}$ by $m + n = 2n - 1$.

$f(v_n) = 2, f(v_{n-1}) = 1, f(v_{n-2}) = 0$

$f(v_i) = (n - i), 1 \leq i < n - 3$.

Then $f$ induces a bijection $f^*: E(Y) \to \{0, 1, 2, \ldots, m - 1\}$.

Hence both vertex labels and edge label are distinct.

Hence the graph $Y$-tree is a Average harmonious graph.

Example 7.1.4. The following example shows that the $Y$-tree graph with 9 vertices is an Average harmonious graph.

![Figure 7.1.2](image)
Theorem 7.1.5. The graph $C_3 \cup P_r^2, (r \geq 4)$ is Average harmonious.

**Proof:** Let $C_3$ be a cycle with 3 vertices and 3 edges and the graph $P_r^2$ with $r$ vertices and $2r - 3$ edges.

Let $V(C_3 \cup P_r^2) = \{v_1, v_2, v_3, u_i, 1 \leq i \leq r\}$

Let $E(C_3 \cup P_r^2) = \{v_1v_2, v_2v_3, v_3v_1, u_iu_{i+1}, 1 \leq i \leq r-1, u_iu_{i+2}, 1 \leq i \leq r-2\}$

$|V(C_3 \cup P_r^2)| = n = r + 3$ and $|E(C_3 \cup P_r^2)| = m = 2r$.

Define an injective function $f: V(C_3 \cup P_r^2) \rightarrow \{0, 1, 2, \ldots, (m + n) = 3r + 3\}$ by

$f(v_1) = 0, f(v_2) = 2, f(v_3) = 4,$

$f(u_i) = 2i + 1, 1 \leq i \leq r.$

Then $f$ induces a bijection $f: E(C_3 \cup P_r^2) \rightarrow \{0, 1, 2, \ldots, m - 1\}$.

Hence both vertex labels and edge label are distinct.

Hence the $C_3 \cup P_r^2$ graph is an Average harmonious graph.

**Example 7.1.6.** The graph $C_3 \cup P_7^2$ is Average harmonious as shown in Figure 7.1.3.
**Theorem 7.1.7.** The comb graph $P_r \odot K_1 (r \geq 1)$ is Average harmonious.

**Proof:** Consider the comb graph with $n = 2r$ vertices and $m = 2r - 1$ edges.

Here $m + n = 4r - 1$ and $m - 1 = 2r - 2$.

Let $V (P_r \odot K_1) = \{ v_i, u_i, 1 \leq i \leq r \}$

Let $E (P_r \odot K_1) = \{ v_i v_{i+1}, u_i v_i, 1 \leq i \leq r - 1, u_r v_r \}$

Define an injective function $f: V (P_r \odot K_1) \rightarrow \{ 0, 1, 2, \ldots, m + n \}$ by

$$f (v_{2i-1}) = 4i - 3, 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor$$

$$f (v_{2i}) = 4i - 2, 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor$$

$$f (u_{2i-1}) = 4i - 4, 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor$$

$$f (u_{2i}) = 4i - 1, 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor$$

Then $f$ induces a bijection $f: E (P_r \odot K_1) \rightarrow \{ 0, 1, 2, \ldots, m - 1 \}$.

Hence the graph $P_r \odot K_1$ is an Average harmonious graph.

**Example 7.1.8.** The graphs $P_5 \odot K_1$ is an Average harmonious graph as shown in Figure 7.1.4.
**Theorem 7.1.9.** The star graph $S_n$ is an Average harmonious graph for all $n$.

**Proof:** Let $S_n$ be a star graph with $n$ vertices and $m = n - 1$ edges, $m + n = 2n - 1$ and $m - 1 = n - 2$.

Consider $v$ as the center vertex.

Let $V(S_n) = \{ v, v_i ; 1 \leq i \leq n - 1 \}$

Let $E(S_n) = \{ vv_i ; 1 \leq i \leq n - 1 \}$

Define an injective function $f : V(S_n) \rightarrow \{0, 1, 2, \ldots, m + n\}$ by

$f(v) = 1$, $f(v_1) = 0$,

$f(v_i) = 2i - 1$, $2 \leq i \leq n - 1$.

Then $f$ induces a bijection $f : E(S_n) \rightarrow \{0, 1, 2, \ldots, m - 1\}$.

Hence the $S_n$ graph is an Average harmonious graph.

**Example 7.1.10.** The $S_7$ graph is an Average harmonious graph as shown in Figure 7.1.5.

![Figure 7.1.5.](image-url)
**Theorem 7.1.11.** The path graph $P_n$, $(n \geq 2)$ is Average harmonious.

**Proof:** Let $P_n$ be a path graph with $n$ vertices and $m = n - 1$ edges.

Let $V(P_n) = \{v_i; 1 \leq i \leq n\}$

Let $E(P_n) = \{v_i v_{i+1}; 1 \leq i \leq n - 1\}$

Define an injective function $f: V(P_n) \to \{0, 1, 2, \ldots, m + n\}$ by

$$f(v_i) = i - 1, 1 \leq i \leq n$$

Then $f$ induces a bijection $f^*: E(P_n) \to \{0, 1, 2, \ldots, m - 1\}$.

Hence the $P_n$ graph is an Average harmonious graph.

**Example 7.1.12.** The $P_7$ graph is Average harmonious as shown in Figure 7.1.6.

![Figure 7.1.6.](https://via.placeholder.com/150)

**Theorem 7.1.13.** The graph $C_3 \cup P_r$, $(r \geq 2)$ is an Average harmonious graph.

**Proof:** Let $C_3$ be a cycle with 3 vertices and 3 edges.

Let $P_r$ be a path with $r$ vertices and $r - 1$ edges.

Let $V(C_3 \cup P_r) = \{v_1, v_2, v_3, u_i, 1 \leq i \leq r\}$

Let $E(C_3 \cup P_r) = \{v_1 v_2, v_2 v_3, v_3 v_1, u_i u_{i+1}, 1 \leq i \leq r - 1\}$

$\mid V(C_3 \cup P_r) \mid = n = r + 3$ and $\mid E(C_3 \cup P_r) \mid = m = r + 2$

Hence $m + n = 2r + 5$ and $m - 1 = r + 1$

Define an injective function $f: V(C_3 \cup P_r) \to \{0, 1, 2, \ldots, m + n\}$ by
\( f(v_1) = 1, f(v_2) = 3, \)

\( f(v_3) = 5, f(u_i) = 0, \)

\( f(u_1) = 2i, 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor \)

\( f(u_{2i+1}) = 2i + 5, 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor \)

Then \( f \) induces a bijection \( f : E(C_3 \cup P_r) \to \{0,1,2,\ldots,m-1\} \).

Hence the \( C_3 \cup P_r \) graph is an Average harmonious graph.

**Example 7.1.14.** The graph \( C_3 \cup P_7 \) is an Average harmonious graph as shown in Figure 7.1.7.

**Theorem 7.1.15.** The graph \( C_3 \circ pK_1 (p \geq 1) \) is Average harmonious.

**Proof:** Let \( v_1, v_2, v_3 \) be the vertices of \( C_3 \) and \( u_1, u_2, \ldots, u_p \) be the new vertices.

Let \( V(C_3 \circ pK_1) = \{v_1, v_2, v_3, u_1, u_2, \ldots, u_p\} \)

Let \( E(C_3 \circ pK_1) = \{v_1v_2, v_2v_3, v_3v_1, v_2u_1, v_2u_2, \ldots, v_2u_p\} \), here \( m = n = p + 3 \).

Define an injective function \( f : V(C_3 \circ pK_1) \to \{0,1,2,\ldots,m + n\} \) by

\( f(v_1) = 0, \)
The edge labels and vertex labels are distinct.

Thus the given graph is an Average harmonious graph.

Example 7.1.16. The graph $C_3 @ 5K_1$ is a Average harmonious graph is shown in Figure 7.1.8.

Theorem 7.1.17. The bistar graph $B_{p,p}$ , $(p \geq 1)$ is Average harmonious.

Proof: Let the bistar graph $B_{p,p}$ with $n = 2p + 2$ vertices and $m = 2p + 1$ edges.

Let $V(B_{p,p}) = \{u,v, u_i, v_i, 1 \leq i \leq p, \}$

Let $E(B_{p,p}) = \{u_iu, v_i v, \ 1 \leq i \leq p, uv \}$, here $m + n = 4p + 3$

Define an injective function $f: V(B_{p,p}) \to \{0, 1, 2, ..., m + n\}$ by

$f(u_1) = 1, f(u) = 0,$

$f(u_i) = 4i - 1, 2 \leq i \leq p.$

$f(v) = 3,$

$f(v_i) = 4i - 2, 1 \leq i \leq p$
Then $f$ induces a bijection $f^* : E(B_{p,p}) \rightarrow \{0, 1, 2, \ldots, m - 1\}$.

The edge labels are given below:

$f^*(uu_i) = 1,$

$f^*(u_iu) = 2i \pmod{m}, \ 2 \leq i \leq p.$

$f^*(uv) = 2,$

$f^*(v_i v) = (2i + 1) \pmod{m}, \ 1 \leq i \leq p.$

Hence the bistar graph is an Average harmonious graph.

**Example 7.1.18.** The graph $B_{5,5}$ is an Average harmonious graph as shown in Figure 7.1.9.

![Figure 7.1.9.](image)

**Theorem 7.1.19.** The graph $C_3 \ast K_{l,r}$ with $n = r + 4$ vertices and $m = r + 4$ edges.

**Proof:** Consider the graph $C_3 \ast K_{l,r}$ with $n = r + 4$ vertices and $m = r + 4$ edges.

Hence $m + n = 2r + 8$ and $m - 1 = r + 3$

Let $V(C_3 \ast K_{l,r}) = \{v_l, v_2, v_3, u_i, \ 1 \leq i \leq r + 1\}$

Let $E(C_3 \ast K_{l,r}) = \{v_l v_2, v_2 v_3, v_3 v_l, u_i v_l, u_i u_{i+1}, \ 1 \leq i \leq r\}$

Define an injective function $f : V(C_3 \ast K_{l,r}) \rightarrow \{0, 1, 2, \ldots, m + n\}$ by

$f(v_l) = 0, \ f(v_2) = 1, \ f(v_3) = 4, \ f(u_i) = 7,$
\[ f(u_2) = 3, \ f(u_3) = 5, \ f(u_i) = 2i - 2, \ 4 \leq i \leq (r + 1). \]

Then \( f \) induces a bijection \( f^* : E(C_3 \ast K_{1,r}) \to \{0, 1, 2, \ldots, m - 1\} \).

The edge are labeled as
\[
\begin{align*}
f^*(v_1v_2) &= 1, \ f^*(v_1v_3) = 2, \\
f^*(v_2v_3) &= 3, \ f^*(v_1u_l) = 4, \\
f^*(u_1u_j) &= j + 3(\text{mod } m), \ 2 \leq j \leq r + 1.
\end{align*}
\]

ence the graph is an Average harmonious graph.

**Example 7.1.20.** The graph \( C_3 \ast K_{1,6} \) is Average harmonious as shown in Figure 7.1.10.

**Theorem 7.1.21.** The Jewel graph \( J_r, (r \geq 1) \) is Average harmonious.

**Proof:** The graph \( J_r \) has \( n = r + 4 \) vertices and \( m = 2r + 5 \) edges.

Let \( V(J_r) = \{u_1, u_2, u_3, u_i, v_i, \ 1 \leq i \leq r\} \).
Edge set $E(J_r) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\} \cup \{u_i v_i, u_3v_i, 1 \leq i \leq r\}$.

Here $m + n = 3r + 9$ and $m - 1 = 2r + 4$

Define an injective function $f : V(J_r) \to \{0, 1, 2, \ldots, m + n\}$ by

$f(u_1) = 0$,  $f(u_2) = 2$,
$f(u_3) = 2r + 6$,  $f(u_4) = 2r + 4$

$f(v_i) = 2(i + 1), 1 \leq i \leq r$.

Then $f$ induces a bijection $f^* : E(J_r) \to \{0, 1, 2, \ldots, m - 1\}$.

Hence the Jewel graph is an Average harmonious graph.

Example 7.1.22. The Average harmonious Jewel graph $J_3$ is shown in Figure 7.1.11.
**Theorem 7.1.23.** A triangular snake $T_r (r \geq 2)$ admits Average harmonious labeling.

**Proof:** Let $V(T_r) = \{ u_i, 1 \leq i \leq r ; v_i, 1 \leq i \leq r - 1 \}$

Let $E(T_r) = \{ u_i u_{i+1} \ if \ 1 \leq i \leq r - 1 $ \n
$u_i v_i \ if \ 1 \leq i \leq r - 1 $ \n
$u_{i+1} v_i \ if \ 1 \leq i \leq r - 1 $ \n

$| V(T_r) | = n = 2r - 1 \text{ and } | E(T_r) | = m = 3r - 3$

Define an injective function $f : V(T_r) \rightarrow \{ 0, 1, ..., m + n \}$ by

$f(u_i) = (i - 1), 1 \leq i \leq r$ \n
$f(v_i) = 2r + 3(i - 1), 1 \leq i \leq r - 1$.

Hence both vertex labels and edge label are distinct.

Therefore $f$ is an injection and the induced function $f^* : E(T_r) \rightarrow \{ 0, 1, ..., m - 1 \}$ is a bijection. It can be observed that the triangle snake graph is an Average harmonious graph.

**Example 7.1.24.** The triangular snake $T_5$ is an Average harmonious labeling graph as shown in Figure 7.1.12.
Theorem 7.1.25. The alternate triangular snake $A(T_r)$ ($r = 2k$, $k \geq 2$) is Average harmonious labeling.

Proof:

Case (i): The triangle starts from the second vertex.

Let $A(T_r)$ be an alternate triangular snake with $n = \frac{3r - 2}{2}$ vertices and $m = 2r - 3$ edges. Here $m + n = \frac{7r - 8}{2}$ and $m - 1 = 2r - 4$

Let $V(A(T_r)) = \{ u_i, 1 \leq i \leq r; \; v_i, 1 \leq i \leq \frac{(r-2)}{2} \}$

Let $E(A(T_r)) = \{ u_i u_{i+1} \; \text{if} \; 1 \leq i \leq r - 1 \}
\{ u_{2i} v_i \; \text{if} \; 1 \leq i \leq \frac{(r-2)}{2} \}
\{ u_{2i+1} v_i \; \text{if} \; 1 \leq i \leq \frac{(r-2)}{2} \}$

Define an injective function $f: V(A(T_r)) \rightarrow \{ 0, 1, \ldots, m + n \}$

by $f(u_i) = (i - 1) , \; 1 \leq i \leq r$

$f(v_i) = 2r - 3 + 2i, \; 1 \leq i \leq \frac{(r-2)}{2}$

Therefore $f$ is an injection and the induced function $f^*: E(A(T_r)) \rightarrow \{ 0,1,\ldots,m-1 \}$
is a bijection. It can be observed that the alternative triangle snake graph is an Average harmonious graph.

Case (ii) The triangle starts from the first vertex.

Let $A(T_r)$ be a alternate triangular snake with $n = \frac{3r}{2}$ vertices and $m = 2r - 1$ edges.

Let $V(A(T_r)) = \{ u_i, 1 \leq i \leq r; \; v_i, 1 \leq i \leq \frac{r}{2} \}$
Let \( E(A(T_r)) = \begin{cases} u_i u_{i+1} & \text{if } 1 \leq i \leq r-1 \\ u_{2i-1} v_i & \text{if } 1 \leq i \leq \frac{r}{2} \\ u_{2i} v_i & \text{if } 1 \leq i \leq \frac{r}{2} \end{cases} \)

Define an injective function \( f: V(A(T_r)) \rightarrow \{0, 1, \ldots, m + n\} \)

\[ f(u_i) = (i - 1), \quad 1 \leq i \leq r \]

\[ f(v_i) = 2(r - 1 + i), \quad 1 \leq i \leq \frac{r}{2} \]

Therefore \( f \) is an injection and the induced function \( f^\#: E(A(T_r)) \rightarrow \{0, 1, \ldots, m - 1\} \)

is a bijection. It can be observed that the alternative triangle snake graph is an Average harmonious graph.

**Example 7.1.26.** The alternate triangular snake \( A(T_8) \) is an Average harmonious labeling graph as shown in Figure 7.1.13.

**Example 7.1.27.** The alternate triangular snake \( A(T_6) \) is an Average harmonious graph as shown in Figure 7.1.14.
**Theorem 7.1.28.** The Jelly fish graph $JF(p, p+2)$, $(p \geq 1)$ is an Average harmonious graph.

**Proof:** Consider the Jelly fish graph $JF(p, p+2)$ with $n = 2p + 6$ vertices and $m = 2p + 7$ edges.

Let $u_1, u_2, u_3, u_4$ be the vertices of the cycle $C_4$.

Let $V(JF(p, p+2)) = \{u_i, 1 \leq i \leq 4, v_j, 1 \leq j \leq p, w_k, 1 \leq k \leq p + 2\}$ and $E(JF(p, p+2)) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_iw_k, 1 \leq k \leq p + 2, u_jv_j, 1 \leq j \leq p\}$

Define an injective function $f: V(JF(p, p+2)) \rightarrow \{0, 1, 2, \ldots, m + n\}$ by $f(u_1) = 0, f(u_3) = 3, f(w_k) = 2k, 1 \leq k \leq p + 2, f(u_2) = 2p + 6, f(u_4) = 2p + 8$ and $f(v_j) = 2p + 8 + 2j, 1 \leq j \leq p$.

Then $f$ induces a bijection $f^*: E(JF(p, p+2)) \rightarrow \{0, 1, 2, \ldots, m - 1\}$

Hence the graph $JF(p, p+2)$ is an Average harmonious graph.

**Example 7.1.29.** The graphs $JF(4,6)$ is an average harmonious graph as shown in Figure 7.1.15.
**Theorem 7.1.30.** The bistar graph $B_{p,p}$ with the subdivision of the central edge $K_2$ is an Average harmonious graph.

**Proof:** Let $u$ and $v$ be the vertices of the graph $K_2$.

Let $w$ be the vertex that divide the edge $K_2$.

The vertices of the bistar subdivision graph are $u, v, w, u_i (1 \leq i \leq p)$ and $v_i (1 \leq i \leq p)$.

The edges are $uu_i, (1 \leq i \leq p)$, $vv_i, (1 \leq i \leq p)$, $uw$ and $vw$.

Hence there are $2p + 3$ vertices and $2p + 2$ edges. Here $m + n = 4p + 5$ and

$m - 1 = 2p + 1$

Define an injective function $f : V[B_{p,p}] \rightarrow \{0, 1, \ldots, m + n\}$ by $f(u) = 1, f(v) = 3$ and $f(w) = 2$.

$f(u_i) = 0$ and $f(v_i) = 4$

Also $f(u) = 4i, \ 2 \leq i \leq p$ and $f(v_i) = 4i + 1, \ 2 \leq i \leq p$.

Define the induced function $f^*: E[B_{p,p}] \rightarrow \{0, 1, \ldots, m - 1\}$ by

$f^*(u_iu) = 1, f^*(u_iu) = 2i + 1 \ (mod \ m), \ 2 \leq i \leq p$

$f^*(uw) = 2, f^*(wv) = 3,$

$f^*(v_i) = 2i + 2 \ (mod \ m), \ 1 \leq i \leq p$

Hence both vertex labels and edge label are distinct.

Thus the bistar subdivision graph is an Average harmonious graph.
Example 7.1.31. The subdivision bistar graph $B_{4,4}$ is an Average harmonious graph as shown in Figure 7.1.16.

![Figure 7.1.16.](image)

Theorem 7.1.32. The complete bipartite graph $K_{p,p}$ is an Average harmonious graph.

**Proof:** A complete bipartite graph is a simple bipartite graph with bipartition of the vertex set $V$ into $X$ and $Y$ in which each vertex of $X$ is joined to each vertex of $Y$.

Let the vertices of the set $X$ be $u_i$, $1 \leq i \leq p$ and the vertices of the set $Y$ be $v_j$, $1 \leq i \leq p$.

The edges are $u_i v_j$, $1 \leq i \leq p$, $1 \leq j \leq p$

Hence there are $n = 2p$ vertices and $m = p^2$ edges.

Define an injective function $f : V[K_{p,p}] \to \{0, 1, \ldots, m + n\}$

by $f(u_i) = 2i - 1$, $1 \leq i \leq p$,

$f(v_1) = 0$ and $f(v_i) = 2(i - 1)p$, $2 \leq i \leq p$.

Example 7.1.33. The complete bipartite graph $K_{4,4}$ is an Average harmonious graph as shown in Figure 7.1.17.

![Figure 7.1.17. The complete bipartite graph $K_{4,4}$](image)
C++ program for finding the Average label of Bi-star graph

#include<iostream.h>
#include<conio.h>

int main()
{

c1rscr();

int ui,i,ui1,ua,a,vj,j,b,vb;

int m,n;

cout<<"enter m n value vertex label";

cin>>m>n;

{

cout<<"u1=1" <<"\n";
}

for(i=2;i<=m;i++)
{

ui=4*i-2;

cout<<"u"<<i<<"="<<ui<<"\t";
}

{

a=m; ua=0;
}
cout<<"u"<<a<<"="<<ua<<"\n";
cout<<"\n";
}

for(j=1;j<=n;j++)
{
    vj=4*j-2;
    cout<<"v"<<j<<"="<<vj<<"\n";
}

{ b=n+1; vb=3;cout<<"v"<<b<<"="<<vb<<"\n";
}

getch();
return 0;
}

Output for the Average harmonious labeling of the Bistar graph $B_{16,9}$

enter m n value vertex label 16 9

u1=1 u2=6 u3=10 u4=14 u5=18 u6=22 u7=26 u8=30 u9=34 u10=38
u11=42 u12=46 u13=50 u14=54 u15=58 u16=62 u17=0

v1=2 v2=6 v3=10 v4=14 v5=18 v6=22 v7=26 v8=30 v9=34 v10=3.
7.2. EVEN AVERAGE HARMONIOUS GRAPHS.

7.2.1. INTRODUCTION.

In this section, the researcher proves the path $P_n$, cycle $C_n$, graph $C_3 \oplus pK_1$, Ladder $L_r$, star graph $S_n$, comb graph $P_r \oplus K$, the wheel graph $W_r$, graph $P_n^2$ and helm graph $H_r$ are even average harmonious graphs.

**Definition 7.2.2.** A function $f$ is called even average harmonious labeling of a graph $G(V,E)$ with $n$ vertices and $m$ edges if $f: V \rightarrow \{0, 1, 2, ..., 2n\}$ is injective and the induced function $f^*: E \rightarrow \{0, 1, ..., (m - 1)\}$ is defined as $f^*(uv) = \left(\frac{f(u) + f(v)}{2}\right) \text{ (mod } m\text{)}$ is bijective, the resulting edge labels should be distinct. A graph which admits an even average harmonious labeling is called an even average harmonious graph.

**Example 7.2.3.** The even average harmonious graph is shown in Figure 7.2.1.

![Figure 7.2.1](image)

**Theorem 7.2.4.** The path $P_n$, $(n \geq 2)$ is an even average harmonious graph, when $n$ is even.

**Proof:** Let $V = \{v_i, 1 \leq i \leq n \}$ and $E = \{v_iv_{i+1}, 1 \leq i \leq (n - 1)\}$
The number of vertices and edges are \( n \) and \( m = (n - 1) \) respectively.

Define an injective function \( f: V(P_n) \rightarrow \{0, 1, 2, \ldots, 2n\} \) by

\[
f(v_k) = 2i - 2, \quad 1 \leq i \leq n.
\]

Then \( f \) induces a bijection

\[
f^*: E(P_n) \rightarrow \{0, 1, 2, \ldots, m - 1\}.
\]

The edge labels are as follows:

\[
f^* (v_i v_{i+1}) = 2^i - 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} - 1 \right\rfloor,
\]

\[
f^* (v_{(n/2)} v_{(n/2) + 1}) = 0,
\]

\[
f^* (v_{(n/2) + j} v_{(n/2) + j + 1}) = 2^j, \quad 1 \leq j \leq \left\lfloor \frac{n}{2} - 1 \right\rfloor.
\]

Hence both vertex labels and edge label are distinct.

Hence the path graph is an even average harmonious graph.

**Example 7.2.5.** The graph \( P_6 \) is an even average harmonious graph as shown in Figure 7.2.2.

![Figure 7.2.2.](image)

**Theorem 7.2.6.** The cycle \( C_n, (n \geq 3) \) is an even average harmonious graph, when \( n \) is odd.

**Proof:** Let \( C_n \) be a cycle with \( n \) vertices and \( m = n \) edges, where \( n \) is an odd integer.

Let \( V(C_n) = \{v_i, 1 \leq i \leq n\} \)

Let \( E(C_n) = \{v_i v_{i+1}, 1 \leq i \leq n - 1, \ v_n v_1\} \)
Define an injective function $f : V(C_n) \rightarrow \{0, 1, 2, \ldots, 2n\}$ by

$$f(v_i) = 2(i - 1), \ 1 \leq i \leq n.$$ 

Then $f$ induces a bijection $f^* : E(C_n) \rightarrow \{0, 1, 2, \ldots, m - 1\}$.

The edge labels are as follows:

- $f^*(v_i v_{i+1}) = (2i - 1) \pmod{m}, \ 1 \leq i \leq \left(\frac{n-1}{2}\right)$,
- $f^*(v_0 v_n) = n - 1$,
- $f^*(\frac{v_{n+1}}{2} \frac{v_{n+1+1}}{2}) = 2j \pmod{m}, \ 1 \leq j \leq \left(\frac{n-3}{2}\right)$,
- $f^*(\frac{v_{n+1}}{2} \frac{v_{n+1+1}}{2}) = 0$

Hence both vertex labels and edge label are distinct.

Hence the cycle graph $C_n$, $(n \geq 3)$ is an even average harmonious graph.

**Example 7.2.7.** The graph $C_7$ is an even average harmonious graph as shown in Figure 7.2.3.
**Theorem 7.2.8.** The graph $C_3 @ pK_1 (p \geq 1)$ is an even average harmonious graph.

**Proof:** Let $v_1, v_2, v_3$ be the vertices of the cycle $C_3$ and $u_1, u_2, \ldots, u_p$ be the new vertices.

Then the vertices of the graph $C_3 @ pK_1$ are $v_1, v_2, v_3, u_1, u_2, \ldots, u_p$.

The edges of the graph $C_3 @ pK_1$ are \{ $v_1v_2, v_2v_3, v_3v_1, v_2u_1, v_2u_2, \ldots, v_2u_p$ \}

$|V| = |E| = p + 3 = m = n$.

Define an injective function $f : V (C_3 @ pK_1) \rightarrow \{ 0, 1, 2, \ldots, 2n \}$ by

$f(v_1) = 0, f(v_2) = 2, f(v_3) = 4$ and

$f(u_i) = 2(i + 2), 1 \leq i \leq p$

Then $f$ induces a bijection $f^* : E(C_3 @ pK_1) \rightarrow \{ 0, 1, 2, \ldots, m - 1 \}$.

The edge labels are given below:

$f^* (v_1v_2) = 1, f^*(v_2v_3) = 3,$

$f^*(v_3v_1) = 2,$

$f^*(v_2u_i) = (3 + i) \pmod{m}, 1 \leq i \leq p$.

Hence both vertex labels and edge label are distinct.

Hence the graph $C_3 @ pK_1$ is an even average harmonious graph.

**Example 7.2.9.** The graph $C_3 @ 4K_1$ is an even average harmonious graph as shown in Figure 7.2.4.

![Figure 7.2.4](image-url)
**Theorem 7.2.10.** The Ladder graph \( L_r \) is even average harmonious, when \( r \) is odd.

**Proof:** Let the vertices of the ladder be \( \{ v_i, u_i; 1 \leq i \leq r \} \)

Let the edges be \( \{ v_i v_{i+1}, u_i u_{i+1}; 1 \leq i \leq r - 1 \} \cup \{ v_i u_i, 1 \leq i \leq r \} \).

The number of vertices and edges are \( n = 2r \), \( m = 3r - 2 \).

The modulo taken is \( m = 3r - 2 \).

**Case:** (i) Label the vertices with even numbers.

Define an injective function \( f : V( L_r ) \to \{ 0, 1, \ldots, 2n \} \)

\[
\begin{align*}
    f(u_{2i}) &= 2(i - 1), \ 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor \\
    f(u_2) &= r + 2i - 1, \ 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor \\
    f(v_{2i}) &= 3r + 2i - 3, \ 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor \\
    f(v_2) &= 2r + 2(i - 1), \ 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor 
\end{align*}
\]

Then \( f \) induces a bijection \( f^* : E( L_r ) \to \{ 0, 1, 2, \ldots, m - 1 \} \).

Hence the graph \( L_r \) is an even average harmonious graph.

**Case:** (ii) Label the vertices with odd numbers.

Define an injective function \( f : V( L_r ) \to \{ 1, 3, 5, \ldots, 2n - 1 \} \), \( n = 2r \).

\[
\begin{align*}
    f(u_{2i+1}) &= 2i - 1, \ 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor \\
    f(u_2) &= r + 2i, \ 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor \\
    f(v_{2i+1}) &= 3r + 2i - 2, \ 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor \\
    f(v_2) &= 2r + (2i - 1), \ 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor 
\end{align*}
\]
Then $f$ induces a bijection $f^*: E(L_r) \to \{0, 1, 2, \ldots, m-1\}$

Hence both vertex labels and edge label are distinct.

Hence the graph $L_7$ is an even average harmonious graph.

**Example 7.2.11.** The graph $L_7$ is an even average harmonious graph as shown in Figure 7.2.5. for case (i)

![Figure 7.2.5, Ladder $L_7$](image)

**Example 7.2.12.** The graph $L_7$ is even average harmonious as shown in Figure 7.2.6. for case (ii)

![Figure 7.2.6, ladder $L_7$](image)

**Theorem 7.2.13.** The star graph $S_n$ ($n \geq 2$) is an even average harmonious graph.

**Proof:** Let $S_n$ be a star graph with $n$ vertices and $m = n - 1$ edges.

Consider $v$ as the center vertex.
Let $V(S_n)=\{v, v_i \mid 1 \leq i \leq n-1\}$

Let $E(S_n)=\{vv_i \mid 1 \leq i \leq n-1\}$

Define an injective function $f: V(S_n) \rightarrow \{1,2,3,\ldots,2n\}$

**Case (i):** Label the vertices with even numbers.

Define a bijective function $f: V(S_n) \rightarrow \{2,4,6,\ldots,2n\}$ by

$$f(v) = 2, \quad f(v_i) = 2(i+1), \quad 1 \leq i \leq n-1$$

Then $f$ induces a bijection $f^*: E(S_n) \rightarrow \{0, 1, 2, \ldots, m-1\}$.

Hence the graph $S_n$ is an even average harmonious graph.

**Case (ii):** Label the vertices with odd numbers.

Define a bijective function $f: V(S_n) \rightarrow \{1,3,5,\ldots,2n-1\}$ by

$$f(v) = 1, \quad f(v_i) = 2i+1, \quad 1 \leq i \leq n-1.$$  

Then $f$ induces a bijection $f^*: E(S_n) \rightarrow \{0, 1, 2, \ldots, m-1\}$

Hence the graph $S_n$ is an even average harmonious graph.

**Example 7.2.14.** The even average harmonious $S_6$ is shown in Figure 7.2.7. for case (i).

![Figure 7.2.7. Star graph $S_6$](image-url)
**Example 7.2.15.** The even average harmonious graph $S_6$ is shown in Figure 7.2.8. for case (ii).

![Figure 7.2.8. Star graph $S_6$](image)

**Theorem 7.2.16.** The comb graph $P_r \odot K_1 \ (r \geq 1)$ is an even average harmonious graph.

**Proof:** Consider the comb graph with $n = 2r$ vertices and $m = (2r - 1)$ edges.

Let $V(P_r \odot K_1) = \{u_i, 1 \leq i \leq r, v_i, 1 \leq i \leq r, \}$

Let $E(P_r \odot K_1) = \{u_i u_{i+1}, 1 \leq i \leq r - 1, u_i v_i, 1 \leq i \leq r\}$

Define an injective function $f: V(P_r \odot K_1) \rightarrow \{0, 1, 2, \ldots, 2n\}$ by

$f(u_{2i-1}) = 8i - 6, \ 1 \leq i \leq \lfloor \frac{r}{2} \rfloor$

$f(u_{2i}) = 8i - 4, \ 1 \leq i \leq \lfloor \frac{r}{2} \rfloor$

$f(v_{2i-1}) = 8i - 8, \ 1 \leq i \leq \lfloor \frac{r}{2} \rfloor$

$f(v_{2i}) = 8i - 2, \ 1 \leq i \leq \lfloor \frac{r}{2} \rfloor$

Then $f$ induces a bijection $f^*: E(P_r \odot K_1) \rightarrow \{0, 1, 2, \ldots, m - 1\}$.

Hence both vertex labels and edge label are distinct.
Hence the graph $P_r \odot K_f$ is an even average harmonious graph.

**Example 7.2.17.** The even average harmonious graph $P_4 \odot K_1$ is shown in Figure 7.2.9.

![Figure 7.2.9, Comb graph $P_4 \odot K_1$](image)

**Theorem 7.2.18.** The wheel graph $W_r \ (r \geq 3)$, is an even average harmonious graph, when $r$ is odd.

**Proof:** The wheel $W_r$ is obtained by joining all nodes of the cycle graph $C_r$ to a further node called the center and contains $r + 1$ nodes and $2r$ edges.

Hence $n = r + 1$ and $m = 2r$.

The modulo taken is $6r$.

Let the vertices of the cycle $C_r$ be $v_1, v_2, \ldots, v_r$ and the center node be $v$.

That is, $V(W_r) = \{ v, v_i ; 1 \leq i \leq r \}$.

Let the edge label of the wheel graph be

$\{ v_i v_{i+1}, 1 \leq i \leq r - 1, v_r v_1 \} \cup \{ v v_i, 1 \leq i \leq r \}$

Define an injective function $f: V(W_r) \to \{ 0, 1, 2, \ldots, 2n \}$, where $2n = 2r + 2$, by

$f(v) = 2, \ f(v_i) = 4(i - 1), \ 1 \leq i \leq r.$
Then $f$ induces a bijection $f^* : E(W_r) \to \{0, 1, 2, \ldots, m - 1\}$.

Hence both vertex labels and edge label are distinct.

Hence the graph $W_r$ is an even average harmonious graph.

**Example 7.2.19.** The even average harmonious graph $W_5$ is shown in Figure 7.2.10.

![Figure 7.2.10, wheel graph $W_5$](image)

**Theorem 7.2.20.** The graph $P_n^2$ $(n \geq 4)$ is even average harmonious, when $n$ is even.

**Proof:** Let $P_n^2$ be a graph with $n$ vertices and $m = 2n - 3$ edges.

The modulo is $2m = 4n - 6$.

**Case:** (i) Label the vertices with even numbers.

Define an injective function $f: V(P_n^2) \to \{0, 1, 2, \ldots, 2n\}$

by $f(v_i) = 2(i - 1); 1 \leq i \leq n$

Then $f$ induces a bijection $f^* : E(P_n^2) \to \{0, 1, 2, \ldots, m - 1\}$.

Hence the graph $P_n^2$ is an even average harmonious graph.

**Case:** (ii) Label the vertices with odd numbers.

Define an injective function $f: V(P_n^2) \to \{0, 1, 2, \ldots, 2n\}$

by $f(v_i) = 2i - 1; 1 \leq i \leq n$
Then $f$ induces a bijection $f^*: E(P_n^2) \rightarrow \{0, 1, 2, \ldots, m - 1\}$.

Hence both vertex labels and edge label are distinct.

Hence the graph $P_n^2$ is an even average harmonious graph.

**Example 7.2.21.** The even average harmonious $P_6^2$ is shown in Figure 7.2.11.

for case (i)

![Figure 7.2.11, The graph $P_6^2$](image1)

**Example 7.2.22.** The even average harmonious $P_6^2$ is shown in Figure 7.2.12.

for case (ii).

![Figure 7.2.12, The graph $P_6^2$](image2)

**Theorem 7.2.23.** The helm graph $H_r, (r \geq 3)$ is an even average harmonious, when $r$ is odd.

**Proof:** Let $v_1, v_2, \ldots, v_r$ be the vertices of the cycle $C_r$ and $v$ as the center vertex.
Let \( u_1, u_2, \ldots, u_r \) be the vertices of the pendant edges.

The helm \( H_r \) with \( 2r + 1 \) vertices and \( 3r \) edges and \( v \) as the center vertex.

That is \( n = 2r + 1 \) and \( m = 3r \).

The modulo taken is \( m = 3r \).

Let \( E(H_r) = \{ vv_i, 1 \leq i \leq r \} \cup \{ v_i v_{i+1}, 1 \leq i \leq r-1, v, v_1 \} \cup \{ u_i v_i, 1 \leq i \leq r \} \).

Define an injective function \( f : V(H_r) \to \{ 0, 1, 2, \ldots, 2n \} \)

by \( f(v) = 2 \),

\[ f(v_i) = 6i - 2, \quad 1 \leq i \leq r, \]

\[ f(u_i) = 6(i-1), \quad 1 \leq i \leq r. \]

Then \( f \) induces a bijection \( f^* : E(H_r) \to \{ 0, 1, 2, \ldots, m - 1 \} \).

Hence both vertex labels and edge label are distinct.

Hence the graph \( H_r \) is an even average harmonious graph.
Example 7.2.24. The graph $H_5$ is an even average harmonious graph as shown in Figure 7.2.13.

Figure 7.2.13, helm graphs $H_5$
C++ program for finding the even average label of path graph $P_n$

```cpp
#include<iostream.h>
#include<conio.h>

int main()
{
    clrscr();
    int vi,i,vj;
    int n;
    cout<<"enter n value vertex label";
    cin>>n;
    for(i=1;i<=n;i++)
    {
        vi=2*(i-1)
        cout<<"v"<<i<<"="<<vi<<"\n";
        cout<<"\n";
    }
    getch();
    return 0;
}
```
Output for the even average harmonious labeling of the graph $P_{20}$

Enter n value - vertex label 20

$v_1=0$  $v_2=2$  $v_3=4$  $v_4=6$  $v_5=8$  $v_6=10$  $v_7=12$  $v_8=14$  $v_9=16$  $v_{10}=18$

$v_{11}=20$  $v_{12}=22$  $v_{13}=24$  $v_{14}=26$  $v_{15}=28$  $v_{16}=30$  $v_{17}=32$  $v_{18}=34$  $v_{19}=36$

$v_{20}=38$. 